Optimization: Examples

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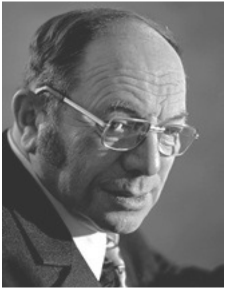

L.V. Kantorovich
(1912-1986)

• Kantorovich
(Soviet mathematician) was awarded
the Nobel Prize in Economics in 1975 for his
contributions to optimal resource allocation.

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where $x \in \mathrm{dom}\,(c)$ and $\mathcal{F} \subseteq \mathbb{R}^n$ represents the domain determined by the constraints.

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• **Implicit constraints.** An algorithm/oracle/subroutine decides whether a given $x \in \mathbb{R}^n$ satisfies the constraints:

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• **Explicit constraints.** The condition is described by a finite set of equations and/or inequalities:

$$\begin{cases} f_i(x) \leq 0, & i \in [k] := \{1, 2, \dots, k\}, \\ g_j(x) = 0, & j \in [\ell], \end{cases} \tag{1.1}$$

where f_i ($i \in [k]$) and g_j ($j \in [\ell]$) are also real-valued functions of n variables. In this case, $\mathcal{F}$ is the solution set of the above system.

Domain of Definition, Feasible Solutions

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• The **domain of definition of the optimization problem** is the intersection of the domains of definition of the objective function and the constraints. For explicit constraints (1.1), this is

$$\mathcal{D} := \text{dom}\,(c) \cap \left(\bigcap_{i=1}^{k} \text{dom}\,(f_i)\right) \cap \left(\bigcap_{j=1}^{\ell} \text{dom}\,(g_j)\right),$$

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• **Feasible solutions** refer to those $x \in \mathcal{D}$ vectors that satisfy the criteria, i.e., $x \in \mathcal{F}$. The set of these x values is denoted by $\mathcal{L}$. Thus,

$$\mathcal{L} := \mathcal{D} \cap \mathcal{F},$$

which, for explicit constraints (1.1), becomes

$$\mathcal{L} = \{x \in \mathcal{D} : f_i(x) \leq 0,\ g_j(x) = 0,\ i \in [k], j \in [\ell]\}.$$

Optimal Value, Optimal Location

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- The **optimal value** associated with the problem is

$$p^* = \inf_{x \in \mathcal{L}} c(x) \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\},$$

where the infimum is ∞ if the set of feasible solutions is empty, and $-\infty$ if the objective function c takes arbitrarily small values on the feasible solutions.

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$$c(x^*) = p^*.$$

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- The vector x_ℓ is a **local optimum** if for every point in its neighborhood, c is at least as large as at x_ℓ:

$$\exists\, \varepsilon > 0,\ \forall\, x : \|x - x_\ell\|_2 < \varepsilon \quad \text{implies} \quad c(x_\ell) \le c(x),$$

where $\| \ \|_2$ is the Euclidean norm defined on $\mathbb{R}^n$,
$\| \ \|_2 \colon \mathbb{R}^n \to [0, \infty)$, $y \mapsto \|y\|_2 := \sqrt{y_1^2 + \cdots + y_n^2}$.

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- In the definition of ε-approximate solution, we include the condition that the optimal value is positive, $p^* > 0$. In problems with negative optimal values, we need to impose the condition $c(x_0) \leq (1 - \varepsilon) p^*$.

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It can be seen that the set of feasible solutions is the open interval

$$\mathcal{L} = \mathbb{R}_{>0} := (0, \infty) =]0, \infty[$$

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- However, there are no ε-approximate solutions.

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- As a result, we obtain

$$p^* = -\frac{1}{e}, \qquad x^* = \frac{1}{e}.$$

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- By equivalent transformation, we mean a formal rearrangement such that the optimal value/location of one problem can be easily determined based on the optimal value/location of the other problem.
- Now (without claiming completeness), let's mention a few possibilities.

Transformations: Exchange of Min/Max

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Maximize	$c(x)$
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• An optimal point of the minimization problem is also an optimal point of the maximization problem.

• If we know the optimal value of the minimization problem, its negative will be the optimal value of the maximization problem.

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$$\frac{x_1}{x_2^2+1} \leq 0 \quad \Longleftrightarrow \quad x_1 \leq 0.$$

$$(x_1+x_2)^2 \leq 0 \quad \Longleftrightarrow \quad (x_1+x_2)^2 = 0 \quad \Longleftrightarrow \quad x_1+x_2=0.$$

Transformations: Substituting Inequalities with Sign Conditions

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Minimize	$c(x)$
subject to	$f_i(x) \leq 0$
	$g_i(x) = 0$

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Minimize	$c(x)$
subject to	$f_i(x) + s_i = 0$
	$g_i(x) = 0$
	$s_i \geq 0$

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- The introduced variables s_i are called *slack variables*.

Transformations: Elimination of Linear Equalities

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- Determining x_0 and F can be done efficiently.

Minimize	$c(x)$
subject to	$f_i(x) \leq 0$
	$Ax = b$

$\equiv$

Minimize	$c(x_0 + Fy)$
subject to	$f_i(x_0 + Fy) \leq 0$

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$\equiv$

Minimize	$m(c(x))$
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Transformations: Substituting the Objective Function with a Monotonic Function: Example

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Minimize	$\|x\|_2^2$
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- In this case, range $c = \mathbb{R}_{\geq 0}$, where $m(x) = x^2$ is a monotonic function.

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Transformations: Substituting the Objective Function with a Monotonic Function: A More Complex Example

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$$\boxed{\begin{array}{ll} \text{Maximize} & x_1x_2 + x_2x_3 + x_3x_1. \\ \text{subject to} & x_1 + x_2 + x_3 = 100, \\ & x_1, x_2, x_3 \geq 0. \end{array}} \equiv$$

$$\boxed{\begin{array}{ll} \text{Minimize} & \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{3}}. \\ \text{subject to} & \frac{x_1 + x_2 + x_3}{3} = \frac{100}{3}, \\ & x_1, x_2, x_3 \geq 0. \end{array}}$$

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Minimize	$\sqrt{\frac{x_1^2+x_2^2+x_3^2}{3}}.$
subject to	$\frac{x_1+x_2+x_3}{3} = \frac{100}{3},$
	$x_1, x_2, x_3 \geq 0.$

- The two sets of constraints are obviously equivalent.

Transformations: Substituting the Objective Function with a Monotonic Function: A More Complex Example (Continued)

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- The two objective functions are:

$c_1(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$, and $c_2(x_1, x_2, x_3) = \sqrt{\frac{x_1^2+x_2^2+x_3^2}{3}}$.

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$$3c_2^2 + 2c_1 = (x_1 + x_2 + x_3)^2 = 100^2.$$

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- Instead of maximizing c_1, we can minimize $100^2 - 2c_1 = 3c_2^2$.

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$$3c_2^2 + 2c_1 = (x_1 + x_2 + x_3)^2 = 100^2.$$

- Instead of maximizing c_1, we can minimize $100^2 - 2c_1 = 3c_2^2$. Then we can apply the strictly monotonic function $m(x) = \sqrt{\frac{x}{3}}$ (on the set of non-negative values taken by the new objective function).

Transformations: Substituting the Objective Function with a Monotonic Function: A More Complex Example (Continued)

- The two objective functions are:

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- The advantage of the new form is evident. We need to minimize the square sum given a fixed arithmetic mean. With the inequality between arithmetic and quadratic means, the optimization question can be solved with elementary mathematics.

Break

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• Below, we present some introductory examples of basic concepts and elementary formalization tricks.

Example I

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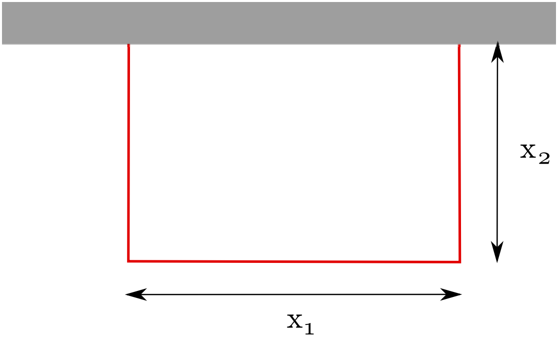

Example I (Continued)

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Maximize	Maximize x_1x_2
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- Applying the inequality between arithmetic and geometric means and taking the constraints into account, we get

$$50 = \frac{x_1 + 2x_2}{2} \geq \sqrt{x_1(2x_2)} \qquad (\geq 0).$$

Example I (Continued)

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- Analyzing the case of equality in the inequality between arithmetic and geometric means is beneficial. Based on this, $(50, 25)$ is not just one optimal point. $(50, 25)$ is the ONLY optimal point.

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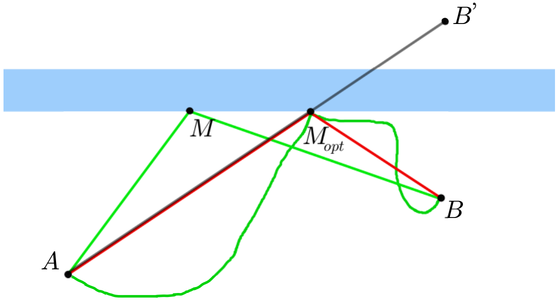

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- Here, coming straight from A and then moving straight to B will be the shortest path for the horse in the problem.

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- Because it is obvious that if the horse does not move in a straight line between points A and M or between M and B, then its path is not optimal.
- Therefore, given points $A(a_1, a_2)$ and $B(b_1, b_2)$, the problem can be formulated as follows:

Minimize $$\sqrt{(a_1 - x)^2 + a_2^2} + \sqrt{(b_1 - x)^2 + b_2^2}.$$

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- Notice that instead of the distance, we can also consider its $\sqrt{\frac{1}{3}}$ times.
- The formalization (similar to a previous example) can be as follows:

Minimize	$\sqrt{\frac{x_1^2+x_2^2+x_3^2}{3}}$
subject to	$\frac{x_1+x_2+x_3}{3} = \frac{100}{3}$
	$x_1, x_2, x_3 > 0.$

Example III (continued)

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We use the inequality between the arithmetic and quadratic means to estimate the original optimal value:

$$\sqrt{\frac{1}{3}}c(x) = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{3}} \geq \frac{x_1 + x_2 + x_3}{3} = \frac{100}{3}.$$

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- Furthermore, its optimal value is

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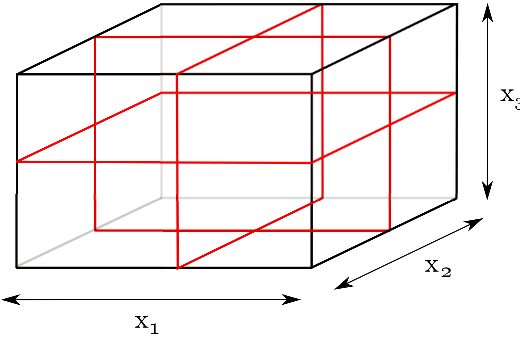

Example IV: Formalization

The formalization is obvious:

Maximize	$2x_1x_2 + 2x_2x_3 + 2x_3x_1$
subject to	$4x_1 + 4x_2 + 4x_3 = 400$
	$x_1, x_2, x_3 > 0.$

Based on previous considerations, this is an equivalent problem to the previous one. Rethinking the equivalence leads to the conclusion that the optimal point is the same as in the previous problem.

$$x^* = \left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right).$$

From this, the optimal value of the original problem is

$$p^* = c(x^*) = \frac{20\,000}{3}.$$

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• With this problem, we encountered the Turán theorem in the Combinatorics course. If we denote the sizes of the three parts as x_1, x_2, and x_3 and observe that we are looking for the maximum among complete 3-partite graphs, then the problem is as follows:

Maximize	$x_1x_2 + x_2x_3 + x_3x_1$
subject to	$x_1 + x_2 + x_3 = 100$
	$x_1, x_2, x_3 > 0$
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- The optimal value is $p^* = 3333$.

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Break

Example VI

Example

Let $d, n \in \mathbb{N}$ and $\ell_1(x), \ldots, \ell_n(x) \colon \mathbb{R}^d \to \mathbb{R}$ be given linear functions. Determine the minimum of the function $c(x) = \max_{1 \leq i \leq n} \ell_i(x)$.

So the formalized problem is:

Minimize	$c(x)$

The problem is global (there are no constraints).

Example VI: Figure

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- The accompanying figure illustrates a general configuration for the case of $d = 1$, $n = 5$.

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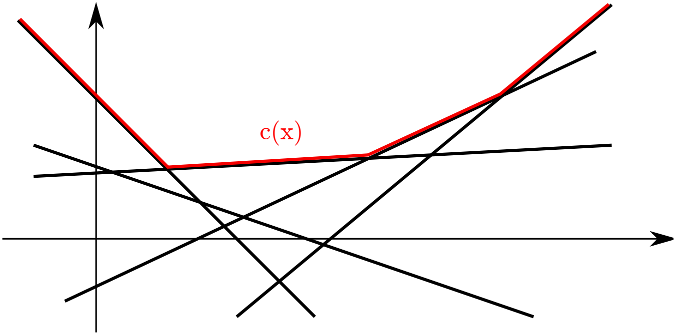
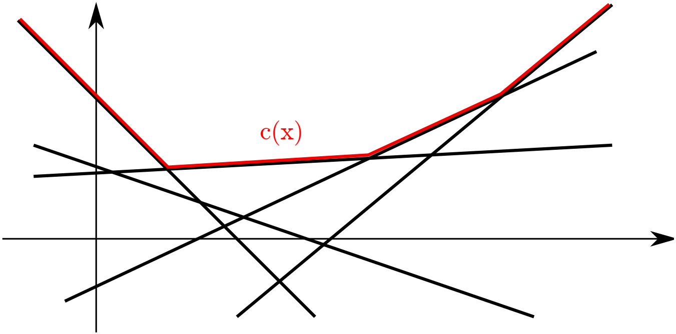


Example VI: Figure

- The accompanying figure illustrates a general configuration for the case of $d = 1$, $n = 5$.



- Even this specific choice gives a faithful representation of the objective function. The maximum of linear functions c is *piecewise* linear. For $d > 1$, this means that the domain of c can be divided into connected parts on which c is linear.

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Minimize	m
subject to	$\ell_1(x) \leq m,$
	$\vdots$
	$\ell_n(x) \leq m,$

where $(x, m) \in \mathbb{R}^d \times \mathbb{R}$.

Example VII: A Class of Problems

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- This form of optimization problem may be familiar from the Operations Research course.

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Linear Programming, LP

Let $A \in \mathbb{R}^{m \times n}$ be a given matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ fixed vectors, and $x \in \mathbb{R}^n$ the unknown vector. The problem:

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Minimize	$c^{\mathsf{T}}x$
subject to	$Ax \preceq b$,

where $Ax \preceq b$ denotes the *componentwise less than or equal* relation between Ax and b in $\mathbb{R}^m$.

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- The LP problem class is extensively discussed in the Operations Research course.
- It is noted that the LP problem is a very central problem. Many algorithms are considered good both in practice and in theory.
- If we formulate a problem as an LP problem, then we are *beyond the hard part*. (Just like in high school, if we reduced our work to solving a quadratic equation during solving equations).
- We finish our work with one of the general LP algorithms (such algorithms are easily accessible with public source code).

Example VIII

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Consider the LP problem where the linear objective function ($c^{\mathsf{T}}x$) is a random variable represented by the vector c. We assume that its expected value, $\overline{c} = \mathbb{E}[c]$, is known, and the constraints are no longer dependent on randomness.

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Minimize the expected value of the objective function.

- Based on the linearity of expected value, $\mathbb{E}[c^{\mathsf{T}}x] = (\mathbb{E}[c])^{\mathsf{T}}x$.
- An LP problem remains our optimization question:

Minimize	$(\mathbb{E}[c])^{\mathsf{T}}x$
subject to	$Ax \preceq b$.

Example IX: Chebyshev Center

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Definition

Sets of points in the form $\mathcal{P} = \{x \in \mathbb{R}^n : a_i^\top x \leq b,\ i = 1, \ldots, k\}$ are called polyhedra. If the polyhedron $\mathcal{P}$ is bounded (i.e., compact: bounded and closed), then it is called a polytope.

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Example: Chebyshev Centers of Polytopes

Given $\mathcal{P}$ polytope.

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- In fact, deciding whether $\mathcal{P}$ is empty is also a central problem.
- In our case, we need to measure how deeply a point in $\mathcal{P}$ is inside $\mathcal{P}$.
- There are many solutions/answers. We discuss one solution associated with Chebyshev.

Example IX: Chebyshev Center (continued)

Definition

$B(c, r) = \{x \in \mathbb{R}^n : |x - c|^2 \leq r^2\}$ is the ball centered at c with radius r.

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- Our fundamental problem: Given $\mathcal{P}$, find a Chebyshev center.

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$$\frac{a^\mathsf{T}}{|a|} p - \frac{b}{|a|}.$$

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There are two types of signed distances. The above one is positive in the half-space $\{x : a^\mathsf{T} x > b\}$ and negative in the complementary (open) half-space.

Example IX: Chebyshev Center (continued)

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- The Chebyshev center problem is equivalent to the following:

Maximize	r
subject to	$a_i^T x + \|a_i\| r \leq b_i, \quad i = 1, \ldots, k$
	$r \geq 0$

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- The Chebyshev center problem is equivalent to the following:

Maximize	r	
subject to	$a_i^{\mathsf{T}} x + \lvert a_i \rvert r \leq b_i,$	$i = 1, \ldots, k$
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- Our first type of condition is equivalent to $\frac{a_i^{\mathsf{T}}}{|a_i|} x + r \leq \frac{b}{|a_i|}$, i.e.,

$$r \leq \frac{b}{|a_i|} - \frac{a_i^{\mathsf{T}}}{|a_i|} x.$$

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- The reformulated optimization problem is an LP problem.

Break

Example X: Least Squares Problem

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- The following problem and its solution methods are encountered in the field of numerical analysis.

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The problem involves minimizing

$$\boxed{\text{Minimize} \qquad \|c - Ax\|}$$

or

$$\boxed{\text{Minimize} \qquad \|c - Ax\|^2}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{k \times n}$ is a real matrix, and $c \in \mathbb{R}^k$.

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- This is an unconstrained optimization problem. It is easily handled based on basic linear algebraic and geometric knowledge.

Example X: Least Squares Problem (continuation)

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Example

Consider a series of measurements (e.g., measurements taken at different times in an experimental laboratory or measurements from a meteorological station at certain location), where $t_1, t_2, \ldots t_N$ and $p_1, p_2, \ldots, p_N$ denote the measurement times and the measured parameters, respectively. We are looking for a "low," d-degree polynomial that is a good "hypothesis" for the changes in p.

Example X: Least Squares Problem (continuation)

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- Let

$$p(x) = x_d x^d + \cdots + x_1 x + x_0$$

be the polynomial, i.e., for the measured vector $p = (p_1, \ldots, p_N)^\mathsf{T}$, we need to minimize the distance in L_2 from

$$x_d(t_1^d, \ldots, t_N^d)^\mathsf{T} + x_{d-1}(t_1^{d-1}, \ldots, t_N^{d-1})^\mathsf{T} + \ldots$$

to p.

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to p.

- In the least squares problem, the objective function is quadratic, unlike in LP where it is linear. We will now examine a common generalization of the two problems.

Example XI: Quadratic Programming

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Quadratic Programming, QP

Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ be symmetric matrices, and let $b \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ be vectors.

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Minimize	$x^{\mathsf{T}}Ax + b^{\mathsf{T}}x$
subject to	$Cx \preceq d$.

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• It is worth noting that the basic QP problem can also be handled (an efficient algorithm is known for it) if the objective function is convex (i.e., A is positive semidefinite).

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• The constraint system of the QP basic problem is linear. However, the objective function is much more general.

• It is worth noting that the basic QP problem can also be handled (an efficient algorithm is known for it) if the objective function is convex (i.e., A is positive semidefinite). If we bring a problem into this form (convex QP), then we are "ready".

Example XII: Stochastic Linear Programming

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- Consider the LP problem in which the linear objective function ($c^{\mathsf{T}}x$) is a vector of random variables c. Assume that its expected value is $\overline{c} = \mathbb{E}[c]$, and its covariance matrix is $\Sigma = \mathbb{E}[(c - \overline{c})(c - \overline{c})^{\mathsf{T}}]$. Furthermore, the constraints do not depend on randomness anymore.

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- We have seen that if we only minimize the expected value of the objective function ($\mathbb{E}[c^\mathsf{T} x] = (\mathbb{E}[c])^\mathsf{T} x$), then we obtain an LP problem.

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- However, in this case, we do not take into account the *risk*.

Example XII: Stochastic Linear Programming (continued)

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Stochastic LP

Minimize	$\mathbb{E}[c^\mathsf{T}x] + \gamma\, \mathbb{V}ar[c^\mathsf{T}x]$
subject to	$Ax \preceq b$
	$Dx = e$

where $\gamma \in \mathbb{R}_{>0}$ is a parameter chosen for the application.

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- With simple transformations,

$$\mathbb{V}ar[c^\mathsf{T} x] = \mathbb{E}[(c^\mathsf{T} x - \mathbb{E}[c^\mathsf{T} x])^2] = \mathbb{E}[(c^\mathsf{T} x - (\mathbb{E}[c])^\mathsf{T} x)^2]$$
$$= \mathbb{E}[((c^\mathsf{T} - \mathbb{E}[c]^\mathsf{T})x)^2] = x^\mathsf{T}\mathbb{E}[(c - \mathbb{E}[c])(c^\mathsf{T} - \mathbb{E}[c]^\mathsf{T})]x = x^\mathsf{T}\Sigma x.$$

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- The convex QP form of the question is now apparent.

Example XIII: Distance between Polyhedra

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Determine the distance between two given polyhedra in n-dimensional Euclidean space!

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Determine the distance between two given polyhedra in n-dimensional Euclidean space!

• Since the polyhedra arise as solution sets of linear inequalities, each polytope corresponds to an inequality system ($C_1 \in \mathbb{R}^{k_1 \times n}$, $C_2 \in \mathbb{R}^{k_2 \times n}$):

$$\mathcal{P}_1 = \{x_1 \in \mathbb{R}^n : C_1 x_1 \preceq d_1\} \quad \text{and} \quad \mathcal{P}_2 = \{x_2 \in \mathbb{R}^n : C_2 x_2 \preceq d_2\}.$$

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• The distance between them is

$$d(\mathcal{P}_1, \mathcal{P}_2) = \inf\{d(x_1, x_2) : x_1 \in \mathcal{P}_1,\ x_2 \in \mathcal{P}_2\},$$

where $d(x_1, x_2) = \|x_1 - x_2\|_2$.

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subject to	$C_1 x_1 \preceq d_1,$
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- To this end, consider the vectors and matrices formed from $d_1 \in \mathbb{R}^{k_1}, d_2 \in \mathbb{R}^{k_2}, x_1, x_2 \in \mathbb{R}^n$ and $C_1, C_2 \in \mathbb{R}^{(k_1+k_2)\times n}$ as follows:

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

where 0 denotes appropriately sized zero matrices.

Example XIII: Distance between Polyhedra (continued)

- Considering the squared distance leads to an equivalent optimization problem:

Minimize	$\|x_1 - x_2\|_2^2$
subject to	$C_1 x_1 \preceq d_1,$
	$C_2 x_2 \preceq d_2.$

- To this end, consider the vectors and matrices formed from $d_1 \in \mathbb{R}^{k_1}, d_2 \in \mathbb{R}^{k_2}, x_1, x_2 \in \mathbb{R}^n$ and $C_1, C_2 \in \mathbb{R}^{(k_1+k_2)\times n}$ as follows:

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where 0 denotes appropriately sized zero matrices. Our constraint system is: $Cx \preceq d$.

Example XIII: Distance between Polyhedra (continued)

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- If we further construct the $n \times n$ identity matrix, then

$$A = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$$

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- Using the above notation, we see that this is a convex QP problem. Based on the above, determining the distance between two polytopes can be efficiently accomplished.

Break

Example XIV: Clique Problem

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- The elements of $\{0,1\}^V$ can be described by the conditions $0 \leq x_v \leq 1$, $x_v \in \mathbb{Z}$ for all v vertices. A 0-1 vector will be the characteristic vector of a clique if at most one of its vertices falls into every pair of disconnected vertices u and v. That is, for all $uv \notin E(G), u \neq v$ vertices, we have $x_u + x_v \leq 1$.

Example XIV: Clique Problem (continued)

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- The well-known $\mathcal{N}P$-hard clique problem can be formulated as follows:

$$\begin{array}{ll} \text{Maximize} & 1^{\mathsf{T}}x, \\ \text{subject to} & 0 \le x_v \le 1,\ x_v \in \mathbb{Z} \text{ for all } v \text{ vertices} \\ & x_u + x_v \le 1 \\ & \qquad \text{for all disconnected vertices } uv \notin E(G), u \neq v. \end{array}$$

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Integer Programming, IP

The basic problem:

Minimize	$c^{\mathsf{T}} x$
subject to	$Ax \preceq b$,
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Example XVI: Matching Problem

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Given a graph G, find the maximum matching among its edges.

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- Formalizing the problem:

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subject to	x is the characteristic vector of the matching.

- Algebraically formalizing the condition yields the following equivalent optimization problem:

Maximize	$1^{\mathsf{T}}x$ (where $x \in \mathbb{R}^{E(G)}$)
subject to	$0 \leq x_e \leq 1,\ x_e \in \mathbb{Z}$
	$\sum_{e: v \in e} x_e \leq 1$ for all v vertices.

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Notation

$$\nu^*(G)$$

denotes the optimal value of the above LP problem.

Example XVI: Matching Problem (continued)

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Theorem

If G is bipartite, then $\nu^*(G) = \nu(G)$.

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Maximize	$1^{\mathsf{T}}x$
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Theorem*

Let G be a bipartite graph. Then

$$\mathrm{conv}\{\chi_M : M \text{ is a matching}\} =$$
$$=\{x \in \mathbb{R}^E : 0 \le x_e \le 1, \sum_{e: v \in e} x_e \le 1 \text{ for all } v \text{ vertices.}\}$$

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- The case of non-bipartite graphs will be studied further later.

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Example

Given a graph G. Determine the clique parameter ($\omega(G)$), i.e., the size of the largest clique.

• We saw that the question can also be formulated as an IP problem. The following theorem gives an alternative, far from obvious, formalization of determining $\omega(G)$.

Example XIV: Clique Problem Revisited: Theorem

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Theorem

Let $A \in \mathbb{R}^{V\times V}$ be the adjacency matrix of graph G, and consider the following problem ($x, 0, 1 \in \mathbb{R}^V$):

$$\begin{array}{ll} \text{Maximize} & x^{\mathsf{T}}Ax \\ \text{subject to} & x \succeq 0 \\ & 1^{\mathsf{T}}x = 1. \end{array}$$

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Maximize	$x^T A x$
subject to	$x \succeq 0$
	$1^T x = 1.$

Then

$$p^* = 1 - \frac{1}{\omega(G)}.$$

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$$x = \left(\frac{\overset{v_1}{n_1}}{N}, \ldots, \frac{\overset{v_k}{n_k}}{N} \right) \in \mathcal{L} \cap \mathbb{Q}^{V(G)}$$

vector, where $k = |V(G)|$ and $n_1, \ldots, n_k \in \mathbb{N}$ numbers form a partition of $N \in \mathbb{N}$.

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- Then assign a graph $\widetilde{G}$ to G as follows:
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◦ If points $v_i, v_j \in V(G)$ are adjacent, then a complete bipartite graph K_{n_i,n_j} is formed between the respective V_i, V_j sets (all possible edges are drawn).

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  ◦ If points $v_i, v_j \in V(G)$ are not adjacent, then there is no edge between the corresponding V_i, V_j sets.

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- By definition, the number of points of $\widetilde{G}$ is

$$|V(\widetilde{G})| = \sum_{i=1}^{k} n_i = N \sum_{i=1}^{k} \frac{n_i}{N} = N$$

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- Based on the construction, the number of edges of $\widetilde{G}$ is

$$|E(\widetilde{G})| = \sum_{\{v_i, v_j\} \in E(G)} n_i n_j = \frac{1}{2} N^2 \sum_{v_i v_j \in E(G)} \frac{n_i}{N} \frac{n_j}{N} = \frac{N^2}{2} x^{\mathsf{T}} A x = \frac{N^2}{2} c(x).$$

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- The largest clique of $\widetilde{G}$ has size $\omega(G)$. Applying Turán's theorem, its number of edges is at most the number of edges of the $\omega(G)$-class/N-point Turán graph ($\omega(G)|N$):

$$|E(\widetilde{G})| \leq |E(T_{N,\omega(G)})| = \binom{\omega(G)}{2}\left(\frac{N}{\omega(G)}\right)^2 = \left(1 - \frac{1}{\omega(G)}\right)\frac{N^2}{2}.$$

Example XIV: Clique Problem Revisited: Proof

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- Summarizing the above, we obtain

$$c(x) = \frac{2}{N^2}|E(\widetilde{G})| \leq 1 - \frac{1}{\omega(G)},$$

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- This proves the missing inequality. Thus, $p^* = 1 - \frac{1}{\omega(G)}$.

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- Among the conditions of the convex QP problem, **only** the positive semidefiniteness of the quadratic form matrix is missing.

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- In the formalization provided by the theorem, a quadratic form needs to be maximized.
- Among the conditions of the convex QP problem, **only** the positive semidefiniteness of the quadratic form matrix is missing.
- Omitting this condition results in a form capable of formalizing hopeless optimization problems.

This is the End!

Thank you for your attention!