Optimization Methods practice session, for Master students

Exercises: Analysis in R^d

Lecturer: Peter Hajnal Week 1 —

1. Convex Functions

Definition. $f : D \to \mathbb{R}$ is convex if $dom(f) := D \subset \mathbb{R}^n$ is convex and for any $x, y \in dom(f)$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

If strict inequality holds for $\lambda \in (0, 1)$, then f is called strictly convex.

1. Exercise. *Let f be a convex function. Prove that if x_0 is a local minimizer of f, then it is also a global minimizer.*

If f is strictly convex, then the minimizer (if it exists) is unique.

2. Exercise. *Every convex function defined on an open set is continuous. Does the statement remain true if the domain is not assumed to be open?*

3. Exercise. *Let $f : D(\subset \mathbb{R}) \to R$ be a convex function, where D is open. Then at every point $x_0 \in D$, f has left and right derivatives. Furthermore, f is differentiable except possibly at countably many points.*

4. Exercise (Jensen inequality, discrete form). *Prove that for a convex function f, for any $x_1, x_2, \ldots, x_N \in dom f$*

$$f\left(\sum_{i=1}^{N} \theta_i x_i\right) \leq \sum_{i=1}^{N} \theta_i f(x_i),$$

where $\theta_i \in [0, 1]$ and $\sum \theta_i = 1$.

Show that if $dom f$ is convex, then the statement is reversible.

5. Exercise (Jensen inequality, continuous form). *Let X be a random variable taking values in $dom f$, where f is convex. Then*

$$f(\mathbb{E}X) \leq \mathbb{E}f(X).$$

Definition. Let $f : D \to \mathbb{R}$.

- The graph of f:

$$gr(f) = \{(x, y) : x \in D, y = f(x)\} \subset \mathbb{R}^{n+1}.$$

- The epigraph of f:

$$epi(f) = \{(x, y) : x \in D, y \geq f(x)\} \subset \mathbb{R}^{n+1}.$$

6. Exercise. *Let $f(x) = x^2$. Which of the following sets are convex?* $\{x \in \mathbb{R} | f(x) \leq 1\}$, $\{x \in \mathbb{R} | f(x) \geq 1\}$, $\{(x, f(x)) | x \in \mathbb{R}\}$, $\{(x, y) | y \geq f(x)\}$. *Formulate general rules and prove them.*

7. Exercise. *Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^d$. Prove that the following are equivalent:*

(i) f *is convex,*

(ii) $epi(f)$ *is a convex set,*

(iii) $f|_\ell : \ell \cap D \to \mathbb{R}$ *is convex for every line* ℓ.

8. Exercise. *Prove that for convex* f,

$$\{x : f(x) \leq \alpha\} \text{ is convex for every real } \alpha.$$

Is this property equivalent to convexity?

Definition. Let $f : D \to \mathbb{R}$. f is differentiable if $D \subset \mathbb{R}^n$ is open and there exists a gradient $\nabla f : D \to \mathbb{R}^n$ such that

$$f(x) = f(x_0) + \nabla f|_{x_0}{}^T (x - x_0) + o(\|x - x_0\|).$$

9. Exercise. *Let $f : D \to \mathbb{R}$ be differentiable. The following are equivalent:*

(i) f *is convex,*

(ii) $f(x) \geq f(x_0) + \nabla f|_{x_0}{}^T (x - x_0)$,

(iii) $f(x) = \max\{\ell(x) : \ell \in \mathcal{A}, \ell(x) \leq f(x)\}$, *where $\mathcal{A}$ is the set of affine functions.*

Definition. f is twice differentiable if $D \subset \mathbb{R}^n$ is open and $\nabla^2 f$ (Hessian) exists everywhere.

10. Exercise. *Let $f : D \to \mathbb{R}$ be twice differentiable on an open convex domain. Then f is convex iff $\nabla^2 f \succeq 0$ everywhere.*

11. Exercise. *Let $f(x) = \frac{1}{x^2}$, $dom(f) = \mathbb{R} - \{0\}$. Compute $f''(x)$. Is f convex?*

12. Exercise. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $dom(f) = \mathbb{R}^n$ defined by*

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

where $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$. Compute ∇f and $\nabla^2 f$.

13. Exercise. *Prove that the following functions are convex:*

(i) $f(x) = ax + b$, $D = \mathbb{R}$,

(ii) $f(x) = a^T x + b$, $D = \mathbb{R}^n$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$,

(iii) $f(x) = e^{ax}$, $D = \mathbb{R}$,

(iv) $f(x) = x^\alpha$, $D = \mathbb{R}_{++}$, $\alpha \in (-\infty, 0] \cup [1, \infty)$,

(v) $f(x) = x^\beta$, $D = \mathbb{R}$, $\beta \in [1, \infty)$,

(vi) $f(x) = x \log x$, $D = \mathbb{R}_{++}$,

(vii) $f(x) = \|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$, $D = \mathbb{R}^n$, $p \geq 1$

(viii) $f(x) = \|x\|_\infty = \max\{|x_k| : k = 1, 2, \ldots, n\}$, $D = \mathbb{R}^n$, $p \geq 1$,

(ix) $f(X) = tr(A^T X) + b$, $D = \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^m$,

(x) $f(X) = \sqrt{\lambda_{max}(X^T X)}$, $D = \mathbb{R}^{n \times m}$,

(xi) $f(X) = \log \det X$, $X \in \S^n_{++}$,

(xii) $f(x) = \|Ax - b\|_2^2$, $D = \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$,

(xiii) $f(x) = \log\left(\sum_{k=1}^n \exp x_k\right)$, $D = \mathbb{R}^n$,

(xiv) $f(x, y) = \frac{x^2}{y}$, $D = \mathbb{R}^2 \cap \{y : y > 0\}$,

(xv) *For $x \in \mathbb{R}^n$, let $f_k(x)$ denote the sum of the k largest coordinates of x,*

(xvi) $f(x) = \max\{a_1^T x + b_1, a_2^T x + b_2, \ldots, a_N^T x + b_M\}$.

Definition. A function $f : D \to \mathbb{R}$ — where $D \subset \mathbb{R}^n$ is a convex set — is called *concave* if $-f$ is convex.

14. Exercise. *Prove that the following functions are concave:*

(i) $f(x) = ax + b$, $D = \mathbb{R}$,

(ii) $f(x) = x^\alpha$, $D = \mathbb{R}_{++}$, $\alpha \in (0, 1)$,

(iii) $f(x) = \log x$, $D = \mathbb{R}_{++}$,

(iv) $f(x) = \sqrt[n]{\prod_{i=1}^n x_i}$, $x \in \mathbb{R}^n_{++}$.

15. Exercise. *The following operations preserve convexity:*

(i) *Multiplication by a nonnegative scalar: if f is convex and $\lambda \geq 0$, then λf is convex.*

(ii) *Summation: if f_1 and f_2 are convex, then so is $f_1 + f_2$.*

(iii) *If $f(x, y_0)$ is convex for every $y_0 \in A$ and $w(y_0) \geq 0$ for all $y_0 \in A$, then*

$$\int_A w(y) f(x, y) dy$$

is convex whenever the integral exists.

(iv) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. If $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$, define*

$$g(x) = f(Ax + b),$$

where $dom(g) = \{x : Ax + b \in dom(f)\}$. Then g is convex.

(v) *If f_1 and f_2 are convex, then $f(x) = \max\{f_1(x), f_2(x)\}$ is convex.*

(vi) Let f_i $(i \in I)$ be convex functions, where $I \subset \mathbb{R}^k$ is convex. Then $f(x) = \sup\{f_i(x) : i \in I\}$ is convex.

16. Exercise. *(i) Let f_1 and f_2 be convex. Is $f(x) = \min\{f_1(x), f_2(x)\}$ necessarily convex?*

(ii) Let $f(x, y)$ be convex. Is $f(x) = \inf\{f(x, y) : y\}$ convex?

Definition. Let $f : D \to \mathbb{R}$. f is lower semicontinuous if for every $x, \{x_i\}_{i=0}^{\infty} \in D$ such that $x_i \to x$ we have

$$f(x) \leq \liminf_{k\to\infty} f(x_k).$$

17. Exercise. *The following are equivalent:*

(i) The sublevel sets $\{x : f(x) \leq a\}$ are closed for every $a \in \mathbb{R}$,

(ii) f is lower semicontinuous,

(iii) epi(f) is closed.

2. Quasiconvex Functions

Definition. Let $f : D(\subset \mathbb{R}^n) \to \mathbb{R}$. The function f is quasiconvex if $dom f = D$ is convex and for every real α

$$S_\alpha = \{x \in dom f : f(x) \leq \alpha\}$$

is convex.

f is quasiconcave if $-f$ is quasiconvex.

18. Exercise. *Prove that every convex function is quasiconvex. Is the converse true? (If yes, prove it; if not, give a counterexample.)*

19. Exercise. *Determine whether the following functions are convex, concave, quasiconvex or quasiconcave:*

(i) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \lceil x \rceil$.

(ii) $f : \mathbb{R}_{++} \to \mathbb{R}$, $x \mapsto \log x$.

(iii) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto xy$.

(iv) $f : \mathbb{R}^2_{++} \to \mathbb{R}$, $(x, y) \mapsto xy$.

(v) $f : dom f(\subset \mathbb{R}^n) \to \mathbb{R}$, $x \mapsto \frac{a^T x + b}{c^T x + d}$, where $dom f = \{x : c^T x + d > 0\}$.

(vi) Let $a \neq b \in \mathbb{R}^n$. Let $\mathcal{D} = \{x : |x - a| < |x - b|\} \subset \mathbb{R}^n$. $f : \mathcal{D} \to \mathbb{R}$, $x \mapsto \frac{|x-a|}{|x-b|}$.

(vii) $X \in \mathcal{S}^n \to \mathbb{R}$, $X \mapsto rk\ X$.

(viii) $X \in \mathcal{S}^n_+ \to \mathbb{R}$, $X \mapsto rk\ X$.

(ix) $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \max\{i : x_i \neq 0\}$ $(\max \emptyset = 0)$.

(x) $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto |\{i : x_i \neq 0\}|$.

(xi) $f : \mathbb{R}^n_+ \to \mathbb{R}$, $x \mapsto |\{i : x_i \neq 0\}|$.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$.

f is $\vee$-*unimodal* if there exists $m \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ such that f is monotonically decreasing on $(-\infty, m)$ and monotonically increasing on (m, ∞). If m is finite then m is a minimizer.

f is $\wedge$-*unimodal* if there exists $m \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ such that f is monotonically increasing on $(-\infty, m)$ and monotonically decreasing on (m, ∞). If m is finite then m is a maximizer.

20. Exercise. *Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is quasiconvex iff it is $\vee$-unimodal. Similarly, it is quasiconcave iff it is $\wedge$-unimodal.*

21. Exercise (Quasiconvex Jensen inequality). *Prove that for quasiconvex f*

$$f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}.$$

Show the converse when $\mathrm{dom} f$ is convex.

Formulate and prove the analogous Jensen inequality for quasiconcave functions.

22. Exercise. *Prove that f is quasiconvex iff its restriction to every line is quasiconvex.*

23. Exercise. *Let f be differentiable and $\mathrm{dom} f$ convex. Show that f is quasiconvex iff for all $x, y \in \mathrm{dom} f$:*

$$f(y) \leq f(x) \Rightarrow (\nabla f)|_x T(y-x) \leq 0.$$