Optimization Methods practice session, for Master students

Exercises: Duality

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1. Duality

1. Exercise (LP with inequalities). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2} \quad & 2x_1 + 3x_2 \\ s.t. \quad & x_1 + x_2 \geq 1, \\ & x_1 + 2x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

2. Exercise (LP with equality). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 = 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

3. Exercise (LP with max). *Let be the primal problem:*

$$\begin{aligned} \max_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ s.t. \quad & x_1 + x_2 \leq 4, \\ & 2x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

4. Exercise (LP with mixed constraints). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1 + 2x_2 + 3x_3 \\ s.t. \quad & x_1 + x_2 + x_3 = 1, \\ & x_1 + 2x_2 \geq 0.5, \\ & x_i \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

5. Exercise (Quadratic program). *Let be the primal problem:*

$$\begin{aligned} \min_{x} \quad & \tfrac{1}{2}x^2 \\ s.t. \quad & x \geq 1. \end{aligned}$$

Derive the Lagrangian dual.

6. Exercise (2D quadratic program). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2} \quad & \tfrac{1}{2}(x_1^2 + x_2^2) \\ s.t. \quad & x_1 + x_2 = 1. \end{aligned}$$

Derive the Lagrangian dual.

7. Exercise (Quadratic with inequality). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + x_2^2 \\ s.t. \quad & x_1 + x_2 \geq 1. \end{aligned}$$

Derive the Lagrangian dual.

8. Exercise (Entropy maximization). *Let be the primal problem:*

$$\begin{aligned} \max_{x_1,x_2} \quad & -x_1 \log x_1 - x_2 \log x_2 \\ s.t. \quad & x_1 + x_2 = 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

9. Exercise (Entropy with moment constraint). *Let be the primal problem:*

$$\begin{aligned} \max_{x_1,x_2} \quad & -x_1 \log x_1 - x_2 \log x_2 \\ s.t. \quad & x_1 + x_2 = 1, \\ & x_1 + 2x_2 = 1.5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

10. Exercise (Log-barrier type problem). *Let be the primal problem:*

$$\begin{aligned} \min_{x} \quad & -\log x \\ s.t. \quad & x \geq 1. \end{aligned}$$

Derive the Lagrangian dual.

11. Exercise (Absolute value via constraints). *Let be the primal problem:*

$$\begin{aligned} \min_{x,t} \quad & t \\ s.t. \quad & -t \leq x \leq t, \\ & x \geq 1. \end{aligned}$$

Derive the Lagrangian dual.

12. Exercise (Simple resource allocation). *Let be the primal problem:*

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & \log x_1 + \log x_2 + \log x_3 \\ s.t. \quad & x_1 + x_2 + x_3 \leq 1, \\ & x_i > 0. \end{aligned}$$

Derive the Lagrangian dual.

13. Exercise LP with redundant variable). *Let be the primal problem:*

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 \\ s.t. \quad & x_1 - x_2 = 0, \\ & x_2 \geq 1. \end{aligned}$$

Derive the Lagrangian dual.

14. Exercise (Quadratic with box constraint). *Let be the primal problem:*

$$\begin{aligned} \min_{x} \quad & x^2 \\ s.t. \quad & 0 \leq x \leq 2. \end{aligned}$$

Derive the Lagrangian dual.

15. Exercise (Three-variable LP). *Let be the primal problem:*

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & x_1 + 2x_2 + 3x_3 \\ s.t. \quad & x_1 + x_2 + x_3 \leq 2, \\ & x_1 + 2x_2 + 3x_3 \leq 3, \\ & x_i \geq 0. \end{aligned}$$

Derive the Lagrangian dual.

2. Convex Conjugate

16. Exercise. *Define the convexity of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. (Rely on intuition.)*

Definition. Let $f(x_1, x_2, \ldots, x_n)$ be an arbitrary function. Define $\widehat{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{dom } f, \\ \infty & \text{otherwise.} \end{cases}$$

17. Exercise. *Prove that f is convex if and only if $\widehat{f}$ is convex.*

Definition. For a function $f : \mathbb{R}^n \to \mathbb{R}$, the (convex / Fenchel / Legendre–Fenchel) conjugate is defined as

$$f^*(y) = \sup\{y^T x - f(x) : x \in \text{dom } f\}.$$

We adopt the convention that dom $f^* = \mathbb{R}^n$ and f^* may take the value $+\infty$ (just like f may).

18. Exercise. *Compute the convex conjugate of the following functions:*

(i) $f(x) = a^T x + b$,

(ii) $f(x) = |x|$,

(iii) $f(x) = x \log x$ *(usually for $x > 0$, with convention at 0),*

(iv) $f(x) = \alpha e^{\beta x}$.

19. Exercise. *Let $f(x_1, \ldots, x_n) = \sum_{i=1}^n g_i(x_i)$. Prove that $f^*(y_1, \ldots, y_n) = \sum_{i=1}^n g_i^*(y_i)$.*

20. Exercise. *Prove that for any function f, its conjugate f^* is a convex function.*

21. Exercise. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that*

(i) $f^{**} \leq f$,

(ii) $f \leq f^{**}$,

and therefore $f = f^{**}$ for convex f.

If f is not necessarily convex, prove that

(iii) $f^* = f^{***}$.

3. Norms in $\mathbb{R}^d$

Definition. A function $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}_+$ is a *norm* if it satisfies:

(i) (Definiteness) $\|v\| = 0 \iff v = 0$,

(ii) (Homogeneity) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{R}$,

(iii) (Triangle inequality) $\|u + v\| \leq \|u\| + \|v\|$.

Definition. The *unit ball* of the norm $\|\cdot\|$ is

$$\mathcal{B}_{\|\cdot\|} = \{v \in \mathbb{R}^d : \|v\| \leq 1\}.$$

22. Exercise. *Prove that $\mathcal{B}_{\|\cdot\|}$ is closed, bounded, centrally symmetric with respect to the origin, and convex.*

23. Exercise. *Let B be a closed, bounded, origin-symmetric, convex set in $\mathbb{R}^d$. For $v \neq 0$ define*

$$\|v\|_B = \sup\{t > 0 : v/t \in B\}$$

(with $\|0\|_B = 0$). Prove that $\|\cdot\|_B$ is a norm.

Definition. For $1 \leq p < \infty$, the ℓ_p-norm is

$$\|v\|_p = \Big(\sum_{i=1}^d |v_i|^p\Big)^{1/p}.$$

The ℓ_∞-norm (Chebyshev norm) is

$$\|v\|_\infty = \max_{i=1,\ldots,d} |v_i|.$$

24. Exercise. *Prove that $\|\cdot\|_p$ is a norm for every $p \in [1, \infty]$.*

25. Exercise. *Show that for every* $v \in \mathbb{R}^d$

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

26. Exercise. *Sketch the unit balls of* $\|\cdot\|_1$*,* $\|\cdot\|_2$ *and* $\|\cdot\|_\infty$ *in* $\mathbb{R}^2$ *and in* $\mathbb{R}^3$*. Describe their shape in general dimension* d*.*

27. Exercise. *Let* $A \in \mathcal{S}^n_{++}$ *(symmetric positive definite). Prove that*

$$\|v\|_A = \sqrt{v^T A v}$$

defines a norm on $\mathbb{R}^n$*.*

4. Dual Norm

Definition. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, its *dual norm* is

$$\|y\|^* = \sup\{|x^T y| : \|x\| \leq 1\}.$$

28. Exercise. *Prove that the dual of the dual norm is the original norm:* $(\|\cdot\|^*)^* = \|\cdot\|$*.*

29. Exercise. *Prove that* $\|\cdot\|_p^* = \|\cdot\|_q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$*.*

30. Exercise. *Let* $f(x) = \|x\|$*. Compute the convex conjugate* f^**.*

31. Exercise. *For* $x = (x_1, x_2)^T \in \mathbb{R}^2$ *define*

$$\|x\| = \max\{|x_1|, |x_2|, |x_1 + x_2|\}.$$

(i) Prove that this is a norm.

(ii) Sketch its unit ball.

(iii) Determine its dual norm.

(iv) Sketch the unit ball of the dual norm.

32. Exercise. *Consider a regular octagon* B *in* $\mathbb{R}^2$ *whose second-neighbor vertices include* $(0,1)$*,* $(-1,0)$*,* $(0,-1)$ *and* $(1,0)$*. Let* $\|\cdot\|$ *be the norm whose unit ball is* B*. Determine the dual norm and sketch its unit ball.*

5. Matrix Norms

Definition. Let $\|\cdot\|^{(n)}$ and $\|\cdot\|^{(m)}$ be norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ is *compatible* (or *induced*) with respect to them if

$$\|Mv\|^{(m)} \leq \|M\| \cdot \|v\|^{(n)} \quad \forall\, M \in \mathbb{R}^{m\times n},\ v \in \mathbb{R}^n.$$

Definition. For $A \in \mathbb{R}^{n\times n}$ define

$$\|A\|_{\max} = n \max_{1\le i,j\le n} |A_{ij}|,$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2} \qquad \text{(Frobenius norm)}.$$

33. Exercise. *(i) Prove that $\|\cdot\|_{\max}$ and $\|\cdot\|_F$ are matrix norms.*

(ii) Show that $\|\cdot\|_{\max}$ is compatible with $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.

(iii) Show that $\|\cdot\|_F$ is compatible with $\|\cdot\|_2$.

Definition. A matrix norm $\|\cdot\|$ is *submultiplicative* if

$$\|AB\| \le \|A\|\cdot\|B\| \quad \forall\, A, B \in \mathbb{R}^{n\times n}.$$

34. Exercise. *Prove that both $\|\cdot\|_{\max}$ and $\|\cdot\|_F$ are submultiplicative.*