Crossing number and its applications

Peter Hajnal

Bolyai Institute, SZTE, Szeged

2023 Fall

Introduction

So far we considered the notion of planar graph.

A graph is planar or non-planar.

Crossing number is a parameter that measure how "far" is a grph G from planarity.

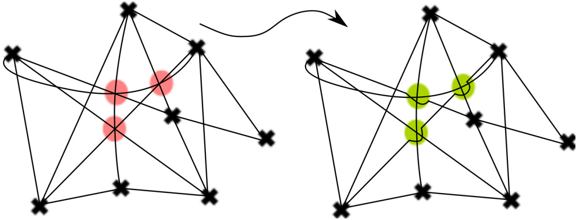
The crsooing number will be a non-negative integer that is 0 iff our graph is planar.

Regular drawing

Definition

Let G be a graph, and λ is one of its drawing. We say that λ is *regular* if there are no three different edge-curves with common inner point.

Regularity is a technical assumption. Any graph has a regular drawing. Any drawing can be made regular by a few local deformation.



A Definition

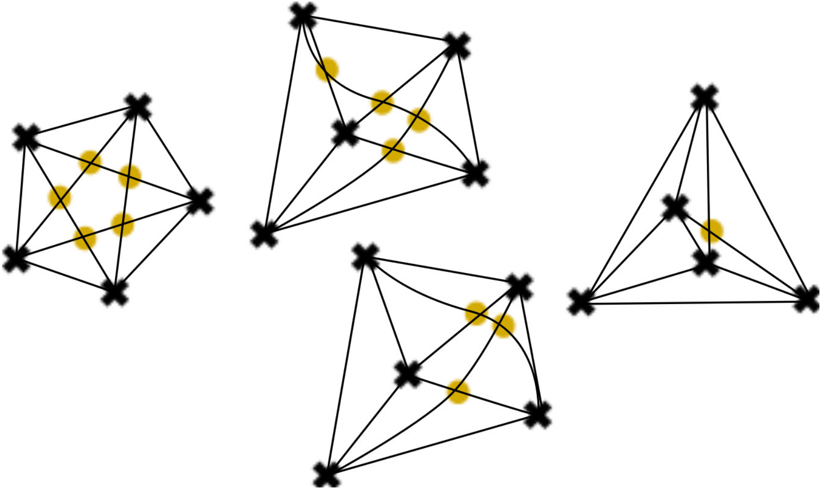
Definition: Crossing number of a drawing

Let G be a graph, and λ be a regular drawing of it.

$x(G, \lambda) = |\{P \in \mathbb{R}^2 - \lambda(V) :\ P \text{ is on more than one edge-curve}\}|.$

An alternative definition can be: Consider an arbitrary drawing. We count points P, that are an inner point of k edge-curves with multiplicity $\binom{k}{2}$.

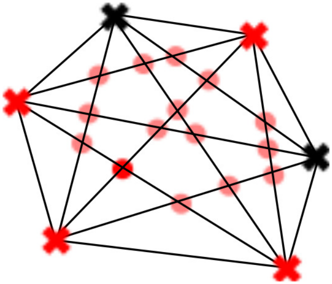
Examples



Several drawings of $G = K_5$, and their crossing number: $x(K_5, \lambda) = 5$, $x(K_5, \lambda') = 4$, $x(K_5, \lambda''') = 3$, $x(K_5, \lambda'''') = 1$.

Example

Le G be K_n, the complete graph on n vertices. Let λ be the drawing where the vertex-points are placed in the vertices og a convex n-gon, and the edges are straight segments. Then $x(K_n, \lambda) = \binom{n}{4}$. Indeed,there is a bijection between the crossings and the 4-tuples of vertices.



The case of K_6.

Example (continued)

Notation

If $R \subseteq G$, then any drawing λ of G can be restricted to R: $\lambda|_R$.

Observartion

Let H be an arbitrary simple graph on n vertices, i.e. $H \subseteq K_n$. Then $x(H, \lambda|_H) \leq \binom{n}{4} = O(n^4)$, where λ is the drawing of the complete graph, we desribed above.

Observation: Loops don't count

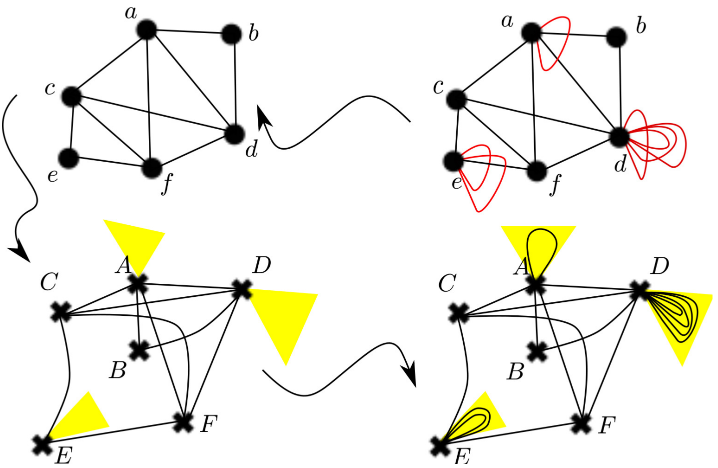
Let G be a graph. We obtain G_0 from G by deleting the loops of G. We can think the other way around: we obtain G from G_0 by adding loops.

Observation

Any drawing λ of G_0 can be extended to a $\widehat{\lambda}$ drawing of G, such a way that the crossing number is not changed: $x(G, \widehat{\lambda}) = x(G_0, \lambda)$.

Consider the drawing of G_0 around a vertex x. In a small enough neighborhood, the edge-curves meeting at x form a star shape. Between the branches there is „enough space" for the nice drawing of arbitrary number of loops.

Loops don't count: On figure

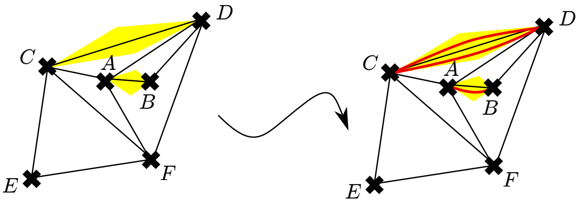


Parallel edges count

Let G be a loopless graph. We obtain a simple graph G_0 from G by keeping one edge between two connected vertices and delete the possible parallel edges to it. We can think the other way around: we obtain G from G_0 by adding parallel edges to an existing edge.

Observation

Given a nice drawing λ of G_0. We can extend it to $\widehat{\lambda}$, a nice drawing of G, i.e if $x(G_0, \lambda) = 0$ then $x(G, \widehat{\lambda}) = 0$.



The Definition

Definition: Crossing number of a graph

$$x(G) = \min\{x(G,\lambda) : \lambda \text{ is regular}\}.$$

Observation

$x(G) = 0$ iff G is planar.

Exercise

$x(K_5) = x(K_{3,3}) = 1.$

For any simple graph G on n vertices $x(G) = O(n^4)$.

Historical remarks

The concept was born in the 1940s, when Pál Turán was working as a forced labor serviceman in a brick factory.

His job was to push mine cars between kilns and railway wagons. The kilns and the loading bays were connected by rails for the mine cars.

The hardest part of the job was to push the wagon when two rails met, when the wagons were jerked.

It was a natural question: Design a rail system that includes n kilns and m loading bays, and and has a minimum number of cross overs.

That is, the question is to find the value of $x(K_{n,m})$. Later the question to determine the value of $x(K_n)$ was raised too.

Although in both cases, the optimal drawings are conjectured, the conjecture is still a central open question.

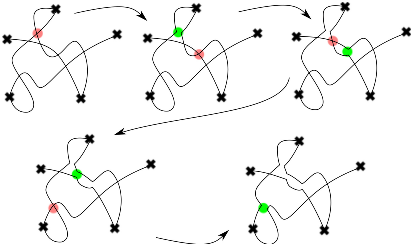
An inportant remark

Assume that e and f are two edges, with common endvertex v. If the two edge-curves cross each other then the drawing is not optimal.

Indeed, from the neighbors of v we approach v. As soon as two edge-curves meet we redraw/switch, we avoid the crossing.

We obtain a drawing of the same graph. Meanwhile, the crossing number cannot increase.

The remark on a figure



Definition

A drawing λ is V-nice iff any two edge-curves that share a common end vertex-point do not cross each other.

Observation

For any drawing λ of a graph G one can find a V-nice drawing λ', that $x(G, \lambda') \leq x(G, \lambda)$.

Break

An easy bound on the crossing number

Reminder

Let G be a simple planar graphs. If $|V| \geq 3$, then $|E| \leq 3|V| - 6$.

Corollary

Let G be a simple graph and λ is its regular drawing. Then

$$x(G, \lambda) \geq |E| - 3|V|.$$

Proof of the easy bound

Let R be a subgraph of G, that $V(G) = V(R)$ and $E(R)$ a maximal edge set with the property that $\lambda|_R$ is nice.

We know that $|E(R)| \leq 3|V|$.

Hence we have at least $|E(G)| - 3|V|$ edges outside R.

For each $e \in E(G) - E(R)$ the edge-curve $\lambda_E(e)$ crosses ate least one edge-curve of $(R, \lambda|_R)$.

We obtain, that

$$x(G, \lambda) \geq |E(G)| - 3|V|.$$

The Crossing Lemma

Theorem (Crossing Lemma)

If G is a simple graph and $|E| \geq 4|V|$, then

$$x(G) \geq \frac{1}{64}\frac{|E|^3}{|V|^2}.$$

The bound $|E| \geq 4|V|$ guarantees that G is not planar, i.e. $x(G) \geq 1$.

First, a Corollary

Corollary

$$x(K_n) \geq \frac{1}{64}\frac{\binom{n}{2}^3}{n^2} = \frac{1}{128}n^4 + O(n^3) = \Omega(n^4).$$

Corollary

$$x(K_n) = \Theta(n^4).$$

The proof the Crossing Lemma I

Let λ be a V-nice drawing of G.

Let $\underline{R}$ be a random plane subgraph, that we obtain by the following random process: for each vertex (independently) we leave it untouched with property p, and delete it with probability $1-p$. The suitable p will be determined later.

We apply the easy bound on $\underline{R}$:

$$x(\underline{R}, \lambda|_{\underline{R}}) \geq |E(\underline{R})| - 3|V(\underline{R})|.$$

The inequality holds for expected values too:

$$\mathbb{E}(x(\underline{R}, \lambda|_{\underline{R}})) \geq \mathbb{E}(|E(\underline{R})|) - 3\mathbb{E}(|V(\underline{R})|).$$

The proof the Crossing Lemma II

The probability that a vertex survives the random process is p. The probability that an edge survives our process is p^2. The probability that two intersecting edges survive the thinning process is p^4.

From this:

$$p^4 x(G,\lambda) \geq p^2|E(G)| - 3p|V(G)|.$$

p will be positive, so we can divide the inequality by p^4:

$$x(G,\lambda) \geq \frac{|E(G)|}{p^2} - \frac{3|V(G)|}{p^3}.$$

The proof the Crossing Lemma III

Let $p = \frac{4|V|}{|E|}$.

$$x(G, \lambda) \geq \frac{1}{16} \frac{|E|^3}{|V|^2} - \frac{3}{64} \frac{|E|^3}{|V|^2} = \frac{1}{64} \frac{|E|^3}{|V|^2}.$$

The claim is proven.

Final remarks

The coefficient $\frac{1}{64}$ is a byproduct of the proof.

With more attention it can be improved, but the optimal value is not known.

Break

A geometric theorem

Definition

Let $\mathcal{P} \subseteq \mathbb{R}^2$ a finite planar point set and $\mathcal{E}$ a finite set of lines on the plane.

$$I(\mathcal{P}, \mathcal{E}) = |\{(P, e) : P \in \mathcal{P},\ e \in \mathcal{E} \text{ and } P\, I\, e\}|,$$

$P\, I\, e$ denotes, that the point P is incident to the line e.

Theorem (Szemerédi—Trotter's theorem)

$$I(\mathcal{P}, \mathcal{E}) \leq 4(|\mathcal{P}||\mathcal{E}|)^{2/3} + 4|\mathcal{P}| + |\mathcal{E}|.$$

The magnitude of the upper bound

$$\mathcal{O}(|P|^{2/3}|\mathcal{E}|^{2/3} + |\mathcal{P}| + |\mathcal{E}|) = \mathcal{O}(\max\{|\mathcal{P}|^{2/3}|\mathcal{E}|^{2/3}, |\mathcal{P}|, |\mathcal{E}|\}).$$

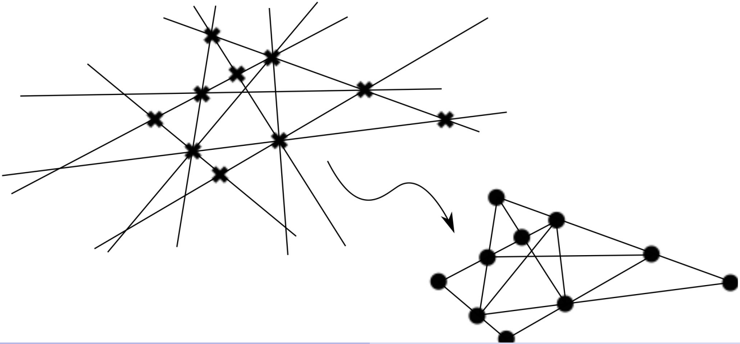
Note that for arbitrary p and e positive integers one can give a set of points $\mathcal{P}$ of size p and a set of lines $\mathcal{E}$ of size e such that number of incidences between them is at least a thousandth of the upper estimate.

That is, the magnitude of the upper estimate is optimal.

The proof I

We can assume that any line $e \in \mathcal{E}$ is incident to at least one point in $\mathcal{P}$.

We construct a simple graph from $\mathcal{P}$ and $\mathcal{E}$: $\mathcal{P}$ forms the vertex set. Two vertices, $P, Q \in \mathcal{P}$ are connected iff there is a line $e \in \mathcal{E}$ that contains both and on e there are no other elements of $\mathcal{P}$ between P and Q.



The proof II

$|V| = |\mathcal{P}|$.

For any $k \geq 1$ if a line contains k points from $\mathcal{P}$, then the contribution of this line to the edge set is $k-1$.

The total number of edges is the sum of these contributions, i.e. $|E| = I(\mathcal{P}, \mathcal{E}) - |\mathcal{E}|$.

Let λ be the drawing of our graph where the vertex-points are defined by $\mathcal{P}$. The edge-curves are the suitable segments of the corresponding line from $\mathcal{E}$.

From geometry it is obvious that $x(G) \leq x(G, \lambda) \leq \binom{|\mathcal{E}|}{2} \leq |\mathcal{E}|^2$.

The proof III

1st Case: $|E| < 4|V|$. $I(\mathcal{P},\mathcal{E}) - |\mathcal{E}| < 4|\mathcal{P}|$.

2nd C:ase $|E| \geq 4|V|$. Then the Crossing Lemma is applicable:

$$|\mathcal{E}|^2 \geq \binom{|\mathcal{E}|}{2} \geq x(G,p) \geq \frac{1}{64}\frac{(I(\mathcal{P},\mathcal{E}) - |\mathcal{E}|)^3}{|\mathcal{P}|^2}.$$

After rearrengment we obtain

$$4|\mathcal{P}|^{2/3}|\mathcal{E}|^{2/3} \geq I(\mathcal{P},\mathcal{E}) - |\mathcal{E}|.$$

In both cases the theorem is proven.

Break

Basic problems

Definition

Let $A, B \subset \mathbb{R}$ be finite set of numbers.
$A + B = \{a + b : \ a \in A$ és $b \in B\}$ and
$A \cdot B = \{a \cdot b : \ a \in A$ és $b \in B\}$.

$A + A$-t, resp, $A \cdot A$ are called the sum-set, resp. product-set of A.

Question: How big and how small can be $|A + A|$ and $|A \cdot A|$, assuming $|A| = n$?

Basic observations: Sum-set

$|A+A| \leq \binom{n}{2} + n.$

If A is a random set of nimbers of size n then the size of $A+A$ is $\binom{n}{2} + n$ almost surely.

We will give a lower bound on $|A+A|$. We can assume that the elements A are $a_1 < a_2 < \ldots < a_n$.

- Using $a_1 + a_1 < a_1 + a_2 < \ldots < a_1 + a_n < a_2 + a_n < \ldots < a_n + a_n$ we have at least $2n-1$ different values in $A+A$.
- If A contains n consequtive elements of an arithmetic progression, then $|A+A| = 2n-1$.

Basic observations: Product-set

$|A \cdot A| \leq \binom{n}{2} + n.$

If A is a random set of numbers of size n then the size of $A \cdot A$ is $\binom{n}{2} + n$ almost surely.

If A contains n consequtive elements of an geometric progression, then $|A \cdot A| = 2n - 1$.

Easy to give a linear lower bound on $|A \cdot A|$.

The fundamental question

In the case of minimization the structure of the extreme sets are completely different (arithmetic and geometric sequences).

Is there a set where the sum set and the multiplication set will be small at the same time?

Question by Pál Erdős: What can we say about $\max\{|A+A|, |A\cdot A|\}$ for n elements number-sets?

Conjecture (Erdős—Szemeredi)

For every positive ε

$$\min_{A\subseteq\mathbb{R},|A|=n} \max\{|A+A|, |A\cdot A|\} = \Omega(n^{2-\varepsilon}).$$

The conjecture is still open today.

Theorem of György Elekes

Theorem (György Elekes)

For large enough n

$$\min_{A\subseteq\mathbb{R},|A|=n} \max\{|A+A|,|A\cdot A|\} \geq \frac{1}{10}n^{5/4},$$

i.e. for any n element set of numbers A we have $\max\{|A+A|,|A\cdot A|\} = \Omega(n^{5/4})$.

Elekes' proof I

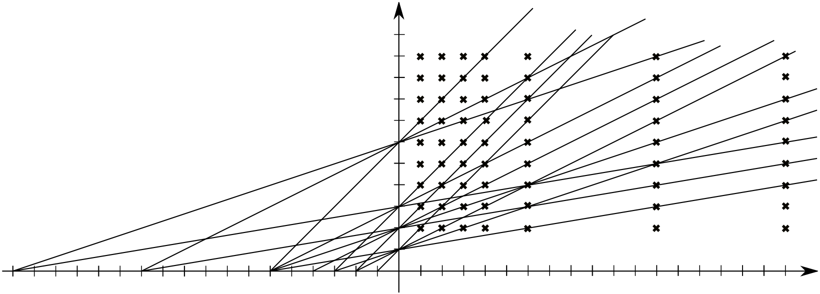
Assume that $0 \notin A$.

We define a planar point set and a set of lines on the plane:

$$\mathcal{P}_A = \{(\pi, \sigma) : \pi \in A \cdot A,\ \sigma \in A + A\},$$

$$\mathcal{E}_A = \{e_{a,a'} :\ y = \frac{1}{a} \cdot x + a',\ a, a' \in A\}.$$

Elekes' proof II



The points and lines in the case of $A = \{1, 2, 3, 6\}$

Elekes' proof III

The parameters of Szemerédi, Trotter's theorem can be easily bounded:

- $|\mathcal{P}_A| = |A \cdot A| \cdot |A + A|$.
- The equation of the line $e_{a,a'}$ is $\frac{1}{a} \cdot y - \frac{1}{a \cdot a'} \cdot x = 1$. It can be seen that the intersections with the axes and (a, a') determine each other. Hence $|\mathcal{E}_A| = |A|^2$.
- The line $e_{a,a'}$ contains $(a \cdot a_1, a_1 + a')$, $(a \cdot a_2, a_2 + a'), \ldots$, where $A = \{a_1, a_2, \ldots\}$. We obtain that $I(\mathcal{P}_A, \mathcal{E}_A) \geq |A||\mathcal{E}_A| = |A|^3$.

Elekes' proof IV

We use the Szemerédi—Trotter Theorem:

$$n^3 = |A|^3 \leq I(\mathcal{P}, \mathcal{E}) \leq 4|A{\cdot}A|^{2/3}\cdot|A{+}A|^{2/3}\cdot(|A|^2)^{2/3}+4|A{\cdot}A||A{+}A|+|A|^2.$$

We know that $|A|^2 = n^2 \leq \frac{1}{3}n^3$, if n large enough. We can assume that $4|A+A||A\cdot A| \leq \frac{1}{3}n^3$, otherwise we obtain a stronger conclusion than the one stated.

The last two terms of the right hand side can be rearranged to the left hand side. There we still have at least $1/3 \cdot n^3$:

$$\frac{1}{3}n^3 \leq 4|A\cdot A|^{2/3}|A+A|^{2/3}\cdot n^{4/3}.$$

The left of the proof is simple arithmetic:

$$\frac{1}{12}n^{5/3} \leq |A\cdot A|^{2/3}|A+A|^{2/3}.$$

$$0{,}15\cdot n^{5/4} \leq \sqrt{|A\cdot A||A+A|} \leq \max\{|A\cdot A|, |A+A|\}.$$

This is the end!

Thank you for your attention!