Ramsey theory

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2023 fall

The Ramsey parameter

Definition

Let $\text{Ramsey}(G) = \max\{\alpha(G), \omega(G)\}$, the Ramsey parameter of graph G.

An alternative description: A subset H of vertices is *homogeneous* if and only if it is either independent or a clique. The Ramsey parameter is the maximum size of a homogeneous set.

Another way of describing it: Color the edges of the graph green, and connect non-adjacent vertices with a red edge. In this way, we obtain a 2-edge-coloring of the complete graph on the vertex set $V(G)$, representing G. A homogeneous set is a set of vertices in which all edges have the same color, or in other words, it is *monochromatic*.

Ramsey Algorithm: Finding a large homogeneous set

(Ramsey Algorithm)

Input: A simple graph G, output: a homogeneous set H.
Initialization: $KF = \emptyset$, $T = V(G)$ // KF is the set of selected vertices. T is the set of surviving vertices.
While $T \neq \emptyset$ **Selection step:** Let $x \in T$ be any surviving vertex.
$KF \leftarrow KF \cup \{x\}$. $N := N_T(x) = \{s \in T : xs \in E\}$,
$\overline{N} := \overline{N}_T(x) = \{s \in T : xs \notin E\} = T - \{x\} - N_T(x)$
$T \leftarrow$ the larger of N and $\overline{N}$.
// If $T = N \neq \emptyset$, we say x is a K-element. If $T = \overline{N} \neq \emptyset$, we say x is an F-element.
$F = \{x \in KF : x \text{ is an } F\text{-element}\}$,
$K = \{x \in KF : x \text{ is a } K\text{-element}\}$.
// $F \cap K = \{z\}$, where z is the last selected element. F is an independent set, K is a clique.
Cleanup: output$\leftarrow$ the larger of K and F. // H is homogeneous.

Analysis of the Ramsey Algorithm

Lemma

Let G be an arbitrary graph with n vertices. Let k be the number of vertices selected by the Ramsey Algorithm. Let ℓ be the size of the output homogeneous set. Then

$$k \geq \log_2 n, \qquad \ell \geq \frac{1}{2} \log_2 n.$$

Proof and consequences

Clearly, $\ell \geq \frac{k+1}{2}$. So it is enough to prove the first inequality.

Let T_i be the set T before the selection of the i-th vertex, and let T_{i+1} be the updated set T after the selection.

It is easy to see that if $|T_i| \geq 2^s$, then $|T_{i+1}| \geq 2^{s-1}$.

The claim follows from this observation.

Corollary

Using the notations from the previous lemma, if $n = 4^e$, then $k \geq 2e$ and $\ell \geq e$.

Corollary

In a graph with 4^k vertices, there always exists a homogeneous set of size k.

Ramsey Numbers

Definition

Let $R(k)$ be the minimum number of vertices such that in every graph with that many vertices, there exists a homogeneous set of size k.

Theorem (Ramsey (1930) and Erdős)

$$\sqrt{2}^k < R(k) < 4^k.$$

Theorem (Campos—Griffiths—Morris—Sahasrabudhe (2023))

$$R(k) < 3.99999^k.$$

Exact Values of Ramsey Numbers

Interesting values are considered for $k \geq 3$ ($k = 1, 2$ trivially have $R(1) = 1$ and $R(2) = 2$).

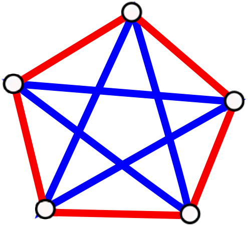
Only a few Ramsey numbers are known: $R(3) = 6$, $R(4) = 18$. For $R(5)$, it is known that $43 \leq R(5) \leq 49$.

The lack of knowledge is even more noticeable for $k = 10$. Currently, we only know that $798 \leq R(10) \leq 23,556$.

$R(3)$ Lower Bound, Construction

Lemma

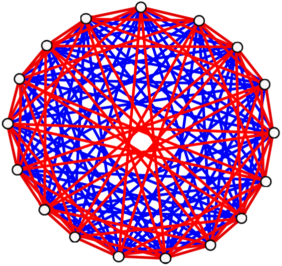
$$R(3) > 5.$$



$R(4)$ Lower Bound, Construction

Theorem

$$R(4) > 17.$$



The set of vertices is $Z_{17} = \{0, 1, 2, 3, \ldots, 16\}$. An edge ij is red if and only if $i - j \in \{-8, -4, -2, -1, 1, 2, 4, 8\}$, where the arithmetic is done modulo 17 ($\mathbb{Z}_{17}$ arithmetic).

The Improvement of Paul Erdős and György Szekeres

This improved the above estimation of Ramsey numbers.

Their algorithm can simultaneously compute an independent set $F(R)$ and a clique $K(R)$ for any vertex set R.

Running the algorithm on G calculates $F(V(G))$ as an independent set and $K(V(G))$ as a clique.

In contrast to previous algorithms, this one does not *discard* vertices.

Erdős—Szekeres Algorithm

Erdős—Szekeres Algorithm

Input: A simple graph G, output: an independent set $F(V)$ and a clique $K(V)$.

Base case of recursion: If $|V| \leq 2$, let both sets be V and a single-element subset.

Recursion: Otherwise, let x be any vertex. Let
$N = \{y \in V(G) - \{x\} : xy \in E\}$. Let
$\overline{N} = \{y \in V(G) - \{x\} : xy \notin E\}$.

$// \ N \dot{\cup} \overline{N} = V(G) - \{x\}$

Recursively call the algorithm on $G|_N$ and $G|_{\overline{N}}$ with $F(N)$ and $K(N)$ being the independent set and clique found in $G|_N$, and $F(\overline{N})$ and $K(\overline{N})$ being the independent set and clique found in $G|_{\overline{N}}$.

Output: $F(V(G))$ is the larger of $F(N)$ and $\{x\} \cup F(\overline{N})$.
$K(V(G))$ is the larger of $\{x\} \cup K(N)$ and $K(\overline{N})$.

Analysis of the Erdős—Szekeres Algorithm

Theorem

Theorem If $|V| \geq \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1}$, then the algorithm finds an independent set of size at least k or a clique of size at least ℓ.

We apply induction on $k + \ell$.

If the values of k or ℓ are at most 2, then the statement is obvious. We assume that $k, \ell \geq 3$.

We know that $|V| \geq \binom{k+\ell-2}{k-1}$ and $|N| + |\overline{N}| = |V| - 1$.

Proof

Then

$$\begin{aligned}|N| + |\overline{N}| =& |V| - 1 \geq \binom{k+\ell-2}{k-1} - 1 \\ &> \left[\binom{(k-1)+\ell-2}{(k-1)-1} - 1\right] + \left[\binom{k+(\ell-1)-2}{k-1} - 1\right].\end{aligned}$$

Thus

$$|\overline{N}| > \binom{(k-1)+\ell-2}{(k-1)-1} - 1 \quad \text{or} \quad |N| > \binom{k+(\ell-1)-2}{k-1} - 1.$$

Proof (Continuation)

If $|\overline{N}| \geq \binom{(k-1)+\ell-2}{(k-1)-1}$, then after the recursive call (by the induction hypothesis) $F(\overline{N})$ is at least $k-1$, or $K(\overline{N})$ is at least ℓ.

Thus, $F(V(G))$ is at least k, as $F(\overline{N}) \cup \{x\}$ is also included in the comparison. $K(V(G))$ is at least ℓ.

The case $|N| \geq \binom{k+(\ell-1)-2}{k-1}$ can be similarly argued.

This completes the justification of the statement.

Asymmetric Ramsey Numbers

Definition

Let $R(k, \ell)$ be the minimum value of $|V|$ such that we can be sure that any simple graph on V contains either an independent set of size k or a clique of size ℓ.

Simple Cases

(0) $R(k, \ell) = R(\ell, k)$,

(i) $R(1, \ell) = 1$,

(ii) $R(2, \ell) = \ell$.

Erdős—Szekeres Inequality

The essence of the proof is summarized by the following lemma.

Lemma: Erdős—Szekeres Inequality

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1).$$

The Legacy of Paul Erdős

• Extreme combinatorics and Ramsey theory are defining themes in Paul Erdős's long research life.
• Erdős's results play a crucial role in shaping and defining these areas.
• An **Erdős joke** illustrates the difficulty of the Ramsey number problem: If an extraterrestrial super-civilization were to arrive on Earth and state that humanity will be spared if they determine the value of $R(5)$, then politicians and economists would need to support mathematicians and computer scientists to combine the power and knowledge of all supercomputers to solve the problem.
◦ If the beings demand the determination of $R(6)$ to avoid violence, then the same support should be given to soldiers and weapon experts to solve the problem.

Multiple Colors Case

The above statement was made for a 2-coloring, but it can be stated and easily proven for a c-coloring.

Accordingly, we can introduce new Ramsey numbers: $R_c(k)$, when working with a palette of size c.

As in the Erdős—Szekeres proof, we can break the symmetry of colors and thus introduce generalized asymmetric Ramsey numbers: $R_c(k_1, k_2, \ldots, k_c)$.

Multiple Colors Case: Proof

Let's only consider the case $c = 3$.

(I): Think of the three colors as red, light blue, dark blue. We are looking for a monochromatic set of size k. Apply the two-color Ramsey theorem for red/blue.

If a monochromatic set is found in the red color, it is monochromatic. However, the monochromatic set found in the blue color, when considering the original color shades, is two-colored. But if the one seeing only blue applies the Ramsey theorem with a size of $R(k)$, then they can be sure of the existence of a monochromatic set of size k.

The general case (3 colors instead of 2) can be handled by induction on the palette size, based on the above idea.

Multiple Colors Case: Proof II

(II): Remember Ramsey's proof. Let $|V| = c^{ck}$ be an ordered set of vertices. Take out the first v_1 vertices. This selection survives those vertices connected to v_1 with an edge of the color most common among the edges around v_1. Repeat this with the surviving vertices until there are surviving vertices.

There will be at least ck selection steps. Among the selected vertices, at least k will be of the same color as the algorithm finds the same color most common.

Theorem

$$R_c(k) \leq c^{ck}.$$

Theorem

$$R_c(k_1, k_2, \ldots, k_c) \leq c^{k_1+k_2+\ldots+k_c}.$$

Coloring r-tuples

The basic Ramsey theorem colored the edges of the complete graph, the two-element subsets of vertices, with two colors. This can be extended to coloring r-element subsets: $c : \binom{V}{r} \to \{\text{red}, \text{blue}\}$.

Of course, now, from a monochromatic set M, we demand that every k-element subset has the same color. That is, M is monochromatic if $c|_{\binom{M}{r}}$ is a constant function.

The corresponding theorem is still true (that is, for a sufficiently large set, the existence of a monochromatic set of size k is inevitable). Accordingly, we can introduce the $R^{(r)}(k)$ and $R^{(r)}(k, \ell)$ Ramsey numbers.

Note: $r = 1$ is also meaningful in a mathematical statement. The case $r = 1$ is essentially the Pigeonhole Principle. The Ramsey theorem can be seen as a generalized Pigeonhole Principle.

2-Coloring triples: Proof

For $r = 1, 2$, we know the cases. Let's consider only the case $r = 3$.

Given a *large* set V, where a function c assigns a color to each triple, we follow the Erdős—Szekeres proof: We introduced the asymmetric $R^{(r)}(k, \ell)$ numbers.

We state the analogue of the Erdős—Szekeres Lemma:

Lemma

$$R^{(3)}(k, \ell) \leq R(R^{(3)}(k-1, \ell), R^{(3)}(k, \ell-1)) + 1.$$

The complete proof is an inductive proof of the finiteness of $R^{(3)}(k, \ell)$.

The train of thought shown represents the inductive step.

Coloring r-sets with c Colors

The two steps can be summarized. We can examine the c-coloring of r-element subsets. For a sufficiently large base set, a monochromatic set of size k is guaranteed in this case as well. The corresponding Ramsey numbers are denoted as $R_c^{(r)}(k)$ and $R_c^{(r)}(k_1, k_2, \ldots, k_c)$.

The details of the development can be carried out by the interested student.

Break

Ramsey Theory

The graph-theoretic Ramsey theorem supports the following philosophy: there is no complete disorder.

No matter how we draw edges among n vertices, there will always be independent sets or cliques of size $\mathcal{O}(\log n)$, meaning an extremely ordered part.

This is unavoidable even if our goal is to create total chaos. A certain local order is inevitable.

This philosophy appears in several mathematical theorems. The corresponding theorems are Ramsey-type theorems.

Due to the many connections, a theory has developed around this philosophy. Various branches of mathematics provide partial results to this theory.

Erdős—Szekeres Theorem

Let n be given, and consider a set $\mathcal{P}$ of n points in the plane such that no three points are collinear ($\mathcal{P}$ consists of points in general position).

From the points of $\mathcal{P}$, we want to select k in such a way that they form the vertices of a convex polygon.

The following theorem states that if $|\mathcal{P}|$ is large enough, then this is guaranteed.

Pál Erdős and György Szekeres

If $\mathcal{P}$ is a set of $R^{(4)}(5, k)$ points in general position, then we can select k points from them in such a way that they form the vertices of a convex polygon.

An Elementary Geometric Lemma

The proof relies on the following simple geometric lemma. The lemma will not be proven. Based on the knowledge of a high school student, the lemma can be easily understood.

Lemma

(i) If there are five points in general position in the plane, then we can choose four of them in such a way that they form a convex quadrilateral.

(ii) If there are k points in the plane such that any four of them form a convex quadrilateral, then the k points are in convex position.

Proof of the Theorem

After this, color the quadrilaterals formed by the elements of $\mathcal{P}$ with two colors in such a way that a quadrilateral receives a red color if the four points in it do not form the vertices of a convex quadrilateral.

The Lemma (i) precisely says that in this case, $\mathcal{P}$ does not contain a monochromatic red set of size five.

Choosing $|\mathcal{P}|$ accordingly, there exists a blue set of size k in $\mathcal{P}$.

Lemma (ii) implies that this blue set forms a convex set of size k.

Happy End Problem

The theorem answered a question posed by Eszter Klein.

In the proof, the elementary geometric statement was noticed by Klein Eszter, who then asked the question: Is it true that for a sufficiently large point set, we can always find a set of k points that forms the vertices of a convex k-gon in our point set?

Erdős Pál named the problem "Happy End" because the question itself might have played a role in the later marriage of Szekeres György and Klein Eszter.

Erdős—Szekeres Numbers

$ESz(n)$ is the minimum number such that for a set of points in general position with that size, we guarantee the existence of a set of n points forming the vertices of a convex polygon.

The following estimate comes from Erdős Pál and Szekeres György:

$$2^{k-2} + 1 \leq ESz(k) \leq \binom{2k-2}{k-1}.$$

(Szekeres György)

Is it true that for any k, $ESz(k) = 2^{k-2} + 1$?

This conjecture/equality has only been proven for $k \leq 6$ (2006).

Theorem, Suk 2017

$$2^{k-2} + 1 \leq ESz(k) \leq 2^{k+o(k)}.$$

Break

Arithmetic Ramsey-Type Theorems

In the following, we consider problems where a set of numbers is given, and its elements are colored. We then take an equation/system of equations and examine whether it can be solved in such a way that the solution forms a monochromatic set.

Our first such theorem will be a lemma. This led to the investigation of the Fermat conjecture. According to this conjecture, the Diophantine equation $x^n + y^n = z^n$ has no non-trivial solutions for $n > 2$. (This conjecture was proven by Wiles in 1994.)

Schur's Theorem

Throughout, when we say that a statement holds for sufficiently large s, we mean

There exists a threshold s_0 such that for all $s \geq s_0$, the statement is true.

We use the language similarly for primes or, for example, perfect squares, or any values taken from an infinite subset of $\mathbb{N}$.

Schur's Theorem

Let $n \in \mathbb{N}^+$. For sufficiently large prime p, the equation

$$x^n + y^n \equiv_p \; z^n$$

has non-trivial solutions, where $x \equiv_p \; y$ means $x \equiv y \mod p$, and a solution (x, y, z) is non-trivial if $x, y, z \not\equiv_p \; 0$.

Schur's Lemma

Of course, the threshold number p depends on n. Before proving the theorem, we need the following lemma, which led to the investigation of the Fermat conjecture.

Schur's Lemma, 1916

Let ν be sufficiently large, and let $c \in \mathbb{N}_+$ be an arbitrary palette size. Take an arbitrary coloring $\varphi : \{1, 2, \ldots, \nu\} = [\nu] \to \{1, 2, \ldots, c\}$. Then the equation

$$x + y = z \text{ , where } x, y, z \in [\nu],$$

has a monochromatic solution.

Proof of the Lemma

Define a coloring of the complete graph on the set $\{0, 1, 2, \ldots, \nu\}$: The color of edge ij is $\varphi(|i - j|)$.

Then, by Ramsey's theorem, if ν is large enough, there will be a monochromatic triple (i.e., a triangle where every edge has the same color). Actually, $\nu = R_c(3)$ is a good bound.

Let h, i, j be the vertices of a monochromatic triangle. Without loss of generality, assume $h < i < j$. We know that

$$\varphi(i - h) = \varphi(j - i) = \varphi(j - h).$$

Then, the values $x = i - h$, $y = j - i$, $z = j - h$ form a suitable solution to the equation.

Proof of the Theorem

For completeness, let's see the proof of the theorem.

Let p be a sufficiently large prime, and consider the multiplicative group of the p elements ($\mathbb{F}_p^*$). We define the subgroup

$$H = \{x^n | x \in \mathbb{F}_p^*\} = \{g^n, g^{2n}, \dots\}$$

of nth powers, where g is a generator of the cyclic group $\mathbb{F}_p^*$.

It is observed that the size of this subgroup, $|H|$, is at least $\frac{p-1}{n}$. Then, $\mathbb{F}_p^*$ decomposes into cosets according to H:

$$\mathbb{F}_p^* = m_1 H \dot\cup m_2 H \dot\cup \dots \dot\cup m_\ell H$$

The number of cosets ℓ is $\ell = \frac{|\mathbb{F}_p^*|}{|H|} = \frac{p-1}{|H|} \leq n$.

Proof of the Theorem (Continued)

Consider $\mathbb{F}_p^* \equiv [p-1] = \{1, 2, \dots, p-1\}$ and color it with the following n-coloring: the elements of each coset $m_i H$ receive the i-th color.

Then, applying Schur's lemma with parameters $\nu = p-1$ and $c = n$, we find a suitable color/coset $(m_i H)$ and suitable elements in this color/in this coset $(x, y, z \in m_i H)$ such that $x + y = z$.

That is, we have $x = m_i {x_0}^n$, $y = m_i {y_0}^n$, $z = m_i {z_0}^n$ and

$$m_i {x_0}^n + m_i {y_0}^n \equiv_p \; m_i {z_0}^n.$$

Dividing by m_i ($m_i \neq 0$), we get

$${x_0}^n + {y_0}^n \equiv_p \; {z_0}^n,$$

where ${x_0}^n$, ${y_0}^n$, ${z_0}^n \in H$, specifically ${x_0}^n$, ${y_0}^n$, ${z_0}^n \not\equiv_p \; 0$.

This gives us the sought non-trivial solutions.

Schur Numbers

As Ramsey's lemma leads to the definition of Ramsey numbers, Schur's lemma also forms the basis of an important definition.

Definition: $Sch(c)$, the c-parameter Schur number

For any $c \in \mathbb{N}_+$, let $Sch(c)$ be the minimum ν such that, for any coloring of $[\nu]$, there exists a monochromatic $\{x, y, z\}$ satisfying $x + y = z$. In other words, $Sch(c)$ is the precise threshold in the *sufficiently large* ν from the lemma.

Further Theorems

Schur's lemma, which will be our true Schur's theorem, sparked further research. Among the achieved results, the following stands out.

van der Waerden's Theorem, 1927

For sufficiently large n, any coloring of $[n]$ will contain a monochromatic arithmetic progression of length k that is not constant (AP).

Once again, it is important to mention the sequence related to the theorem, which describes the concept of *sufficiently large* in the theorem.

Definition

The smallest n for which the above theorem holds is denoted as $W_c(k)$.

Break

Positional Games

The simplest form of positional games is: a two-player game where players take turns occupying still available positions on a board, with the goal of forming some winning configuration.

Example: Tic-Tac-Toe

The board is a 3×3 grid, and winning configurations include rows, columns, and the two diagonals.

Example: Gomoku/Five in a row

The board is an infinite plane grid. Winning configurations consist of five adjacent positions either horizontally, vertically, or diagonally.

Generalized Tic-Tac-Toe Board

Definition

$U_k^d = \{\text{set of positions}\} = \{1, 2, \ldots, k\}^d$.

This means we have two parameters: k is the *width* of the board, and d is the *dimension* of the board.

So, a position can be described by a d-dimensional vector, where each coordinate ranges from 1 to k.

This convention is natural. For example, in the original Tic-Tac-Toe game, positions can be identified as $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$.

Generalized Tic-Tac-Toe Winning Positions

Definition

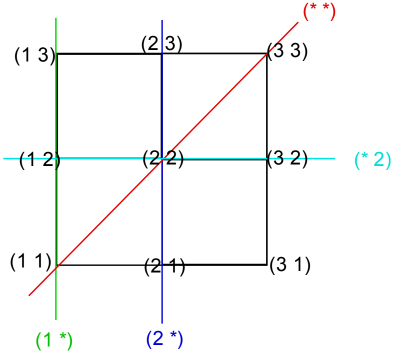
Let $e \in \{*, 1, \ldots, k\}^d \setminus \{1, 2, \ldots, k\}^d$, and associate with it a line $\mathcal{L}_e = \{P_1, P_2, \ldots, P_k\}$, where $P_i = P_i(e)$ denotes the position obtained by replacing the asterisks in e with i.

In other words, a line consists of positions whose coordinates are fixed outside an index set S and take the same value inside S. U_k^d contains k positions on each line.

The U_k^d board has $(k+1)^d - k^d$ lines in total.

Example

In the following figure, an example is shown for $k = 3$ and $d = 2$. On the $(1\ *)$ line, the positions are $(1, 1)$, $(1, 2)$, and $(1, 3)$. The $(2\ *)$ line contains the positions $(1, 1)$, $(2, 2)$, and $(3, 3)$. The other diagonal won't form a line. In this case, there are a total of 7 lines.



Generalized Lines, Subspaces

Definition

In the U_k^d table, an e-dimensional subspace can be described by a vector $a \in \{*_1, *_2, \ldots, *_e, 1, 2, \ldots, k\}^d$ where each indexed asterisk appears at least once. The elements of the subspace $\mathcal{A}_a$ described by this vector are obtained by replacing the $*_i$'s with the same element from $\{1, 2, \ldots, k\}$ (independently for different indices).

Thus, an e-dimensional subspace occupies k^e positions. For $e = 1$, the 1-dimensional subspace corresponds to a line.

Hales—Jewett Theorem (1963)

Hales—Jewett Theorem (1963)

For every k (table width), every c (palette size), and sufficiently large d (dimension), the positions of the U_k^d table can be arbitrary colored with c colors and the existence of a monochromatic line is guaranteed.

This can also be interpreted as follows: on the table U_k^d, for sufficiently large dimension d, if c players share the positions/color them, then there cannot be a tie, i.e., one of the players contains a winning set of positions/line within their color class.

Hales—Jewett Theorem ⇒ van der Waerden's Theorem

Let k be the length of the arithmetic sequence sought in van der Waerden's theorem.

In the Hales—Jewett theorem, this corresponds to (as the table width) a dimension d. Let $n = k^d$.

Consider the set $\{0, 1, \ldots, n-1\}$ and express its elements in base k. If, during the conversion, we pad the digit sequences with leading zeros to make them of length d, we establish a bijection

$$\{0, 1, \ldots, n-1\} \longleftrightarrow \{0, 1, \ldots, k-1\}^d$$

between our numbers and the positions on the table.

The coloring of van der Waerden's theorem corresponds to a Hales—Jewett-style coloring of our table, where the Hales—Jewett theorem guarantees the existence of a monochromatic line corresponding to a k-length arithmetic sequence.

Beginning of the Proof

Definition

The minimal dimension for which the above theorem holds, parametrized by k and c, is denoted as $HJ_c(k)$. These are the Hales–Jewett numbers for given k and c.

Hales–Jewett theorem $\equiv$ Hales–Jewett numbers are finite.

The proof is by complete induction on k, i.e., the table width.

For the case $k = 2$, observe that the set of positions (expressed in a monotonically increasing sequence) $00\ldots000$, $00\ldots001$,$00\ldots011$, ..., $01\ldots111$, $11\ldots111$ forms a set (of length $d + 1$) such that any two elements form a line. If $d \geq c$, then the pigeonhole principle guarantees two monochromatic elements, i.e., a monochromatic line.

Structure of the Inductive Step

The inductive step: Assume that the theorem holds for k (HJ-Statement(k)) and we need to prove it for $k + 1$ (HJ-Statement($k + 1$)).

This is the challenging part. We break it into two parts. We introduce an intermediate statement, denoted as: Statement($k + \frac{1}{2}$).

The proof proceeds as follows:

$$\text{HJ-Statement}(k) \Rightarrow \text{Statement}(k + \frac{1}{2}) \Rightarrow \text{HJ-Statement}(k).$$

Towards Statement($k+\frac{1}{2}$): Structure of a Subspace

Increase the width by 1. Identify the elements of our subspace with the positions of U^e_{k+1}. Select the following subset

$$U^e_{k+1} \supseteq \{(a_1, a_2, \ldots, a_e) : \text{ if } a_i = k+1 \text{ , then } \forall j > i,\ a_j = k+1\} \overset{\text{notation}}{=} S^e_{k+1}.$$

Thus, we can obtain S^e_{k+1} as follows

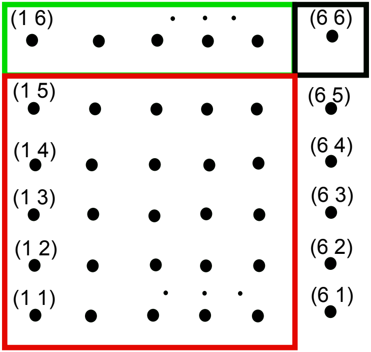
$$S^e_{k+1} = \bigcup_{i=0}^{e} S^e_{k+1}(i),$$

where $S^e_{k+1}(i)$ contains numbers with the first $e-i$ digits at most k, followed by i digits equal to $k+1$.

Note that the above definition requires that the order of our e asterisks is fixed.

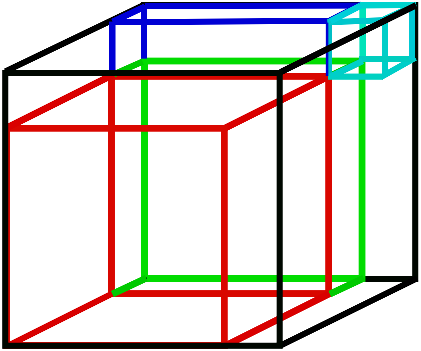
Example

$k = 6$ and $e = 2$. The black square corresponds to $S_6^2(2)$, as in this case everywhere a_1 to 6 must be 6. The green rectangle represents $S_6^2(1)$, and the red square corresponds to $S_6^2(0)$. The non-framed part does not satisfy the condition because it has 6 in the first position, but the next position is less than 6.



Example

In the following figure, $e = 3$ is illustrated. The red cube represents $S_k^3(0)$, the green box represents $S_6^2(1)$, the blue box represents $S_6^2(2)$, and the light blue cube represents $S_6^2(3)$.



Description of Statement$(k+\frac{1}{2})$

Definition

A subspace is *nice* if all $S_{k+1}^{e}(i)$ subsets are monochromatic.

Note that the $S_{k+1}^{e}(i)$ subsets $(i = 0, 1, 2, \ldots, e)$ do not cover the entire subspace. There are no coloring conditions for the uncovered part. The parts designated by different i's are independent. Each of them must be monochromatic, but the different parts can have different colors (or the same color).

From Statement($k + \frac{1}{2}$) to HJ-Statement($k + 1$)

Let e be chosen as the palette size in HJ-Statement($k + 1$) and work in a sufficiently large dimension for Statement($k + \frac{1}{2}$).

Statement requires the monochromaticity of $e + 1$ sets.

By the pigeonhole principle, there will be two that have the same color. The proof of Hales—Jewett statement comes from the fact that the union of any two $S^e_{k+1}(i)$ sets contains a line. (This is easily verified after studying examples.)

HJ-Statement(k) $\Rightarrow$ Statement($k+\frac{1}{2}$)

We prove Statement($k+\frac{1}{2}$) by induction on e.

For $e = 1$, it follows easily: the positions of U^d_{k+1} contain the narrower U^d_k table, where our assumption guarantees a monochromatic line.

This line becomes part of the larger table ($*$ can now take the value of $k+1$). Thus, in the larger table, the appropriate line is an extension of the narrow but monochromatic line by one position. Monochromaticity may be lost, but we still obtain a nicely colored line/1-dimensional subspace.

Jump from e to $e+1$

We are looking for the sufficiently large dimension as $d' + d''$, where both terms are appropriately large.

Take an arbitrary coloring. We need to find the nicely colored $e+1$-dimensional subspace.

Each position will have a first d' coordinate, this is the *beginning* of the position, and it will have a last d'' coordinate, the position's *end*. (Our table is the product of two smaller dimensional tables.)

Fix the beginning of the position. The possibilities for fixing are identified with the positions of $U_{k+1}^{d'}$.

For a fixed beginning, the possible ends are identified with the positions of $U_{k+1}^{d''}$. In this, each end (with the fixed beginning) describes a colored position in the entire table. Thus, fixing corresponds to a colored $U_{k+1}^{d''}$ table.

Jump from e to $e+1$ (Continuation)

There are $c^{(k+1)^{d''}}$ possibilities for coloring $U_{k+1}^{d''}$. Each of these can be considered as a *super-color*.

That is, $U_{k+1}^{d'}$ has a super-coloring. In this, there is a nicely colored line (see the case $e=1$). The selection of the line: asterisk the first d' coordinates and fix.

The nicely colored line is a subset of $S_{k+1}^{1}(0)$, i.e., every element (position's beginning) has the same super-color, i.e., the same colored $U_{k+1}^{d''}$ table belongs to it.

d'' should be large enough so that there is a nicely colored e-dimensional subspace in it. Selecting this subspace: asterisk the last d'' coordinates.

We claim that this is nicely colored. This can be easily verified.

Break

Density vs. Structure

The graph theoretical Ramsey theorem discusses the arbitrary red/blue coloring of the edges of a complete graph. We divide the $\binom{n}{2}$ edges into two categories. The majority constitutes at least $\frac{1}{2}\binom{n}{2}$ edges.

It arises whether this set of monochromatic edges already guarantees the formation of a large monochromatic set in this color. The first thought is immediately refuted by Turán's theorem. More than half of the edges can be red without forming a monochromatic triangular subset.

If a larger monochromatic set is our goal, then more edges can be specified while avoiding the large monochromatic set. The validity of Ramsey's theorem is of a structural nature. If the red edges avoid forming a large monochromatic set, then the complementary set (the blue edges) cannot have a similar structure.

Summary

Theorem	Structure to be Colored	Monochromatic Substructure to be Found	Maximum Size of Possible Color Class
Ramsey Theorem	Edges of a complete graph with n vertices	Edges of a complete graph with 3 vertices	$K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, the n-vertex bipartite Turán graph
Ramsey Theorem	Edges of a complete graph with n vertices	Edges of a complete graph with k vertices	$T_{n,k-1}$, the n-vertex, $k-1$ part Turán graph
Schur	$[n]$	$\{x, y, x+y\}$	I. Example: odd numbers. II. Example: $[n] \setminus [\lfloor n/2 \rfloor]$.
van der Waerden	$[n]$	AP of length k (non-constant)	???

Erdős—Turán Theorem

Paul Erdős and Pál Turán conjectured that ??? is not good for the case of the van der Waerden theorem, meaning that a *significant* part of $\{1, 2, \ldots, n\}$ cannot contain an arithmetic sequence of length k.

Thus, the van der Warden theorem is a kind of justification for density. Which is much stronger than the usual combinatorial proof of Ramsey theorems.

Definition

$$r_k(n) = \max\{|R| : R \subseteq [n],\ R\text{-contains no AP of length } k\}.$$

(Erdős Pál—Turán Pál, 1936)

$r_k(n) = o(n)$, if $k \geq 3$.

Results

Erdős Pál—Turán Conjecture: For every positive ε, if n is large enough, then $r_k(n) \leq \varepsilon n$.

(Roth's Theorem, 1956)

$$r_3(n) = o(n).$$

Later, Endre Szemerédi proved the case of four-term arithmetic progressions, followed by the general case.

(Szemerédi's Theorem, 1975)

For every $k \geq 3$, the conjecture holds. That is,

$$r_k(n) = o(n).$$

Results II

After the proof of the conjecture, the exploration of the topic became even more vibrant. We only outline the most outstanding results.

The theorem was re-proven several times:

- 1977 Fürstenberg. His proof uses ergodic theory.
- 2001 Gowers. His proof employs strong combinatorial number-theoretical results and Fourier techniques. The Fourier method was introduced by Roth, but its successful application required additional brilliant ideas.

Gowers' Estimate

Gowers' new proof is remarkable because the original combinatorial and later ergodic-theoretic proofs necessarily did not provide estimates for the $r_k(n)$ numbers. The application of the Fourier method, however, provides effective estimates. Thus, as a byproduct, the following estimate for the van der Waerden numbers was obtained.

(Gowers' Estimate)

$$W_2(k) \leq 2^{2^{2^{2^{2^{k+9}}}}}.$$

Green and Tao Theorem

All the power of the aforementioned methods and more was needed for the following result to emerge.

Green—Tao Theorem

For every positive integer k, there exists an arithmetic progression of length k among the primes.

Terence Tao received the Fields Medal in 2006. In the award justification, the above result was highlighted. The reason for the theorem is again of a density nature.

Green and Tao Theorem (Density Version)

(Green—Tao Theorem, Density Version)

Let $P_n = \{2, 3, 5, 7, 11, p_6, \ldots, p_n\}$ be the set of the first n primes. Let ϵ be any (small) positive constant. If $A \subset \mathbb{N}$ satisfies $|A \cap P_n| \geq \epsilon n$ for infinitely many n, then A contains an arithmetic progression of length k for every positive integer k.

Thank you for your attention!