Planar graphs

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Faces with short boundary

Observation

Let v be an isolated node in a nicely drawn graph.
The vertex-point P_v, corresponding to v is an inner point of a face τ_0 in the (induced) nice drawing of $G - v$. There is a corresponding face, τ in the nice drawing of G. One might include P_v in the boundary of τ, but it does not contribute to the lemgth of it.

If $E(G) = \emptyset$, then the drawing of G contains one face of length 0.

Now on we assume that we have no isolated nodes.

Faces with short boundary (continued)

Observation

If the boundary of a face is a length 1 walk, then there is only one bounding edge, a loop. The edge-curve of the bounding edge is a closed Jordan curve.

Observation

Ha egy tartomány határa egy 2 hosszú séta, akkor az vagy egy párhuzamos élpár, vagy egy él oda-vissza bejárása. A második esetben az egész gráf két pont és egy összekötő él által alkotott gráf.

Faces in bipartite plane graph

It is staright forward, that in a planar bipartite each face has an even length.

Furthermore, in a bipartite graőh any closed walk has even length.

Observation

Observation

(i) If G is a simple connected plane graph on at least 3 nodes then each of its face-boundary has length at least 3.

(ii) If G is a simple connected, bipartite, plane graph on at least 3 nodes then each of its face-boundary has length at least 4.

Euler's theorem

Euler's theorem

Let G be a connected plane graph (λ is a nice drawing). Then

$$|T(G,\lambda)| - |E(G)| + |V(G)| = 2,$$

where $T(G,\lambda)$ is the set of faces.

1st proof: Induction

G is connected, so we can think about it as a spanning tree and some extra edges added to it.

Let h be the number of extra edges ($h = |E| - (|V| - 1)$). We use induction on h.

If $h = 0$, then G is a tree: $|T(G, \lambda) = 1|$ and $|E(G)| = |V(G)| - 1$.

Let $G \to G^+ := G + e$:

$$|V(G^+)| = |V(G)|, |E(G^+)| = |E(G)| + 1, |T(G^+, \lambda^+)| = |T(G, \lambda)| + 1,$$

where λ^+ is the original drawing extended by the new edge.

2nd Proof: Dualization

Introduce the dual G^* graph (λ^* is its nice drawing).

G is connected, hence it has a spanning tree, T.

Let

$$F = \{e \in E(G) : e \in E(T)\} = E(T), \quad F^* = \{e^* \in E(G) : e \notin E(T)\}.$$

It os obvious that $|F| + |F^*| = |E(G)|$ and $|F| = |V(G)| - 1$.

Theorem

F^* is the edge set of a spanning tree in G^*.

From the Theorem we know that
$|F^*| = |V(G^*)| - 1 = |T(G, \lambda)| - 1$ and Euler's theorem is proven.

Euler's theorem: Second form

The Theorem is a little bit 'eclectic': It mixes topology/geometry and combinatorics.

Corollary

Let G be a simple planar graph, and assume $|V(G)| \geq 3$. Then

(i) $|E(G)| \leq 3|V(G)| - 6$,

(ii) furthermore if G is bipartite, then $|E(G)| \leq 2|V(G)| - 4$.

Proof: (i)

We can assume that G is connected! Let λ be a nice drawing of G.

Add up the lengths of the faces.

First, we get $2|E|$.

Second, we have $|T(G,\lambda)|$ terms, eacg is at least 3.

The two arguments are consistent:

$$2|E(G)| \geq 3|T(G,\lambda)|.$$

From Euler's theorem

$$3|T(G,\lambda)| = 3|E(G)| - 3|V(G)| + 6.$$

The rest is simple algebra.

Proof: (ii)

The same proof works. The only difference that each face has a boundary at least 4. The rest is an exercise.

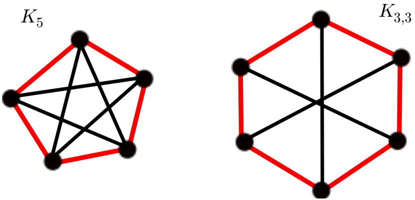
Corollary of Euler's Theorem

Theorem

K_5 and $K_{3,3}$ are non-planar graphs.

1st proof

We start with a partial drawing of our graphs.



The red cycles and their drawings are unique. The red edge-curves divide the plane into an inner and an outer face. The missing/black edges are in one of those faces.

Without the loss of generality we can assume that the majority of the black edges are drawn in the inner face.

We get contradiction.

2nd proof

Proof by contradiction.

K_5: Apply the second form of Euler's theorem. We obtain contradiction: $|E| = 10$ and $3|V| - 6 = 3 \cdot 5 - 6 = 9$.

$K_{3,3}$: Apply the second form of Euler's theorem. We obtain contradiction: $|E| = 9$ and $2|V| - 4 = 2 \cdot 6 - 4 = 8$.

Break

Operation: Deleting an edge

Definition

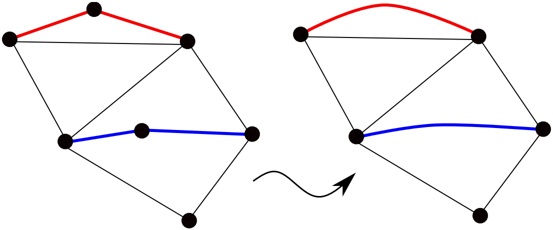
Let G be a graph, $e = xy \in E$ is an edge of it.

$G - e$ (or $G \setminus e$) denotes the graph, that we obtain from G by deleting e.

Operation: Merging two edges

Definition: $G(e \wr f)$

Let G be a graph, and $e = xa, f = ay$ two edges of it, that meet in a, a vertex of degree 2.
When merging e and f we obtain a graph by deleting e, f, a and adding a new edge: xy.



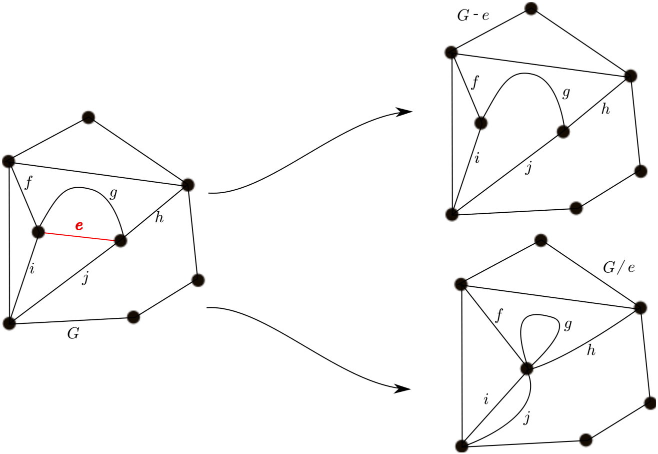
Operation contraction

Definition

Let G/e denote the graph we obtain by contracting e in the graph G:

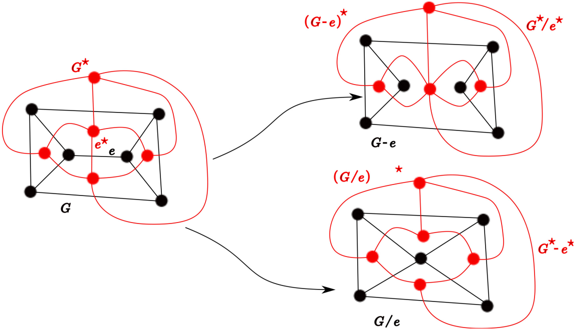
- $V(G/e) = (V(G) - \{x, y\} \dot{\cup} \{[e]\}$,
- $E(G/e) = E(G) \setminus \{e\}$,
- $I(G/e)$ is the natural incidence.

The operations in pictures



We emphasize an edge in G by coloring it red. We show the graphs we obtain by deleting the red edge and the graph we obtain by contracting it.

The next two figures show two operations described above: deleting and contracting an edge. These two operations are illustrated on the graph G and its dual G^*.



Relations between the operations

Claim

(i) $(G-e)^* = G^*/e^*$,

(ii) $(G/e)^* = G^* - e^*$.

Observation

$G(e \wr e') \simeq G/e \simeq G/e'$.

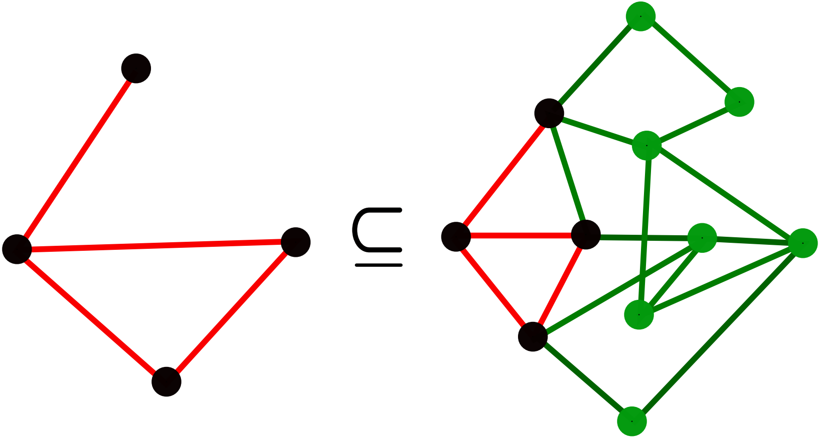
Subgraphs, topological subgraphs, minors

Definition

Let G be a graph.

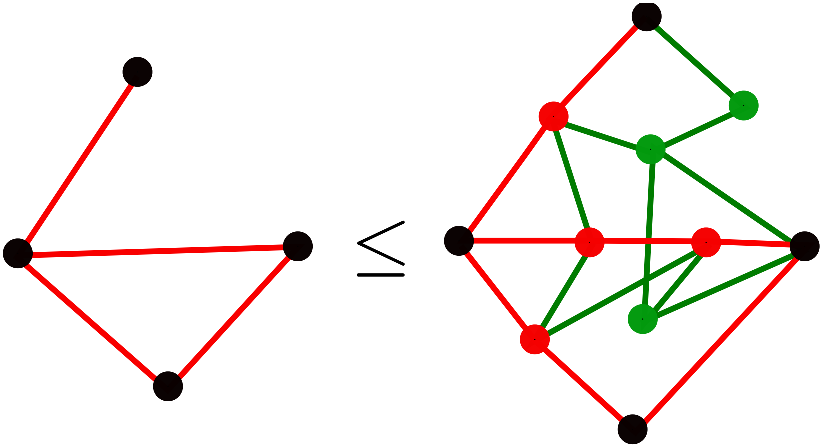
a) If R can be obtained from G by deleting edges/vertices, then R is referred as a subgraph of G: $R \subseteq G$.
b) If T can be obtained from G by deleting edges/vertices and merging edges, then T is referred as a topological subgraph of G: $R \subseteq G$: $T \leq G$.
c) If M can be obtained from G by deleting edges/vertices and contacting edges, then M is referred as a minor of G: $M \preceq G$.

Subgraph: example



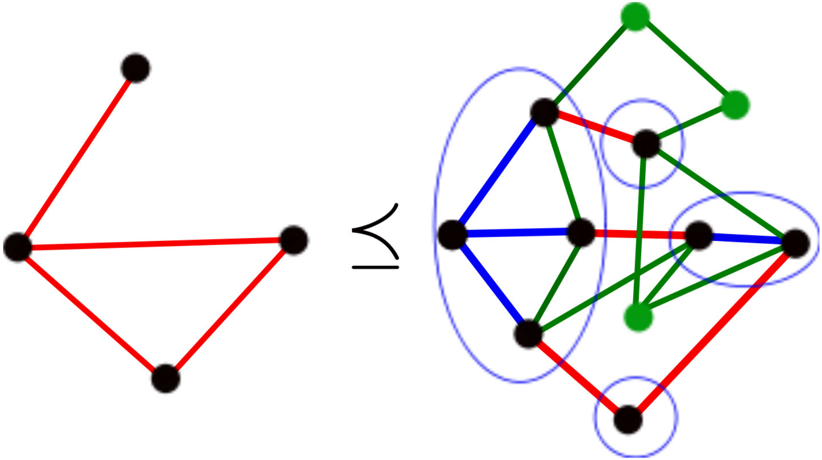
The red graph R is a subgraph of the graph G, since by deleting the green edges and vertices we get the graph R.

Topological subgraph: example



The red graph T is a topological subgraph of the graph G, since deleting the green edges and vertices, and merging the edges marked in red we get the graph T.

Minor: example



The red graph M is a minor of the graph G, since deleting the green edges and vertices, and contracting the edges marked in blue we get the graph M.

Relations

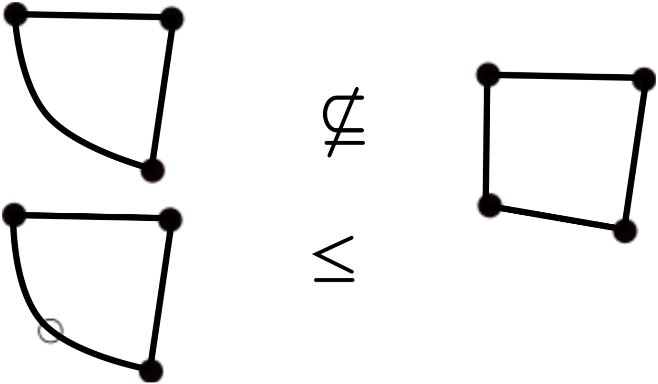
A subgraph is a topological subgraph too.

A topological subgraph is a miner too.

Formally
$$G \supseteq R \Rightarrow G \geq R \Rightarrow G \succeq R.$$

The reverese directions are false.

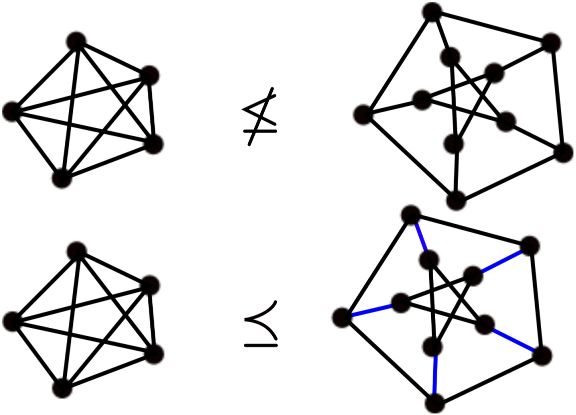
Example for topological subgraph that is not a subgraph



C_3 can be obtained from C_4 by merging e and e', i.e. C_3 is the topological subgraph of C_4.

C_4 has more vertices than C_3. In the case of a subgraph one would have to use vertex deletion, which would result in a graph with a vertex if degree less than 2. So C_3 is not a subgraph of C_4.

Example for a minor that is not a topological subgraph



K_5 can be obtained from the Petersen graph by contracting the edges marked in blue, so K_5 is minor in the Petersen graph.
K_5 is not a topological subgraph of the Petersen graph, since the Petersen graph graph has degree 3, but the degree of the vertices of K_5 is 4.

An important observation

Observation

If G is a planar graph, furthermore $R \subseteq G$; $T \preceq G$; $M \preceq G$, then R, T, M are planar too.

Corollary

If G is planar, then

(i) K_5 and $K_{3,3}$ cannot be a subgraph of G,
(ii) K_5 and $K_{3,3}$ cannot be a topological subgraph of G,
(iii) K_5 and $K_{3,3}$ cannot be a minor of G,

Many further examples of non-planar graphs.

Break

Theorems of Kuratowski and Wagner

Theorem

The following three properties are equivalent:

(i) G is planar.

(ii) G doesn't contain K_5 or $K_{3,3}$ as a topological subgraph ($G \not\succcurlyeq K_5; K_{3,3}$).

(iii) G doesn't contain K_5 or $K_{3,3}$ as a minor ($G \not\succeq K_5; K_{3,3}$).

(i)⇔(ii) is Kuratowski's Theorem, and (i)⇔(iii) is Wagner's Theorem.

Proof of Wagner's theorem (sketch)

Proof by contradiction: Assume that there exista a G non-planar graph, that has neither K_5, nor $K_{3,3}$ minor. Assume that G is counterexample, where $|V| + |E|$ is minimal. If "we make G smaller", that it won't be a counterexample. The rest of the proof is "contradiction hunting".

Lemma

G is 3-connected simple graph.

Lemma

If H is 3-connected and $|V(H)| > 4$, the for a suitable edge e the graph H/e remains 3-connected.

We do not prove these technical tools.

Corollaries of the Lemmas

Corollary

Let H be a simple, 3-connected graph with at least 5 vertices. Then we can find an edge $xy \in E(H)$, that the graph $H - \{x, y\}$ is 2-connected.

Corollary

Let G be the minimal counterexample. Then we can find an edge $xy \in E(G)$, that the graph G/e is 3-connected, and the graph $G - \{x, y\}$ is 2-connected.

We know that G/e is not a counterexample, it can be drawn nicely. $[e]$ is a vertex-point in a face of the plane graph $G - \{x, y\}$, that is bounded by a cycle C.

Corollaries of the Lemmas (continued)

Let $P = N(x) \cap V(C)$, $K = N(y) \cap V(C)$, where $N(x)/N(y)$ is the neighborhood of x/y in $G - e$.

We refer to the elements of P as red vertices and the elements of K as blue vertices.

It is important to see that $P \cap K \neq \emptyset$ can also occur, i.e. the two colors are not two exclusive categories.

The following two notions and a lemma help us to arrive at the to end of the proof.

Arcs, separability

Definition: Arc of a cycle

Let C be a cycle with two vertices u and v. The two vertices define two closed arcs: $[u, v]^\frown$ and $[v, u]^\frown$. The two arcs can be considered as the vertex sets of the two uv paths. The intersection of the two arcs (sets of vertices) is $\{u, v\}$. Let $(u, v)^\frown$ be $[u, v]^\frown - \{u, v\}$.

Definition: Separability on a cycle

Let A and B be two subsets of the vertex set of the cycle C. We say that A and B are separable iff there exist $u, v \in V(C)$ for which $A \subseteq [u, v]^\frown$ and $B \subseteq [v, u]^\frown$.

The Main Lemma

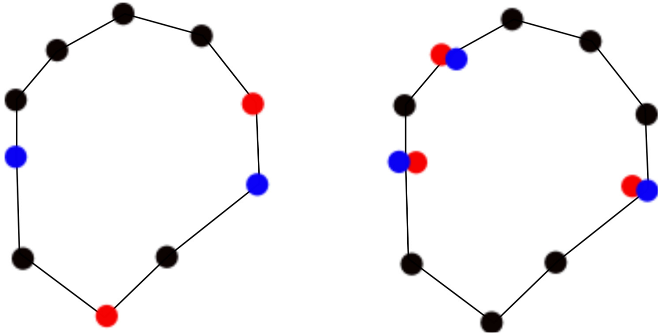
The Main Lemma

Let C be a circle and A and B be two finite subsets of the circle. A and B are not separable if and only if one of the following two possibilities is satisfied.

(i) There exist $a, a' \in A$ and $b, b' \in B$ four different vertices alternating on the cycle, i.e the arcs $(a, a')^{\curvearrowright}$ contains exactly one of the two points b and b'.

(ii) $A = B$ and $|A| = |B| = 3$.

The two obstructions of separability on picure

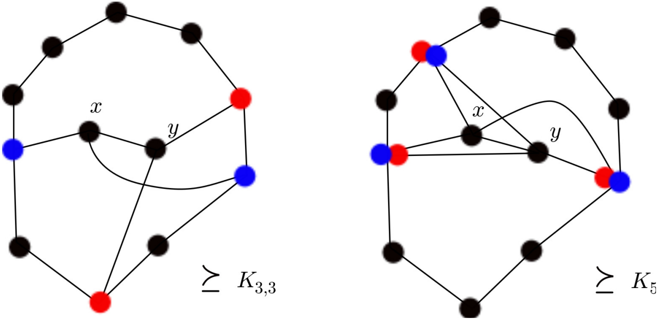


The proof of the Main Lemma is elementary, the interested students can prove it.

Proof of Wagner's Tehorem: The end I

1st case: P and K are non-separable. By the Main Lemma we must see one of the obstructions.

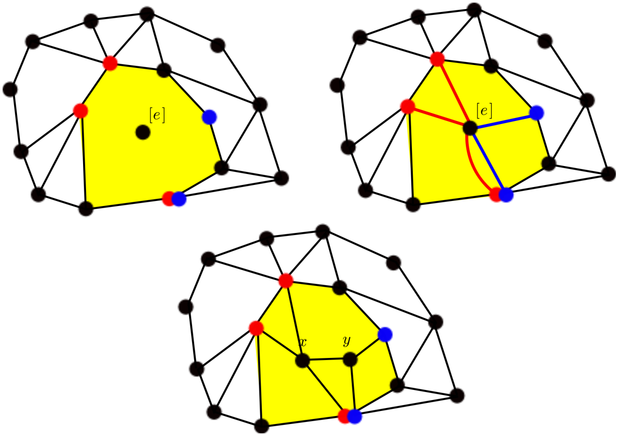
Proof of Wagner's Tehorem: The end II



We see that on the left hand side $K_{3,3}$, and on the right hand side K_5 appears as a minor. A contradiction.

Proof of Wagner's Tehorem: The end III

2nd case: P and K are separable along the circle C.



In this case a nice drawing of G/e and later a nice drawing of G can be easily constructed, contraction.

Break

Theorems on drawings of graphs

Finally, some theorems are stated without proof.

Theorem (Fáry's theorem)

If G is a simple plane graph, then it can be drawn such that every edge curve is a straight segment.

Theorem (Tutte's theorem)

If G is a simple 3-connected plane graph, then can be drawn such that every curve is a straight line segment, and every bounded face is a convex polygon.

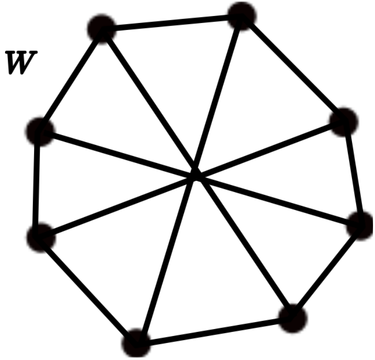
Theorem (Steinitz's theorem)

A graph G is an edge graph of a convex polyhedron if amd only if G is 3-connected simple graph.

Wagner's sturcture theorem

Wagner's sturcture theorem

A graph G doesn't contain K_5 as a minor iff it can be constructed from planar graphs and W, the Wagner graph with operations vertex/edge deletion and gluing along a clique of size at most 3.



Wagner's Theorem, 4CT and Hadwiger's conjecture

Corollary, Wagner's coloring theorem

If G doesn't contain a K_5 as minor, then its chromatic numer is at most 4.

Hadwiger's conjecture

If the graph G does not contain K_{k+1} as a minor, then G has a chromatic number at most k.

An equivalent formulation is:

Hadwiger's Conjecture

If the graph G is not k-colorable, then it contains a K_{k+1} minor.

Results toward Hadwiger's conjecture

Hadwiger's conjecture is straight forward if $k = 2$.

The case $k = 3$ is not complicated.

The case $k = 4$ is proven (Wagner's structure theorem and 4CT).

As k increases the conjecture is getting harder.

The case $k = 5$ is proven.

The proof is based on 4CT, but still it is very complicated.

This is the end!

Thank you for your attention!