Optimization Methods practice session, for Master students

Exercises: Geometry in R^d-ben

Lecturer: Peter Hajnal Week 1—

1. Linear subspaces

Definition. For $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$, the expression

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N$$

— where the α_i are real numbers — is a linear combination of our points/vectors. The numbers α_i are the coefficients of the linear combination. The linear combination is trivial if all its coefficients are 0.

Definition. For $x, y \in \mathbb{R}^d$, $\ell(xy) = \{\alpha x + (1-\alpha)y : \alpha \in \mathbb{R}\}$ is the line connecting x and y.

1. Exercise. *Let $A \subset \mathbb{R}^d$. Then the following are equivalent:*

(i) A is closed under forming linear combinations.

(ii) $\underline{0} = (0, 0, \ldots, 0) \in A$, and moreover A contains the line connecting any two of its elements.

Definition. A set $L \subset \mathbb{R}^d$ is called a linear subspace if it satisfies any (equivalently all) of the conditions in the previous exercise.

2. Exercise. *Prove that the intersection of linear subspaces is a linear subspace.*

3. Exercise. *Let $R \subset \mathbb{R}^d$. Then there exists a unique set $M = lin(R)$ in $\mathbb{R}^d$ such that*

(i)$_M$ $M \supset R$,

(ii)$_M$ M is a linear subspace,

(iii) Every set T satisfying properties (i)$_T$ and (ii)$_T$ contains M.

Definition. The set M defined in the previous exercise is called the linear hull of the set R.

Definition. Let $\mathcal{M}_n$ be the space of $n \times n$ matrices. $\mathcal{M}_n \simeq \mathbb{R}^{n^2}$. If we think of the elements of $\mathbb{R}^{n^2}$ as matrices, we emphasize this by the notation $\mathbb{R}^{n \times n}$. Let $\mathcal{S}_n$ be the set of symmetric $n \times n$ matrices in $\mathbb{R}^{n \times n}$.

4. Exercise. *Prove that $\mathcal{S}_n$ is a linear subspace. Determine the dimension of $\mathcal{M}_n$ and $\mathcal{S}_n$.*

2. Affine subspaces

Definition. For $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$, the linear combination

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N$$

is called an affine combination of our points if $\alpha_1 + \alpha_2 + \ldots + \alpha_N = 1$.

Definition. A set $A \subset \mathbb{R}^d$ is called an affine subspace if it is closed under forming affine combinations.

5. Exercise. *The following are equivalent:*

(i) A is an affine subspace,

(ii) A contains the line connecting any two distinct points of A.

6. Exercise. *Let $R \subset \mathbb{R}^d$. Then there exists a unique set $M = aff(R)$ in $\mathbb{R}^d$ such that*

(i)$_M$ $M \supset R$,

(ii)$_M$ M is an affine subspace,

(iii) Every set T satisfying properties (i)$_T$ and (ii)$_T$ contains M.

Definition. The set M defined in the previous exercise is called the affine hull of the set R.

7. Exercise. *Prove that every linear subspace is an affine subspace. Give an example showing that the converse implication is false.*

3. Convex sets

Definition. For $x, y \in \mathbb{R}^d$, $xy = \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$ is the segment xy.

Definition. A set $C \subset \mathbb{R}^d$ is convex if for every $x, y \in C$ we have $xy \subset C$.

8. Exercise. *Let $R \subset \mathbb{R}^1$. R is convex if and only if it is empty, an interval, a ray, or the whole line.*

9. Exercise. *Prove that $D \subset \mathbb{R}^d$ is convex if and only if for every line ℓ, $D \cap \ell$ is convex.*

10. Exercise. *To an open square in the plane we add some of its boundary points. Characterize those subsets of the boundary whose addition yields a convex set.*

11. Exercise. *To an open circular disk in the plane we add some of its boundary points. Characterize those subsets of the boundary whose addition yields a convex set.*

12. Exercise. *What cardinalities can a convex set in $\mathbb{R}^n$ have?*

13. Exercise. *Prove that the following sets in $\mathbb{R}^n$ are convex:*

(o) *empty set and* $\mathbb{R}^n$*,*

(i) *halfspaces,*

(ii) *balls,*

(iii) *ellipsoids,*

(iv) *strips (the set of points lying between two parallel halfspaces),*

(v) *parallelepipeds.*

Definition. $\|.\| : \mathbb{R}^n \to \mathbb{R}$ is a norm if

(i) $\|x\| \geq 0$ and equality holds if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

14. Exercise. *Let* $\|.\|$ *be an arbitrary norm on* $\mathbb{R}^n$*. Prove that the following sets are convex:*

(i) *For* $x_0 \in \mathbb{R}^n$ *and* $r \in \mathbb{R}_+$*,* $\{x : \|x - x_0\| \leq r\}$*,*

(ii) $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq t\}$*.*

Definition. $\mathcal{S}$, $\mathcal{S}_+$ and $\mathcal{S}_{++}$ denote respectively the sets of symmetric, symmetric positive semidefinite, and symmetric positive definite matrices in $\mathbb{R}^{n \times n}$. The notation for a matrix M being positive semidefinite is $M \succeq 0$. The notation for a matrix M being positive definite is $M \succ 0$. In general, $N \preceq M$ means $0 \preceq M - N$.

15. Exercise. *Every affine subspace is a convex set. Give an example showing that the converse implication is false.*

16. Exercise. $\mathcal{S}$*,* $\mathcal{S}_+$ *and* $\mathcal{S}_{++}$ *are convex. Among these sets, which are affine subspaces, which are linear subspaces?*

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Definition. For $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$, the linear combination

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N$$

is called a convex combination of our points if $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ and the coefficients are non-negative.

17. Exercise. *Let* $D \subset \mathbb{R}^d$*. Prove that the following are equivalent:*

(i) *D is convex (according to the original definition, with any TWO points it contains all their convex combinations),*

(ii) *D is closed under forming convex combinations, i.e., if* $p_1, p_2, \ldots, p_N \in D$ *and* q *is a convex combination of the points* $\{p_i\}_{i=1}^N$*, then* $q \in D$*,*

(iii) For every line ℓ, $D \cap \ell$ is convex.

18. Exercise. *Let $Z \subset \mathbb{R}^d$ be a closed set. Prove that the following are equivalent:*

(i) Z is convex,

(ii) Z is closed under taking midpoints, i.e., if $p_1, p_2 \in Z$ and q is the midpoint of the segment p_1p_2 ($q = \frac{p_1+p_2}{2}$), then $q \in Z$,

(iii) Z can be expressed as an intersection of closed halfspaces,

(iv) Z can be expressed as an intersection of open halfspaces,

(v) Z can be expressed as an intersection of halfspaces.

19. Exercise. *Let $Z \subset \mathbb{R}^d$ be a bounded closed set. Let H be an open halfspace containing Z. Then there exists a closed halfspace H_0 that is contained in H and still contains Z. Is the statement true without the boundedness condition?*

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Definition. Let $R \subset \mathbb{R}^d$. Let $R^{cl} = cl(R)$ be the closure of the set R. Let $R^{int} = int(R)$ be the set of interior points of R. Let $R^{relint} = relint(R)$ be the set of interior points of R considered as a subset of $aff(R)$, i.e., the relative interior of R.

20. Exercise. *Let C be a convex set. Prove:*

(i) $cl(C) = cl(relint(C))$,

(ii) $relint(C) = relint(cl(C))$.

21. Exercise. *Let C and $\widetilde{C}$ be two non-empty convex sets. Then the following are equivalent:*

(i) $relint(C) = relint(\widetilde{C})$,

(ii) $cl(C) = cl(\widetilde{C})$,

(iii) $relint(C) \subset \widetilde{C} \subset cl(C)$.

22. Exercise. *Let $C \subset \mathbb{R}^d$ be a convex set, and let $\{C_i\}_{i \in I}$ be convex sets. Then the following sets are also convex:*

(i) $\cap_{i \in I} C_i$,

(ii) $C_1 + C_2 = \{x + y : x \in C_1, y \in C_2\}$,

(iii) $\lambda C = \{\lambda x : x \in C\}$ for every $\lambda \in \mathbb{R}$,

(iv) C^{cl},

(v) C^{int},

(vi) $C^{rel\text{-}int}$,

(vii) $\pi_A(C)$, *where* π_A *is the projection onto an affine subspace* A,

(viii) $f(C)$ *and* $f^{-1}(C)$, *where* $f : \mathbb{R}^d \to \mathbb{R}^e$ *is an affine map.*

23. Exercise. *Let* $f_{a,b}(x) = a + bx + x^2$. *Is the set* $\{(a,b) \mid f_{a,b}(0.37) \geq 1\}$ *convex? Is the set* $\{(a,b) \mid f_{a,b}(x) \geq 1$ *for all* $x \in [-1,1]\}$ *convex?*

24. Exercise. *Let* $p_{x_1,\ldots,x_m}(t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt$. *Is the set* $\{x \in \mathbb{R}^m : |p_{x_1,\ldots,x_m}(t)| < 1$ *for all* $|t| \leq \pi/3\}$ *convex?*

25. Exercise. *Prove that the set* $\{x \in \mathbb{R}^m : x_1A_1 + x_2A_2 + \ldots + x_mA_m \preceq B\}$ *is convex, where* $A_1, \ldots, A_m, B \in \mathcal{S}_n$.

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26. Exercise. *Let* C *be a convex set in* $\mathbb{R}^d$.

(i) *Let* $z \notin C$. *Then there exists a hyperplane* H *such that for the two closed halfspaces* Γ^+ *and* Γ^- *determined by* H *we have* $C \subset \Gamma^+$ *and* $z \in \Gamma^-$.

(ii) *Let* $z \notin C^{cl}$. *Then there exists a hyperplane* H *such that for the two open halfspaces* Γ^+ *and* Γ^- *determined by* H *we have* $C \subset \Gamma^+$ *and* $z \in \Gamma^-$.

27. Exercise. *Let* $R \subset \mathbb{R}^d$. *Then there exists a unique set* $M = conv(R)$ *in* $\mathbb{R}^d$ *such that*

(i)$_M$ $M \supset R$,

(ii)$_M$ M *is convex,*

(iii) *Every set* T *satisfying properties (i)*$_T$ *and (ii)*$_T$ *contains* M.

Definition. The set M defined in the previous exercise is called the convex hull of the set R.

28. Exercise. *If* R *is compact, then* $conv(R)$ *is also compact.*

4. Convex cones

Definition. For $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$, the linear combination

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N$$

is called a conic combination of our points if the coefficients α_i are non-negative.

Definition. A set $K \subset \mathbb{R}^d$ is a cone if it is closed under forming conic combinations.

29. Exercise. *Let* $R \subset \mathbb{R}^d$. *Then there exists a unique set* $M = cone(R)$ *in* $\mathbb{R}^d$ *such that*

(i)$_M$ $M \supset R$,

(ii)$_M$ M *is a cone,*

(iii) Every set T satisfying properties (i)$_T$ and (ii)$_T$ contains M.

Definition. The set M defined in the previous exercise is called the cone generated by the set R.

30. Exercise. *Determine which of the following sets are convex, affine subspaces, cones, linear subspaces.*

(i) lines,

(ii) hyperplanes,

(iii) halfspaces.

Consider the position of the above set classes.

31. Exercise. *Let F be a closed halfspace defined by a hyperplane passing through the origin. Prove that F is a cone.*

32. Exercise. *Prove that an affine subspace is a cone if and only if it is also a linear subspace.*

33. Exercise. *Prove that $T = \{x : x_i \geq 0$ for all $i\}$ is a cone. Is $\{x : x_i > 0$ for all $i\}$ a cone?*

Prove that $\mathcal{S}_+$ is a cone. Is $\mathcal{S}_{++}$ a cone?

34. Exercise. *Prove that $int(\mathcal{S}_+) = \mathcal{S}_{++}$.*

Definition. Let $h \in \mathbb{R}[x_1, x_2, \ldots, x_d]$ be a homogeneous polynomial. h is hyperbolic with respect to the vector $e \in \mathbb{R}^d$ if $h(e) > 0$ and for every $x_0 \in \mathbb{R}^d$ the polynomial $\widetilde{h}_{x_0}(t) = h(x_0 - te)$ has all its roots real.

Let $H_e(h) = \{x_0 : \text{all roots of } h_{x_0}(t) \text{ are positive}\}$.

35. Exercise. *Let $h(x) = x_1 x_2 \ldots x_d$. Then $h(x)$ is a hyperbolic polynomial with respect to the vector $e = (1, 1, \ldots, 1)$. Determine the set $H_e(h)$.*

36. Exercise. *Let $h(X) = \det(X)$ for $X \in \mathcal{S}_n$ (viewed as a polynomial). Then $h(X)$ is a hyperbolic polynomial with respect to the vector $e = I$. Determine the set $H_I(\det)$.*

37. Exercise. *Let h be a hyperbolic polynomial. Then H_e is the interior of a convex cone.*

38. Exercise. *Prove that the intersection of cones is also a cone.*

39. Exercise. *Prove that the following are equivalent:*

(i) K is a cone generated by a finite set of points/vectors.

(ii) K is the intersection of finitely many closed halfspaces defined by hyperplanes passing through the origin.

Definition. A cone that can be obtained in the above way is called a *finitely generated cone*.

40. Exercise. *Prove that the following are equivalent:*

(i) P is the intersection of finitely many closed halfspaces,

(ii) P is the sum of a polytope and a finitely generated cone.

5. Polyhedra, polytopes

Definition. Let $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$. Then a set of the form

$$\mathcal{P} = \{x : Ax \succeq b\}$$

is called a polyhedron. That is, a polyhedron is the solution set of a system of finitely many linear inequalities.

41. Exercise. *Write the sets of points in $\mathbb{R}^3$ representing a cube, a tetrahedron, and an octahedron as the solution set of a system of inequalities. We may choose the shape and position of our solids. Let k be the number of inequalities used in the description. What is the minimal value of k?*

Also write the 4-dimensional analogues of the above solids.

42. Exercise. *Solve the previous exercise in dimension d.*

43. Exercise. *Let $P \subset \mathbb{R}^d$ be a polyhedron.*

(i) Let π_A be the projection onto an affine subspace A. Then $\pi_A(P)$ is also a polyhedron.

(ii) Let $\alpha : \mathbb{R}^d \to \mathbb{R}^e$ be an affine map. Then $\alpha(P)$ is also a polyhedron.

(iii) Let $\alpha : \mathbb{R}^e \to \mathbb{R}^d$ be an affine map. Then $\alpha^{-1}(P)$ is also a polyhedron.

44. Exercise. *Let O_d be the d-dimensional octahedron. In a previous exercise we determined as the intersection of how many halfspaces O_d can be represented.*

Find a dimension $e > d$ and a polyhedron $\widehat{O} \subset \mathbb{R}^e$ such that

(i) $\widehat{O}$ is the intersection of $3d$ halfspaces,

(ii) O_d can be obtained as a projection of $\widehat{O}$.

45. Exercise. *Prove that polyhedra are closed and convex sets.*

46. Exercise. *Prove that a closed convex set is unbounded if and only if it contains a ray.*

47. Exercise. *Prove that if a closed convex set $\mathcal{P}$ is unbounded, then from every point of $\mathcal{P}$ there starts a ray lying in $\mathcal{P}$.*

48. Exercise. *Prove that if a closed convex set $\mathcal{P}$ contains a ray with direction vector v, then for every point of $\mathcal{P}$ the ray starting from that point with direction v lies in $\mathcal{P}$.*

Definition. A polyhedron $\mathcal{P}$ is a polytope if it is bounded.

49. Exercise. *The following are equivalent:*

(i) P is the convex hull of a finite set of points,

(ii) P is a polytope (bounded and the intersection of finitely many closed halfspaces).

50. Exercise. *How many halfspaces may be needed to represent the convex hull of n points? (We are not necessarily in 2 dimensions.)*

51. Exercise. *As the convex hull of how many points can a bounded intersection of n halfspaces be represented? (We are not necessarily in 2 dimensions.)*

52. Exercise. *Let $A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$. Prove that if the non-empty polyhedron $\mathcal{P} = \{x : Ax \succeq b\}$ is a polytope, then $k > n$.*

That is, for $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, a non-empty polyhedron $\mathcal{P} = \{x : Ax \succeq b\}$ is necessarily unbounded.

Definition. $H = \{x : nx = \alpha\}$ is a supporting hyperplane of the polyhedron $\mathcal{P}$ if $H \cap \mathcal{P} \neq \emptyset$ and $\mathcal{P} \subset H_{\leq} = \{x : nx \leq \alpha\}$.

Caution: The above definition is sensitive to the equation describing the hyperplane. $H' : -nx = -\alpha$ defines the same set of points, but $H'_{\leq} = \{x : -nx \leq -\alpha\}$ is completely different from the previous one.

53. Exercise. *Let $\mathcal{P}$ be a polyhedron and $x_0 \in \mathbb{R}^n$. Prove that the following are equivalent:*

(i) There exists a supporting hyperplane passing through x_0.

(ii) In every neighbourhood of x_0 there exist points belonging to $\mathcal{P}$ and points not belonging to $\mathcal{P}$.

Definition. The points described in the previous exercise are the boundary points of the polyhedron; their set is denoted by $\partial\mathcal{P}$.

Definition. Let $x \in \partial\mathcal{P}$. Then

$$C_x = \{n : \text{ there exists } \alpha \text{ such that } nx = \alpha \text{ is a supporting hyperplane}\}.$$

54. Exercise. *For $x \in \partial\mathcal{P}$, C_x is a cone.*

55. Exercise. *Let $\mathcal{P}$ be a polyhedron and $x \in \mathcal{P}$. The following are equivalent:*

(i) There exists a supporting hyperplane that contains only the point x from $\mathcal{P}$.

(ii) $x \in \partial\mathcal{P}$ and C_x has an interior point.

(iii) If a segment ab lying in $\mathcal{P}$ contains x, then necessarily $a = b = x$.

(iv) x cannot be written as a convex combination of other points of $\mathcal{P}$.

Definition. Points with the above property are called extreme points/vertices of $\mathcal{P}$. The set of these points is denoted by ext $\mathcal{P}$.

56. Exercise. *Let P be a closed bounded set. Then ext P is non-empty.*

57. Exercise. *Let P be a closed bounded convex set. Then $P =$ conv $(\{P : P \in$ ext $\mathcal{P}\})$.*

58. Exercise. *Let P be a closed bounded convex set. P is a polytope if and only if ext P is a finite set.*