Optimization Methods practice session, for Master students

Exercises: Linear algebra in R^d

Lecturer: Peter Hajnal 2026

1. Matrices

Definition. Let S be a finite set of rows and O a finite set of columns. Then $M : S \times O \to \mathbb{R}$ is a (real) matrix. The elements of $S \times O$ form a rectangular array of positions. The real numbers $M(s, o)$ are the entries of the matrix ($s \in S$, $o \in O$).

If $(S, \leq)$ and $(O, \leq)$ are ordered sets, then the rows and columns of the corresponding table should be arranged according to this order.

If $|S| = |O|$, we speak of a square matrix. In general, for square matrices we assume $S = O$. The entries $M(s, s)$ in a square matrix are the diagonal entries.

Usually $S = [n] = \{1, 2, \ldots, n\}$ and $O = [m] = \{1, 2, \ldots, m\}$.

Definition. A submatrix of M is a matrix obtained by deleting some rows and columns. If the remaining row and column index sets are the same, we speak of a symmetric submatrix.

If the submatrix has $S = O = \{1, 2, \ldots, k\}$, then it is called a leading principal submatrix.

The determinant of a square submatrix of M is called a minor of M. If the row and column index sets of the submatrix are equal, its determinant is called a symmetric minor.

If the rows and columns of the submatrix are $\{1, 2, \ldots, k\}$, then the determinant is a leading principal minor.

Definition. Vectors are ALWAYS treated as COLUMN vectors when regarded as matrices.

Definition. Let $M \in \mathbb{R}^{n \times m}$. Define

$$\text{column-rank} = \max\{k : \text{there exist } k \text{ linearly independent column vectors in } M\},$$

$$\text{row-rank} = \max\{k : \text{there exist } k \text{ linearly independent row vectors in } M\},$$

$$\text{det-rank} = \max\{k : \text{there exists a } k \times k \text{ square submatrix with non-zero determinant}\}.$$

1. Exercise. *Prove that for every $M \in \mathbb{R}^{n \times m}$*

$$\textit{column-rank}(M) = \textit{row-rank}(M) = \textit{det-rank}(M).$$

Definition. The common value above is called the rank of the matrix M. Notation: rk (M).

2. Eigenvalues

Definition. Let $M \in \mathcal{S}^n$ (symmetric $n \times n$ matrices). The characteristic polynomial of M is

$$p_M(t) = \det(M - tI).$$

The roots of p_M are the eigenvalues of M.

2. Exercise. *The roots of* $\widetilde{p}_M(t) = \det(M + tI)$ *are the negatives of the eigenvalues of* M*.*

3. Exercise. *If* $M \in \mathcal{S}^n$*, then all eigenvalues are real.*

4. Exercise. *Prove that for* $M \in \mathcal{S}^n$*, the rank rk* (M) *equals the number of non-zero eigenvalues (counted with multiplicity).*

Jelölés. For $M \in \mathcal{S}^n$, let

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$$

be the eigenvalues of M arranged in decreasing order.

5. Exercise. *Let* $M \in \mathcal{S}^n$*. Prove that*

$$\lambda_1(M) = \sup\{x^T M x : x \in \mathbb{R}^n,\ \|x\| = 1\} = \max\{x^T M x : x \in \mathbb{R}^n,\ \|x\| = 1\},$$

and

$$\lambda_n(M) = \inf\{x^T M x : x \in \mathbb{R}^n,\ \|x\| = 1\} = \min\{x^T M x : x \in \mathbb{R}^n,\ \|x\| = 1\}.$$

3. Positive Definite / Positive Semidefinite Matrices

Definition. A matrix $M \in \mathcal{S}^n$ is positive semidefinite if the bilinear form $x^T M x$ is nonnegative for every $x \in \mathbb{R}^n$. Notation: $M \succeq 0$ (where $0 \in \mathcal{S}^n$) or $M \in \mathcal{S}^n_{++}$.

$M \in \mathcal{S}^n$ is positive definite if $x^T M x > 0$ for every $x \neq 0$. Notation: $M \succ 0$ or $M \in \mathcal{S}^n_{++}$.

6. Exercise. *For* $M \in \mathcal{S}^n$*, the following are equivalent:*

(i) $M \succ 0$,

(ii) all eigenvalues of M *are positive,*

(iii) all coefficients of $\widetilde{p}_M$ *are positive,*

(iv) there exists a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ *such that* $M = A^T A$*,*

(v) every symmetric minor of M *is positive,*

(vi) for every leading principal submatrix F *of* M*, all coefficients of* $\widetilde{p}_F$ *are positive,*

(vii) every leading principal minor of M *is positive.*

7. Exercise. *For* $M \in \mathcal{S}^n$*, the following are equivalent:*

(i) $M \succeq 0$,

(ii) *all eigenvalues of* M *are nonnegative,*

(iii) *all coefficients of* $\widetilde{p}_M$ *are nonnegative,*

(iv) *there exists* $A \in \mathbb{R}^{k\times n}$ *(for some* k*) such that* $M = A^T A$,

(v) *every symmetric minor of* M *is nonnegative,*

(vi) *for every leading principal submatrix* F *of* M*, all coefficients of* $\widetilde{p}_F$ *are nonnegative.*

8. Exercise. *Is the following condition equivalent to the properties in the previous problem?*

(vii) *every leading principal minor of* M *is nonnegative.*

9. Exercise. *Let* $M \in \mathcal{S}^n$ *with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_n$*. Prove that*

$$tr\ (M) = \sum_{i=1}^{n} \lambda_i, \qquad \det(M) = \prod_{i=1}^{n} \lambda_i.$$

10. Exercise. *Let* $A, B \in \mathcal{S}^n$ *with* $A \succ 0$ *and* $B \succeq 0$*. Prove that* $A + B \succ 0$.

11. Exercise. *Let* $M \in \mathbb{R}^{n\times n}$ *(not necessarily symmetric). Prove that* $M^T M$ *is always positive semidefinite.*

12. Exercise. *Let* $M \in \mathcal{S}^n \succ 0$*. Show that* M^{-1} *is also positive definite.*

13. Exercise. *Prove that if* $M \in \mathcal{S}^n \succeq 0$ *and* $v \in \mathbb{R}^n$*, then* $v^T M v = 0$ *if and only if* $Mv = 0$.

14. Exercise. *Let* $A \in \mathbb{R}^{m\times n}$ *with* $m \geq n$*. Prove that*

$$rk\ (A^T A) = rk\ (A) = rk\ (AA^T).$$

15. Exercise. *Let* $M \in \mathcal{S}^n$ *and let* $\lambda_k(M)$ *be its* k*-th largest eigenvalue. Prove that*

$$\lambda_k(M) = \max_{\dim V = k} \ \min_{x \in V, \|x\|=1} x^T M x$$

(the minimax principle / Courant–Fischer theorem).

16. Exercise. *Let* $A, B \in \mathcal{S}^n$ *with* $A \succeq B \succeq 0$*. Prove that* $\lambda_k(A) \geq \lambda_k(B)$ *for each* $k = 1, \ldots, n$ *(Weyl's monotonicity theorem for eigenvalues).*