Higher order connectivity of graphs

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Flows: Reminder

In the course *Algorithms and Their Complexity*, the theory of flows was discussed (the necessary definitions can be found there). The following theorem is the fundamental theorem of flows.

Theorem (the main theorem of flows)

Let $\mathcal{H}$ be a network and f be a flow in it. Then the following are equivalent:

(i) f is a maximum value flow in the network $\mathcal{H}$.

(ii) There is no augmenting path with respect to f in the network $\mathcal{H}$.

(iii) There exists a source/sink cut in $\mathcal{H}$ with capacity equal to the value of f.

The main theorem of flows: Consequences

Consequence: Maximum Flow-Minimum Cut Theorem, MFMC Theorem

Let $\mathcal{H} : (\overrightarrow{G}, c, s, t)$ be a network. Then

$$\max\{\mathrm{val}(f) : f \text{ is a flow in } \mathcal{H}\} = \min\{c(\mathcal{V}) : \mathcal{V} \text{ is a source/sink cut in } \mathcal{H}\}.$$

Another consequence of the fundamental theorem is the Ford-Fulkerson algorithm.

Integral Flow Theorem

If every edge in network $\mathcal{H}$ has an integer capacity ($c : E(\overrightarrow{G}) \to \mathbb{Z}$), then there exists an optimal flow in which every edge carries an integer amount of material.

Uniform Networks

Let $\overrightarrow{G}$ be a directed graph with source/sink nodes s and t. If we set the capacity of every edge to be 1, we obtain a network $\mathcal{H}_{\overrightarrow{G}}$.

Easy observation/exercise

Let $\overrightarrow{G}$ be an arbitrary directed graph with two distinguished nodes s and t. Let $\mathcal{H}_{\overrightarrow{G}}$ be the following network: $(\overrightarrow{G}, c \equiv 1, s, t)$.

(i)

$$\max\{\text{val}(f) : f \text{ is a flow in } \mathcal{H}_{\overrightarrow{G}}\} = \max\{k : P_1, P_2, \ldots, P_k \text{ are edge-disjoint } \overrightarrow{st}\text{-paths in } \overrightarrow{G}\}$$

(ii)

$$\min\{c(\mathcal{V}) : \mathcal{V} \text{ is a source/sink cut in } \mathcal{H}_{\overrightarrow{G}}\} = \min\{|S| : S \subset E(G) \text{ is a source}\to\text{sink separating edge set}\}.$$

Menger's First Theorem

The MFMC theorem and the observation provide a purely graph-theoretical theorem:

Menger's Theorem

Let $\overrightarrow{G}$ be an arbitrary directed graph with two distinguished nodes s and t. Then

$$\max\{k : P_1, P_2, \ldots, P_k \text{ are edge-disjoint } \overrightarrow{st} \text{ paths in } \overrightarrow{G}\} =$$
$$\min\{|S| : \; S \subset E(G) \text{ is a source/sink separating edge set}\}.$$

Menger's Theorems for Directed Graphs

Menger's Theorems

Let $\overrightarrow{G}$ be an arbitrary directed graph with two distinguished nodes s and t. Then

(i)

$$\max\{k : P_1, P_2, \ldots, P_k \text{ edge-disjoint } \overrightarrow{st} \text{ paths in } \overrightarrow{G}\} = \min\{|S| : S \subset E(G) \text{ is a source/sink separating edge set}\}.$$

(ii)

$$\max\{k : P_1, P_2, \ldots, P_k \text{ internally node-disjoint } \overrightarrow{st} \text{ paths in } \overrightarrow{G}\} = \min\{|U| : U \subset V(\overrightarrow{G}) - \{s, t\} \text{ is a source}\rightarrow\text{sink separating node set}$$

Menger's Theorems for Undirected Graphs

Menger's Theorems for Undirected Graphs

Let G be an arbitrary undirected graph with two distinguished nodes s and t. Then

(i)

$$\max\{k : P_1, P_2, \ldots, P_k \text{ edge-disjoint } st \text{ paths in } G\} = \min\{|S| : \ S \subset E(G) \text{ is a source/sink separating edge set}\}.$$

(ii)

$$\max\{k : \ P_1, P_2, \ldots, P_k \text{ internally node-disjoint } st \text{ paths in } G\} = \min\{|U| : \ U \subset V(G) - \{s, t\} \text{ is a source/sink separating node set}\}.$$

A Note

In the case of internally node-disjoint paths, if there exists an $\overrightarrow{st}$ or st edge, then the theorem is uninteresting.

There is no suitable separating set U, and the paths P_i may be the same one-edge path (without internal nodes).

That is, the optimum of both optimization problems is ∞. In this case, it is worthwhile to assume the absence of edges between s and t.

k-Edge Connectivity

Definition

Let k be a positive integer. A graph G is *k-edge-connected* (shortened as k-edge-connected) if removing any set of fewer than k edges results in a connected graph.

For every set $F \subseteq E(G)$ with $|F| < k$, the graph $G - F$ is connected.

The condition must hold even for $F = \emptyset$, i.e., our base graph must be connected. Connectivity should be preserved when any proper but not *large* set of edges is removed.

k-(Vertex) Connectivity

Definition

A graph G is *k-(vertex) connected* (shortened as $k$vc), if removing any set of fewer than k vertices results in a connected graph and $|V(G)| > k$.

For every set $U \subseteq V(G)$ with $|U| < k$, the graph $G - U$ is connected, and $|V| > k$.

The technical condition for the vertex count serves to ensure that the graph is sufficiently large: after removing the *not too large* vertex set mentioned in the definition, at least two vertices should remain.

Examples

Example

Trees are not 2-edge-connected if they have edges.

Example

Cycles are 2-connected (if they have at least three vertices) and therefore 2-edge-connected, but they are not 3-connected.

Example

Among graphs with $k+1$ vertices, only the complete graph is k-connected.

Connections

The following diagram summarizes the relationships between various connectivity concepts. Graph classes not derivable from the diagram are not included.

$$
\begin{array}{ccccccccccc}
1\text{vc} & \Leftarrow & 2\text{vc} & \Leftarrow & 3\text{vc} & \Leftarrow & \dots & \Leftarrow & k\text{vc} & \Leftarrow \\
\Updownarrow^{*} & & \Big\Downarrow & & \Big\Downarrow & & & & \Big\Downarrow & \\
\text{vc} & & & & & & & & & \\
\Updownarrow & & & & & & & & & \\
1\text{ec} & \Leftarrow & 2\text{ec} & \Leftarrow & 3\text{ec} & \Leftarrow & \dots & \Leftarrow & k\text{ec} & \Leftarrow
\end{array}
$$

The horizontal connections are obvious from the definitions. The vertical arrows represent a somewhat more challenging relationship. The starred equivalence is only partially true. In 1-vertex-connectedness, the condition of having at least two vertices is essential; this is not a requirement for connectivity. The other vertical implications follow from the lemma below.

Lemma

Lemma

Let e be any edge of graph G and v be any vertex. Let $k \geq 2$.

(a) If G is k-edge-connected, then $G - e$ is $(k-1)$-edge-connected.

(b) If G is k-vertex-connected, then $G - v$ is $(k-1)$-vertex-connected.

(c) If G is k-edge-connected, then $G - v$ can have any number of components.

(d) If G is k-vertex-connected, then $G - e$ is $(k-1)$-vertex-connected.

Characterization of Higher Connectivity

Theorem

(i) A graph G is k-edge-connected if and only if, for any two of its vertices, there exist k pairwise edge-disjoint paths between them.

(ii) A graph G is k-vertex-connected if and only if, for any two of its vertices, there exist k paths, whose internal vertices form pairwise disjoint sets (Path system is *vertex-independent*), and $|V(G)| > k$.

Proof: Trivial Direction

One direction of each statement is straightforward: the existence of the required paths ensures the corresponding connectivity.

Indeed: Suppose that after the appropriate reduction of our graph, we obtain a non-connected graph between two vertices — x and y.

Applying the condition to x and y, the guaranteed path system between x and y is in our graph. Removing the edges/vertices must eliminate each of them. Due to the independence of the paths, this cannot happen.

Proof: Non-trivial Direction (i)

Let G be a graph, and $x, y \in V$ be any two vertices, with k given.

Assume that G is k-edge-connected, and apply Menger's theorem.

$$k \leq \min\{|L| : \ L \subseteq E(G), G - L \text{ does not have an } xy \text{ path}\} =$$
$$= \max\{l : P_1, \ldots, P_l \text{ pairwise edge-disjoint } xy \text{ paths in } G\}$$

Thus, there exist k pairwise edge-disjoint xy paths in G.

Proof: Non-trivial Direction (ii)

Assume that G is k-vertex-connected.

Let P be the set of xy edges, and let p be its cardinality. The edges in P are vertex-independent xy paths.

If $p \geq k$, then the statement holds. If $p \leq k-1$, then $G-P$ is $(k-p)$-vertex-connected.

We show that there exist $k-p$ vertex-independent xy paths in $G-P$.

Apply the undirected, vertex-independent version of Menger's theorem (x and y are not connected in $G-P$):

$$k-p \leq \min\{|U| : \ U \subseteq V(G) \setminus \{x,y\}, \ G-P-U \text{ does not have an } xy$$
$$= \max\{l : \text{ vertex-independent } xy \text{ paths in } G-P\}$$

Hence, there exist $k-p$ vertex-independent xy paths in $G-P$. Adding the elements of P as 1-length xy paths, we obtain k vertex-independent xy paths in G.

Connectivity Parameters

Definition

The connectivity parameters of graph G:

$$\kappa_e(G) = \begin{cases} \max\{k : \; G \text{ is } k\text{-edge-connected}\}, & \text{if } G \text{ is connected} \\ 0, & \text{if } G \text{ is not connected} \end{cases}$$

$$\kappa(G) = \begin{cases} \max\{k : \; G \text{ is } k\text{-vertex-connected}\}, & \text{if } G \text{ is connected} \\ 0, & \text{if } G \text{ is not connected} \end{cases}$$

Observation

For every graph G, the following hold:

$$\kappa_e(G) = \min_{x,y \in E(G)} \max\{k : \ P_1, \ldots P_k \text{ pairwise edge-disjoint } xy \text{ paths in } G\}$$
$$= \min_{x,y \in E(G)} \min_{\mathcal{V} \ xy \text{ cut}} |E(\mathcal{V})| = \min_{\mathcal{V} \text{ cut}} |E(\mathcal{V})|,$$

where $\mathcal{V} = \{S, T\}$, $S \cup T = V(G)$, $S \cap T = \emptyset$, $S, T \neq \emptyset$.

Algorithmic Remarks

Theorem

$\kappa_e(G)$ and $\kappa(G)$ can be efficiently calculated with a flow algorithm.

Theorem

Calculating $\max_{\mathcal{V} \text{ cut}} |E(\mathcal{V})|$ is *hard*, an $\mathcal{NP}$-complete problem.

Break

Minimal k-edge-connected graphs

Definition

Let G be a graph, k a positive integer. G is called minimal k-edge-connected if

(i) k-edge-connected, and

(ii) for any edge e, $G - e$ is not k-edge-connected.

For $k = 1$, minimal k-edge-connected graphs are trees.

If G is minimal k-edge-connected, then it has no loops.

If G is k-edge-connected and has at least two vertices, then every vertex has degree at least k.

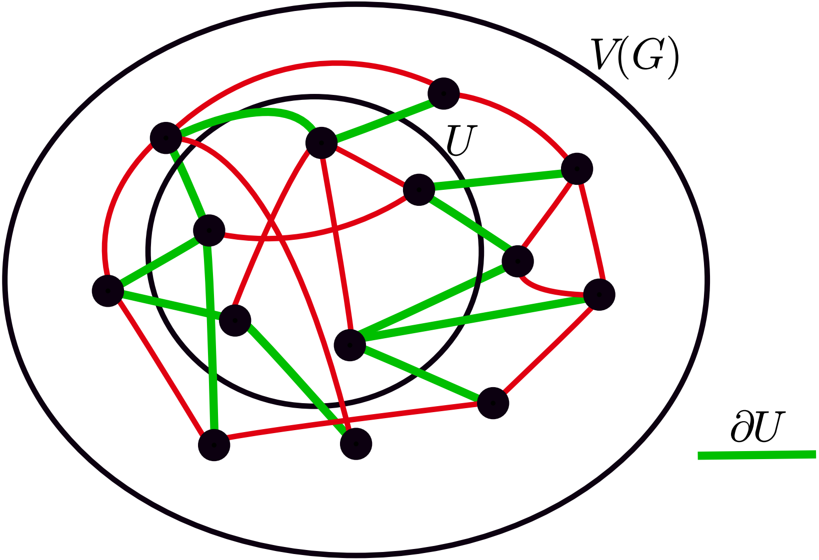
Boundary of a Vertex Set: Definition

Notation

The boundary of $U \subseteq V(G)$:

$$\partial U = \{xy \in E(G) : \; x \in U \text{ and } y \notin U, \text{ or } x \notin U \text{ and } y \in U\}$$

Boundary of a Vertex Set: Image



Boundary of a Vertex Set: Image

$\partial U = \partial \overline{U}$, where $\overline{U} = V(G) \setminus U$.

If G has no loops, then for any $x \in V(G)$, $d(x) = |\partial\{x\}|$.

G is k-edge-connected if and only if the boundary of any proper, non-empty subset of $V(G)$ contains at least k edges.

Mader's Theorem

Mader's Theorem

Let k be a positive integer, G a minimal k-edge-connected graph with $|V(G)| \geq 2$. Then the following hold:

(i) G has a k-degree vertex.

(i)$^+$ G has at least two k-degree vertices.

Definition

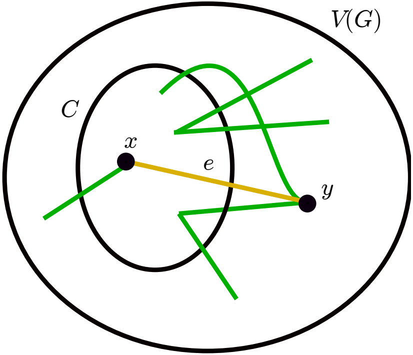
k a positive integer, G a minimal k-edge-connected graph. A set $P \subseteq V(G)$ is called a precise set if its boundary contains exactly k edges.

The statement (i) of the theorem is equivalent to the existence of a singleton precise set in G.

Observation

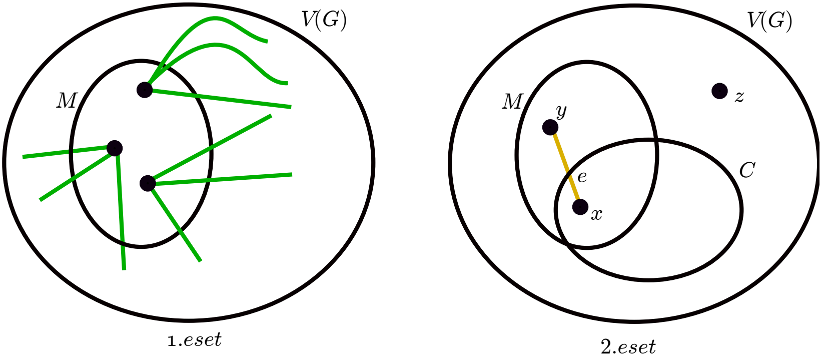
Observation

If, for any $e = xy \in E(G)$, $G - e$ is not k-edge-connected, then there exists a separating set $C \subset V(G)$ such that $|\partial_{G-e} C| < k$. In this case, C is a precise set in G and separates x and y.



Proof of Mader's Theorem (i): Cases

Let M be a minimal precise set in G, i.e., a precise set such that none of its proper subsets is precise. We claim that M is a singleton set.



Proof of Mader's Theorem (i): Case 1

Case 1: *No edge crosses within M.*

In this case, the following equality holds:

$$k = |\partial M| = \sum_{m \in M} |\partial\{m\}| = \sum_{m \in M} d(m)$$

Since every vertex in G has degree at least k, M can only be a singleton set.

Proof of Mader's Theorem (i): Case 2

Case 2: At least one edge crosses within M.

Let xy be such an edge. Since G has no loops, x and y are distinct vertices.

M is precise, so $M \neq V(G)$.

Let $z \in V(G) \setminus M$.

Due to the observations, there exists a precise set $C \subseteq V(G)$ that separates x and y. Without loss of generality, we can assume $z \notin C$; if z was an element of C, we could replace C with $\overline{C}$.

Proof of Mader's Theorem (i): Submodularity

Lemma

$$|\partial(A \cap B)| + |\partial(A \cup B)| \leq |\partial(A)| + |\partial(B)|.$$

Both sides count edges.

Let $e = xy \in E(G)$.

There are eight cases. In all cases the right hand side counts e at least as many times as the left hand side.

Proof of Mader's Theorem (i): Case 2

Apply the lemma to M and C.

By our choices, $M \cap C \neq \emptyset$ and $M \cup C \neq V(G)$.

$$k + k \leq |\partial(M \cap C)| + |\partial(M \cup C)| \leq |\partial M| + |\partial C| = 2k$$

The first and last terms in the inequality are equal, so all our estimates are tight, in particular, $|\partial(M \cap C)| = k$.

Since x and y belong to different subsets of C, $M \cap C$ is a proper precise subset of M.

This contradicts the minimality of M, so the second case is not possible.

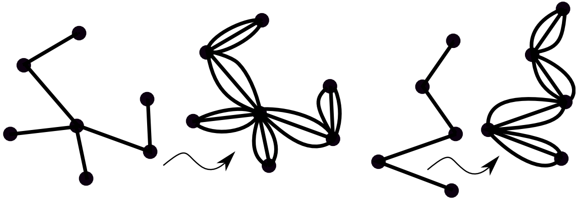
(ii) Let P be a precise set in G. In this case, $\overline{P}$ is also precise. P and $\overline{P}$ each have a minimal precise subset for containment, let these be M_1 and M_2. These are two different singleton precise sets in G.

Example

Let $m \geq 2$ be an integer. If we replace each edge in a tree T with m parallel edges, we obtain a minimal m-edge-connected graph.

In particular, if T is a path of length at least one, then we have exactly two vertices with degree m.

The figure below illustrates the case of $m = 3$.



Break

Lovász's Lifting Lemma

Lovász's Lifting Lemma

Let G be a graph, $u \in V(G)$, $G_0 = G - u$, $k \geq 2$ an integer. Suppose that the number of edges between u and G_0 is even and positive, and u satisfies the following condition:

(L) If U is a nontrivial subset of $V(G_0)$, then $|\partial_G U| \geq k$.

Then, there exist two edges $e = ux$ and $f = uy$ incident to u such that the graph $\widetilde{G} = G - e - f + xy$ also satisfies condition (L).

Lovász's Lifting Lemma in Pictures

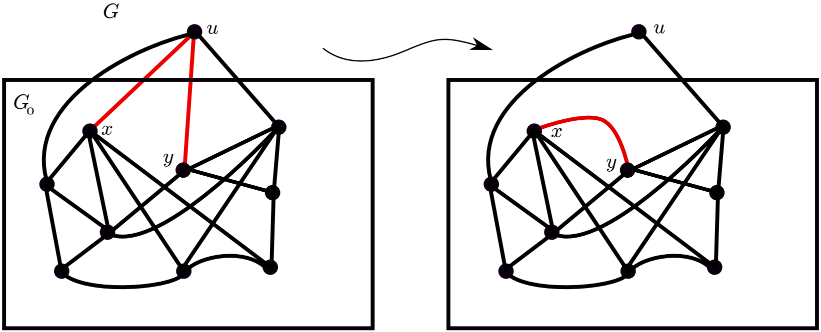
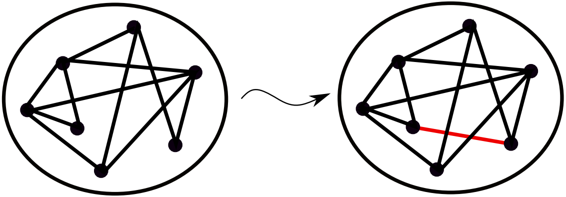
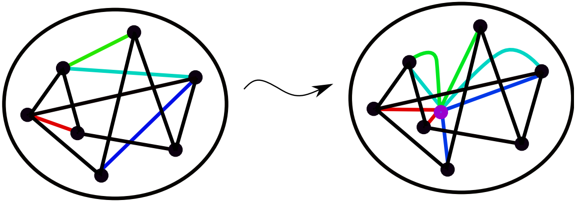


Figure: In the figure, the red edges are being exchanged. If an edge already exists between x and y, we add a new edge parallel to the existing xy edges.

Edge addition: We add a new edge between two vertices of G: $G \to G^+$.





Contraction of $k/2$ edges: We remove $k/2$ edges from G, replace their endpoints with new vertices, and then identify the $k/2$ new vertices: $G \to \widetilde{G}$.

Observation

If G is k-edge-connected, then G^+ and $\widetilde{G}$ are also k-edge-connected.

For G^+, this is obvious. For $\widetilde{G}$, we need to verify that the boundary of any non-empty proper subset of $V(G)$ has at least k elements. This is sufficient to check for subsets that do not contain the new vertices. This is a straightforward task.

Observation

Let G_0 be the graph with one vertex and no edges.

Assume that G can be built in the following way:

$$G_0 \to G_1 \to \ldots \to G_l = G,$$

where for every $i = 0, \ldots, l-1$, the $G_i \to G_{i+1}$ operation is either edge addition or contraction of $k/2$ edges.

Then G is k-edge-connected.

Application of the Lifting Lemma: Growth of 2ℓ-edge-connected graphs

Our goal is to prove the converse of the observation above.

Theorem

If k is a positive even number, and G is a k-edge-connected graph, then G can be built from G_0 (see above) using the previous two operations.

Proof of the Enhanced Lifting Lemma

Let G and k be given. We prove the statement by induction on the number of edges.

G_0 and all graphs with at most one edge can be trivially constructed.

Let G be a k-edge-connected graph with at least two vertices. Assume that all graphs with at most $|E(G)|-1$ edges can be constructed.

G is not minimally k-edge-connected. Then G has an edge e such that $G-e$ is k-edge-connected. Since $|E(G-e)|=|E(G)|-1$, by the induction hypothesis, $G-e$ can be constructed. Thus, G can be constructed by adding back the edge e to $G-e$.

From now on: G is minimally k-edge-connected, $|V(G)|\geq 2$.

Completion of the Proof

In this case, G has a vertex u with degree k.

By applying the lifting lemma to this vertex $k/2$ times, we lift the edges, and then remove u.

Thus, by the lemma, we obtain a graph H that is k-edge-connected and has fewer edges than G. Therefore, H can be constructed.

If we contract the edges in $E(H) \setminus E(G)$ to a single vertex u, we obtain the graph G.

The Sharpened Lifting Lemma

We prove the following slightly stronger version of Lovász's lemma:

Lemma$^+$

Let G be a graph, $u \in V(G)$, $G_0 = G - u$, $k \geq 2$ an integer. Suppose that the number of edges between u and G_0 is even and positive, and G_0 satisfies

(L) If U is a nontrivial subset of $V(G_0)$, then $|\partial_G U| \geq k$.

Then, for any edge $e = ux$, there exists an edge $f = uy$ such that the graph $\widetilde{G} = G - e - f + xy$ also satisfies property (L).

Beginning of the Proof

Let G, u, k, and $e = ux$ be given.

Let's try the edge $f = uy$. Let $\widetilde{G} = G - e - f + xy$. Suppose that $\widetilde{G}$ does not have property (L). Then, there exists a set $C_f \subseteq V(G_0)$ that is a counterexample, meaning $|\partial_{\widetilde{G}} C_f| < k$.

If C_f separates x and y, then $|\partial_{\widetilde{G}} C_f| = |\partial_G C_f| \geq k$, which is a contradiction.

The Proof

Assume $u \notin C_f$. If C_f separates x and y, or $x, y \notin C_f$, then C_f would not be a counterexample. Thus, $x, y \in C_f$.

Then $k > |\partial_{\widetilde{G}} C_f| = |\partial_G C_f| - 2$, so $|\partial_G C_f| \leq k + 1$. Let $\overline{C_f}$ be $V(G_0) \setminus C_f$.

Let d be the number of edges between u and G_0, d_1 be the number of edges between u and C_f, d_2 be the number of edges between u and $\overline{C_f}$, and d_3 be the number of edges between C_f and $\overline{C_f}$.

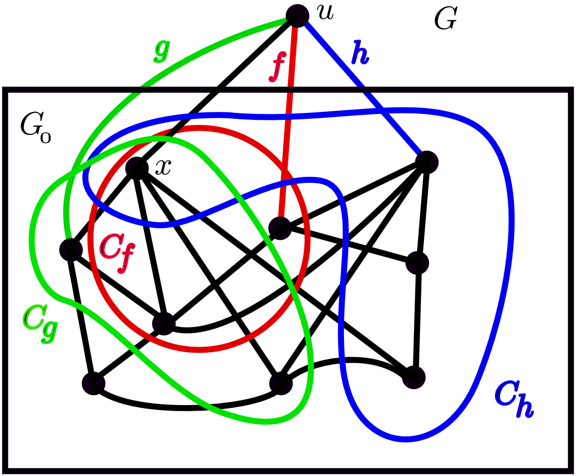
Since G has property (L), we have $d_2 + d_3 = |\partial_G \overline{C_f}| \geq k$, and $d_1 + d_3 = |\partial_G C_f| \leq k + 1$. Since $d_1 + d_2 = d$ is even, we have

$$d_1 \leq d_2. \tag{1}$$

Thus, at most half of the edges incident to u can go to the counterexample set C_f.

Iteration

Repeat the procedure for other edges.



System of Counterexample Sets

Either we find a suitable edge uy, or we obtain a set of counterexample sets $\mathcal{C}$ such that $\bigcup_{C\in\mathcal{C}} C$ contains the neighborhood of u.

Let's thin out the $\mathcal{C}$ system in such a way that this property holds, but with the minimal number of counterexample sets.

Let $\mathcal{C}_0$ be the obtained system. Due to (1), it cannot be the case that $\mathcal{C}_0$ consists of only two counterexample sets: Otherwise, at most half of the edges incident to u could extend to both sets in a way that the edge ux is included in both, and the two sets together still cover the neighborhood of u. This is clearly impossible.

Lemma

Lemma

For any graph H and sets $A, B, C \subseteq V(H)$, the following inequality holds:

$$|\partial(A \cap B \cap C)| + |\partial(A \cap \overline{B} \cap \overline{C})| + |\partial(\overline{A} \cap B \cap \overline{C})| + |\partial(\overline{A} \cap \overline{B} \cap C)| \leq |\partial A| + |\partial B| + |\partial C|$$

The proof of the lemma (like proving submodular inequalities) involves simple calculations. We need to check for each edge how many times it contributes to the left and right sides. Each edge contributes at least as much to the right side as to the left side.

Conclusion of the Proof

Let $C_1, C_2, C_3 \in \mathcal{C}_0$. Apply the lemma to these, with the additional observation that the edge ux is counted once on the left side but three times on the right side:

$$\begin{aligned}
|\partial(C_1 \cap C_2 \cap C_3)| + |\partial(C_1 \cap \overline{C_2} \cap \overline{C_3})| + |\partial(\overline{C_1} \cap C_2 \cap \overline{C_3})| + \\
|\partial(\overline{C_1} \cap \overline{C_2} \cap C_3)| \leq & |\partial C_1| + |\partial C_2| + |\partial C_3| \\
\leq & (k+1) + (k+1) + (k+1) - 2
\end{aligned}$$

Each of the four terms in the starting four-term sum involves the intersection of three sets, all of which are non-empty (the first has x as an element, the others are empty due to the minimality of $\mathcal{C}_0$). Thus, due to property (L), each term is at least k. Summing up, we have $4k \leq 3k + 1$, i.e., after sorting, $k \leq 1$.

This is a contradiction because we assumed $k \geq 2$. Thus, one of the edges uy satisfies the lemma.

Thank you for your attention!