Edge colorings of graphs

Peter Hajnal

Bolyai Institute, University of Szeged, Hungary

2023 fall

Introduction

Definition

A function $c : E(G) \to P/\mathbb{N}^+$ is an edge coloring of the graph G.
c is a k-edge-coloring of G iff $|P| = k / c(E(G)) \subseteq \{1, 2, \ldots, k\}$.

Definition

c is a proper edge coloring of G if for each vertex x the incident edges to it have $d(x)$ different colors.

Notation

$$\chi_e(G) := \min\{k \in \mathbb{N}^+ : \ G \text{ has a proper } k\text{-edge-coloring}\}.$$

Loops are obstructions for proper edge colorings. In this unit we assume that our graphs have no loops.

Edge colorings and degrees

Reminder

$\Delta(G) := \max_{x \in V(G)} d(x)$, the maximum degree of G.

Observation

$$\Delta(G) \le \chi_e(G).$$

Example

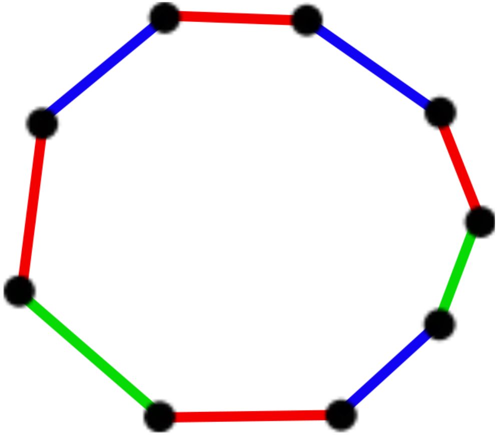


Figure: C_{2k+1} is an odd cycle ($k \in \mathbb{Z}^+$) (on the Figure $k = 4$). Easy to see that $\Delta(C_{2k+1}) = 2$ and $\chi_e(C_{2k+1}) = 3$.

Example

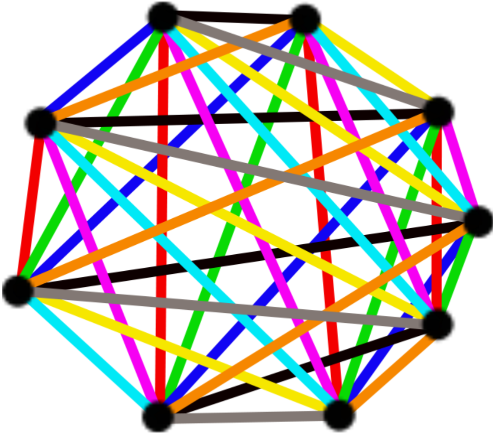


Figure: K_{2k+1}, a complete graph on a vertex set of odd site. It is easy to see that $\Delta(K_{2k+1}) = 2k$ and $\chi_e(K_{2k+1}) = 2k + 1$.

Example

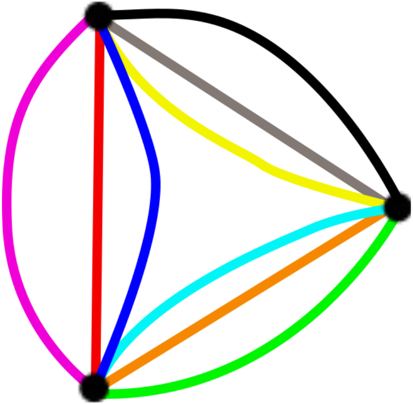


Figure: T_k is the graph with three vertices and any of two different vertices are connected by k parallel (on the Figure $k = 3$). Any two edges are adjacent, hence $\Delta(T_k) = 2k$ and $\chi_e(T_k) = 3k$.

The fundamental theorems

Theorem of Shannon

Let G be a loopless graph. Then

$$\chi_e(G) \leq \frac{3}{2}\Delta(G).$$

Theorem of Vizing

Let G be a simple graph. Then

$$\chi_e(G) \leq \Delta(G) + 1.$$

The first idea of the proofs: greediness

We assume that we have a $c_0 : E_0(\subset E(G)) \to P$ proper partial edges coloring. c_0 can be a coloring of an algorithm, or an induction hypothesis of a mathematical proof.

We would like to extend E_0 to $E(G)$. We can assume that $E_0 = E(G) - \{e\}$. The extension of the coloring to e is an induction step of a mathematical proof or a step in a coloring algorithm.

For each vertex v let S_v the set of free colors around v, i.e. the set of colors from P, that is not used on edges incident to v. We have $|S_v| = |P| - d_v$, where d_v is the number of colored edges incident to v.

If uv is an uncolored edges and $S_u \cap S_v \neq \emptyset$, then we can take any color $\gamma \in S_u \cap S_v$, and use it to color e. We call this step the greedy coloring of c_0 (the originally colored edges are not changed).

The second idea of the proofs: augmentation

We take two colors γ and γ' from our palette. Let $G_{\gamma,\gamma'}$ be the graph formed by $V(G)$ and all edges that are given color γ or γ'.

The maximal degree of this subgraph is at most 2. Hence its components are cycles and paths. The cycle components are cycles of even length, colored alternately with γ and γ'.

Take a path component P (let x and y the two endvertices of the path, we assume that $x \neq y$). Exchanging colors γ, γ' along P (the edges not on P are not recolored) is a modification of our original coloring.

The set of colored edges is not changed. S_x and S_y are changed. The new sets of free colors around x and y are $S_x \Delta \{\gamma, \gamma'\}$, $S_y \Delta \{\gamma, \gamma'\}$. This is a real change.

Break

The proof of Shannon's theorem

The case of $\Delta(G) \leq 1$ is obvious. For the further discussion we assume $\Delta(G) \geq 2$.

We have $P = \{1, 2, \ldots, \lfloor 3\Delta(G)/2 \rfloor\}$. Let $e = uv$ be an arbitrary edge and assume that c_0 is a proper edge coloring of $G - e$. Our goal is to find a proper edge coloring of the whole graph G.

We know that

$$|S_x| \geq |P| - \Delta(G) \geq \lfloor 3\Delta(G)/2 \rfloor - \Delta(G) = \lfloor \Delta(G)/2 \rfloor \geq 1$$

is true for every proper coloring with palette P (complete or partial).

Even more, if around u, and v there is uncolored edge, then

$$|S_u|, |S_v| \geq \lfloor \Delta(G)/2 \rfloor + 1.$$

The case of $S_u \cap S_v = \emptyset$: The greedy case

Observation

If $S_u \cap S_v \neq \emptyset$, we can use the greedy extension to obtain a proper edge coloring of the whole graph G.

The case of $S_u \cap S_v = \emptyset$: Augmentation

Take any color $\alpha \in S_u$. In our case $\alpha \notin S_v$. So we must have an edge vw with color α.

Observation

$w \neq u \neq v \neq w$. // Don't forget that we can have parallel edges!

Observation

If $S_v \cap S_w \neq \emptyset$, then we can recolor the edge vw with color $\kappa \in S_v \cap S_w$.

After the augmentation the only change in the $S_v \leftarrow S_v \Delta \{\alpha, \kappa\}$ and $S_w \leftarrow S_w \Delta \{\alpha, \kappa\}$.

We can recolor uv with color α, and we are done.

The case of $S_u \cap S_v = \emptyset$ and $S_v \cap S_w = \emptyset$: $S_u \cap S_w \neq \emptyset$

Now we assume that $S_u \cap S_v = \emptyset$ and $S_v \cap S_w = \emptyset$.

After a little arithmetic we have $|S_u| + |S_v| + |S_w| > |P|$:

If $\Delta(G) = 2k$ or $\Delta(G) = 2k + 1$ ($k \in \mathbb{N}$), then

$$\begin{aligned}
|S_u| + |S_v| + |S_w| \geq & \left(\left\lfloor \frac{\Delta(G)}{2} \right\rfloor + 1 \right) + \left(\left\lfloor \frac{\Delta(G)}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{\Delta(G)}{2} \right\rfloor \\
= & 3k + 2 > \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor \\
= & \begin{cases} 3k, & |\Delta(G)| = 2k \\ 3k + 1, & |\Delta(G)| = 2k + 1. \end{cases}
\end{aligned}$$

This implies that $S_u \cap S_w = \emptyset$ is impossible.

The case of $S_u \cap S_v = \emptyset$, $S_v \cap S_w = \emptyset$, $S_u \cap S_w \neq \emptyset$

Let $\beta \in S_u \cap S_w$. In the present case $\beta \notin S_v$. Specially there is an edge vs that is colored β. Easy to check that s is different from u, v, w.

We know that $S_v \neq \emptyset$, hence for a suitable color $\gamma \in P$ we have $\gamma \in S_v$.

By our assumptions $\gamma \notin S_u, S_w$.

The case of $S_u \cap S_v = \emptyset$, $S_v \cap S_w = \emptyset$, $S_u \cap S_w \neq \emptyset$ (cont'd)

We consider the component of $G_{\beta,\gamma}$ that contains the vertex w.

This must be a path P: At vertex w the color β is free, hence there is no edge of color β is incident to w. P starts from w with an edge, colored γ.

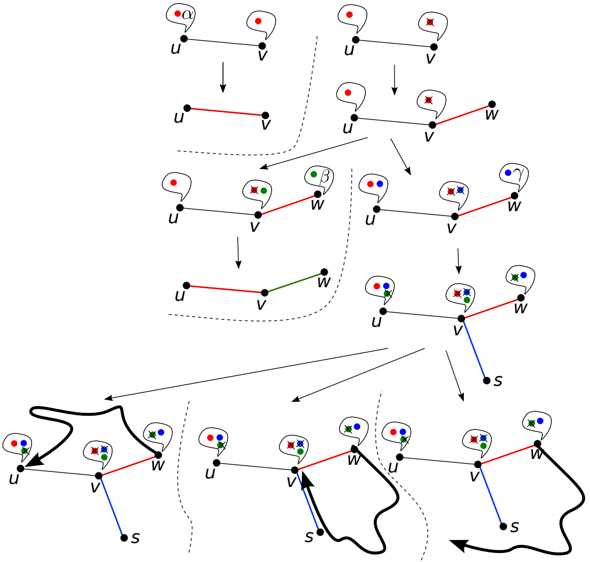
We have several possibilities for the last vertex of P:

(1) u, and the last edge has color γ.

(2) v, and the last edge has color β.

(3) $x \neq u, v, w, s$.

The proof on Figure



The end of the proof: Augmentation

Along P exchange colors β, γ.
We obtain a proper (partial) edge coloring.

In the case of (1) or (3) we can recolor the edge *vw* with color γ, α becomes a free color around *v*, the edge *uv* can be colored with it.

In the case of (2) β becomes a free color around *v*, the edge *uv* can be colored with it.

Break

The proof: Preliminary steps

Now we assume that G is simple: If u and v are adjacent then there is only one edge connecting them.

We have the palette $P = \{1, 2, \ldots, \Delta(G) + 1\}$.

We assume again that c_0 is a proper edge coloring of $G - e$, where $e = uv$ is the only uncolored edge in G.

We know that for each vertex x the set of free colors around it, S_x is a non-empty set.

We can assume that $S_u \cap S_v = \emptyset$.

The proof of Vizing's theorem

Let $\alpha \in S_u$, so $\alpha \notin S_v$, hence there is an edge vu_1 that has color α.

$u \neq u_1$ (our graph is simple). For the further discussion we use the notation $u = u_0$.

We can assume that $S_{u_1} \cap S_v = \emptyset$.

Indeed, if the set above is non-empty then we can recolor the edge $u_1 v$, hence α becomes a free color for uv and we are done.

The proof of Vizing's theorem (cont'd)

Let $\alpha_2 \in S_{u_1}$ (in the further discussion α is also mentioned as α_1).

Assume that $\alpha_2 \neq \alpha_1 = \alpha$. We can conclude that $\alpha_2 \notin S_v$. In this case we have a neighbor u_2 of the vertex v, such that the color of vu_2 is α_2.

We continue our process until we got stuck. At the end we found $u_0, u_1, \ldots, u_\ell$ different vertices, and $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ different colors.

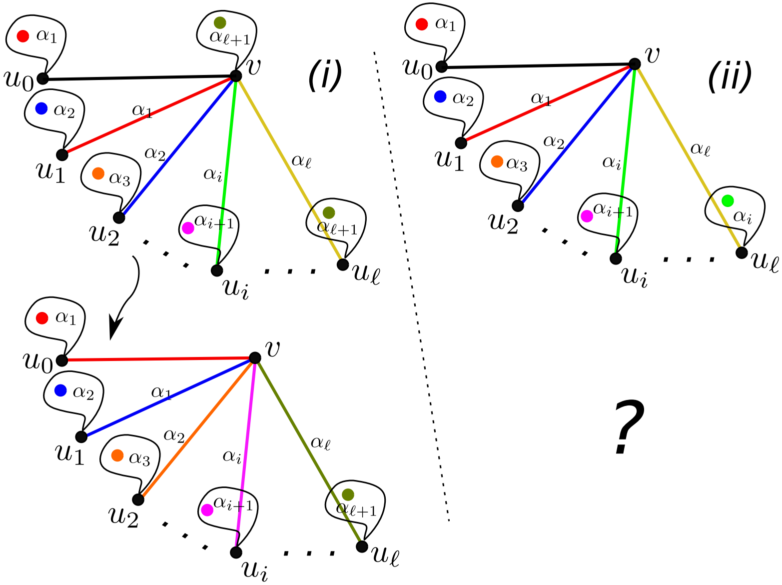
The stoppage

The stoppage in the process is necessary, since the graph is finite. How can this happen? There are two possibilities:

(i) We find a new free color $\alpha_{\ell+1}$ at vertex u_ℓ, but there is no edge of color $\alpha_{\ell+1}$.

(ii) At vertex u_ℓ each possible color $\alpha_{\ell+1}$ can't be different from the previously chosen α_j's. Let i be the index such that $\alpha_{\ell+1} = \alpha_i$.

Stoppage on Figure



(i)

The edge vu_ℓ can be recolored with color $\alpha_{\ell+1}$, at the same time each edge vu_i can get the color α_{i+1} $(i = 0, 1, \ldots, \ell - 1)$.

Specially the edge uv will be colored.

The only problem is case (ii).

(ii)

Let $b \in S_v$. We can assume that the color β is not in any set S_{u_j} (see case (i)).

Consider the component of the graph $G_{\alpha_i\beta}$, that contains the vertex u_ℓ. This component P is a path

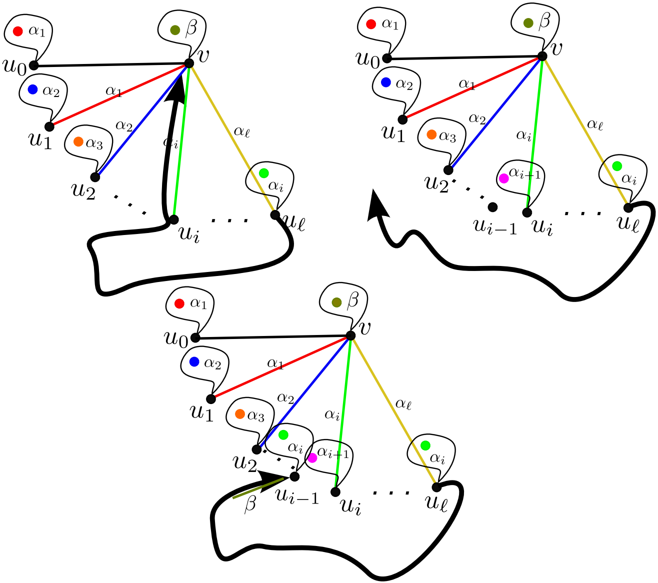
We have three cases:

(iia) P reaches u_i, then it crosses the edge $u_i v$ of color α_i, and P ends at v,

(iib) P reaches u_{i-1}, through an edge of color β and ends there,

(iic) P doesn't reach neither u_i, nor u_{i-1}.

The cases in a Figure



The end of the proof

Execute the augmentation: exchange colors α_i/β.

In the case of (iia) the new color of vertex $u_i v$ will be β. At the same time we can "rotate" the colors of the edges $u_1 v$, $u_2 v$, ..., $u_i v$ as we dis in the case (i). This rotation colors the edge uv.

In the case (iib) β will be a free color around vertex $u_{i-1}v$. We can execute the rotation of (iia).

In the case (iic) we can recolor the edge vu_ℓ with color β. The rotation of (i) works again.

We are done with all cases. The proof of Vizing's theorem is complete.

Break

Theorem of Kőnig

Recall Kőnig's theorem from BSc. An easy consequence is the following theorem:

Theorem

If G is a bipartite graph then

$$\chi_e(G) = \Delta(G).$$

Complexity

Based on the previous theorems it might seem that the edge coloring of a given simple graph is easier than the vertex coloring problem.

That is not true. For those who know some complexity theory the following theorem explain this.

Theorem

We consider the EDGE-COLORING problem: Given a G simple graph, decide whether the value of $\chi_e(G)$ is $\Delta(G)$ or $\Delta(G) + 1$.

The EDGE-COLORING problem is $\mathcal{NP}$-complete.

This is the end!

Thank you for your attention!