Face coloring of plane graphs

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The 4-color-conjecture

Coloring problems in graph theory are closely related applications.

The beginning of their research started with puzzle in the XIXth century.

The puzzle was the so called 4-color problem.

Today it is a theorem: 4-color-theorem, or simply 4CT.

Drawing graphs

ρ is a *drawing of a graph G*, iff $\rho = (\rho_V, \rho_E)$, where:

(i) $\rho_V : V(G) \to \mathbb{R}^2$ is a 1-1 map, i.e. we assign a geometrical point to each node of our graph (different points to different vertices). // We will refer to the points assigned to a vertex as vertex-points.

(ii) $\rho_E : E(G) \to \mathcal{J}$, where $\mathcal{J}$ is the set of continuous, non-selfintersecting plane curves. We assume that for an edge $e = xy$ the curve $\rho_E(e)$ connects $\rho_V(x)$ and $\rho_V(y)$ and it doesn't contain any other vertex-point. // We will refer to the curves assigned to an edge as edge-curves.

$(+)$ We assume that any two different edge-curves have finitely many common points. These are common endpoints or points where the two edge-curves transversally meet.

Plane graphs

Refreshing memory

A drawing is nice iff two different edge-curves can share only common endpoint.

A graph is planar graph iff it has a nice drawing.

There are graphs, that are not planar.

Definition: Plane graphs

A plane graph is a pair (G, ρ), where G is a graph and ρ is a nice drawing it.

Faces

A nice drawing od a graph divide the plane into regions.

Definition: Faces of a nice drawing

We introduce a relation on the set of the point on the plane, that are not covered by edge-curves: $P \sim Q$, there is continuous curve connecting P and Q and not meeting any edge-curve.

This is an equivalence relation. Its equivalence classes are planar point-sets, the faces of the drawing.

Theorem

Let G be a cycle-free graph and a nice drawing λ of it. In this case there is only one face

Example/A fundamental theorem: C_n

Theorem

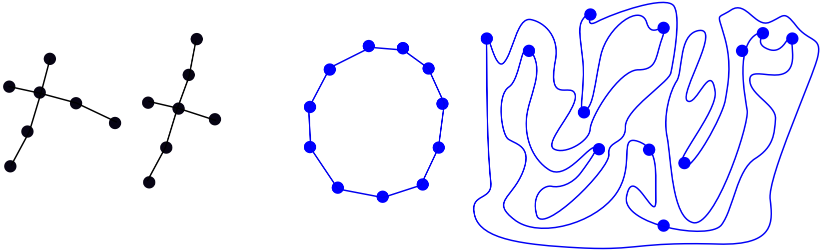
Let C_n be the cycle-graph on n vertices ($|E| = n$) and its nice drawing λ. Then we have exactly two faces: a bounded ($\equiv$ inside) and a non-bounded ($\equiv$ outside). Furthermore the drawing topologically unique.

The second claim is the so called Jordan—Schönflies theorem. It is a very complicated statement.

Without uniqueness the claim is the famous Jordan curve theorem.

We accept this Theorem, we don't prove it.

Example



On the left hand side the two drawings of the same tree are different (why?). On the right hand side we see two topologically equivalent drawings of the same cycle.

Boundary of a face

Definition: Bounding edge of a face

An edge is a bounding edge of a face iff any neighborhood of an inner point of it intersects the face.

Think about a drawing as an overview of system of fences. Think ourselves as living in one of the faces. Put one of our hands on the fence and start walking like that. We will return to our starting point, our walk (edges we touched) will be closed. It is possible that we didn't walk through a bounding edge. Then we take an edge not crossed so far and define a closed walk as before. We proced as far as all bounding edges are included in the boundary.

Definition: Boundary of a face

The boundary of a face is the set of walks, we described above. The length of a boundary is the sum of the length of the walks, contained in the boundary.

Observations

Assume that our graph doesn't have an isolated node.

Observation

If G is not connected, then it has a face with more than 1 walk in the boundary.
Moreover, if G is connected then the boundary of each face is a single walk.

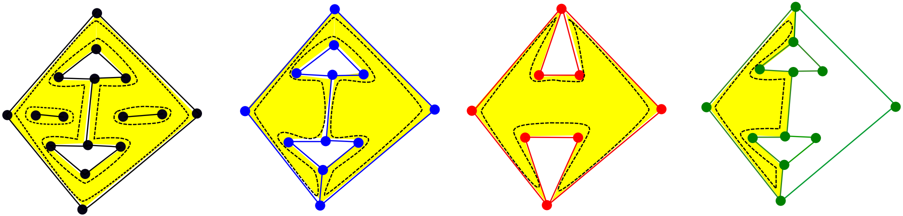
Observation

If G is connected and e is a cut-edge, then the boundary of e is a walk that traverses this edge twice.
Moreover, if G is connected and it has no cut-edge, then the boundary of each face is a single tour (no edge is repeated).

Observation

If G is connected and v is a cut-vertex, then it has a face, that its boundary travereses v more than once.
Moreover if G is connected and it has no cut-vertex, then the boundary of each face is a cycle.

Examples



The figure shows four drawings, each contains an emphasized yellow face.

Break

Plane graphs: Dualization

Let (G, λ) be a connected plane graphs.

In each face in the "middle" we mark a special point, the "capital of the face".

For each edge-curce in the "middle" of it we mark a special point, a "border crossing".

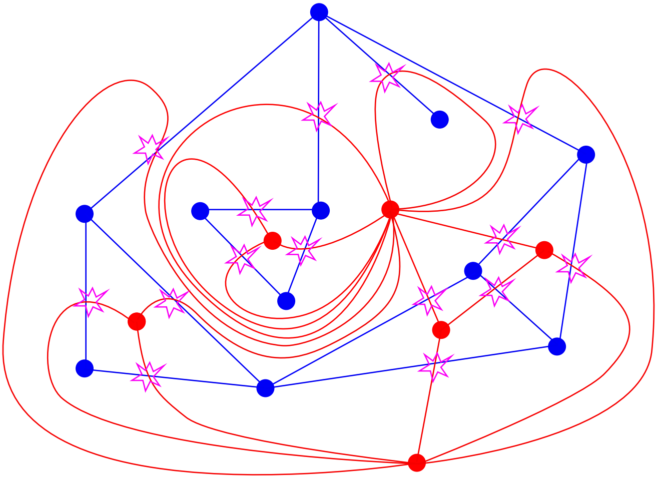
At each border crossing we put two persons standing back to back each other and facing different sides of the edge-curve.

Each person "builds" a path/half-edge from its position to the capital of the face he/she is seeing. Easy to see that this is possible to do such a way that the half-edges are disjoint.

Definition

Let (G^*, λ^*) be the plane graph we obtain after merging the two half-edges meeting at each border crossing to an edge-curve.

Example



On the figure we see purple star that match the edges of the original graph and the edges of the dual graph.

Loops, cut-edges, duality

Let τ be a face of (G, λ), and τ^* the corresponding vertex in G^*. The bounding edges of τ have corresponding edges in G^*. They are exactly the edges incident to τ^*.

A cut-edge e in the original graph corresponds to a loop e^* in the dual graph.

A loop e^* in the dual graph adds 2 to the degree of the correspoonding dual vertex τ^*. The original edge e add 2 to the length of the boundary of τ.

The length of the boundary of face τ is the same as the degree of the dual vertex τ^*.

Dictionary between G and G^*

ORIGINAL	DUAL
G plane graph	G^* plane graph
faces	vertices
edges	edges
two faces with common bordering edge	two adjacent vertices
face coloring	vertex coloring
proper face coloring (for any edge the two faces on the two sides of it get different colors)	proper vertex coloring
condition for proper face coloring: no edge	condition for proper vertex coloring: no loop

Dictionary between G and G^*

ORIGINAL	DUAL
vertices	faces
set of edges, that adjacent to a vertex	edges bounding a face
degree	length of the boundary
4-color-theorem ($4CT$): Faces of any 2-edge-connectred plane graph can be legally colored with 4 colors	4-color-theorem ($4CT$): Any looples planar graph can be legally vertex colored with 4 colors
We can assume: G 3-regular	We can assume: Each face is a triangle

4-color-theorem: Classical form

Definition

A *map* is a graph G, that is nicely drawn on the plane, furthermore it doesn't have a cut-edge (i.e. 2-edge-connected).

4CT: face coloring version

Every map has a proper face coloring with 4 colors.

The following special case is equivalent to the "full version":

4CT: face coloring version, 3-regular case

Let (G, λ) be a map, where G is 3-regular. G has a proper face coloring with 4 colors.

4-color-theorem: Graph theoretical form

4CT: vertex coloring version

Any planar, looples graph has a proper vertex coloring with 4 colors, i.e. $\chi(G) \leq 4$.

The following special case is equivalent to the "full version":

4CT: triangulated vertex coloring version

Let (G, λ) be a loopless graph with a nice drawing If each face is a triangle, then $\chi(G) \leq 4$.

4-color-theorem: Final observations

Observation

If G is a 3-regular, looples graph, then we can find a proper 4-coloring using greedy algorithm.

Observation

We can find a proper 4-coloring of the faces of triangulated map.

Break

4CT as an edge coloring problem

Theorem

The following two claims are equivalent:

(i) For any G, a 3-regular, 2-edge-connected planar graph $\chi_e(G) = 3$.

(ii) 4CT.

Edge coloring theorem $\Rightarrow$ 4CT

It is enough to prove 4CT for a 3-regular, 2-edge-connected planar graph G.

We assume that the edge set of G is a disjoint union of M_1, M_2, M_3, three perfect matchings.

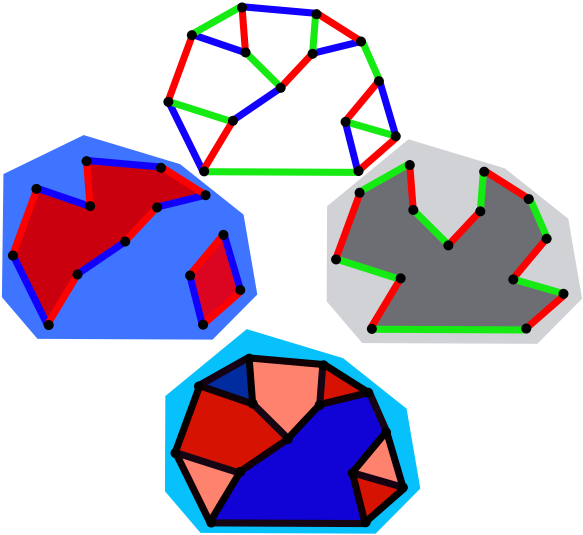
Let $M_1 + M_2$ be the spanning subgraph of G with edge set $M_1 \cup M_2$.

$M_1 + M_2$ is a 2-regular graph, i.e. its components are cycles, nicely drawn on the plane.

It is easy to see that the faces of $M_1 + M_2$ can be legally colored with two colors (red/blue).

We can do the same for $M_1 + M_3$. The two colors can be chosen as dark/light.

Proof by picture



The end of the proof

The two colorings above give two coloring of the same plane.

We can consider that as coloring with 4 colors: dark rad, light red, dark blue, light blue.

This is a legal face coloring of the given plane graph.

4CT $\Rightarrow$ edge coloring theorem

Let G be a 3-regular, 2-edge-connected planar graph G. We assume that faces is legally colored with 4 colors $(1, 2, 3, 4)$.

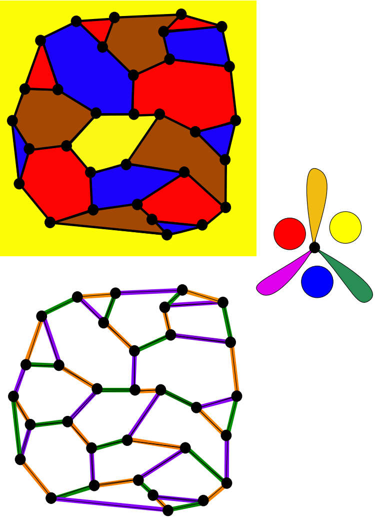
Let

$$M_1 := \{e \in E(G) | \ e \text{ on the two sides we see colors } 1, 2 \text{ or } 3, 4\},$$

$$M_2 := \{e \in E(G) | \ e \text{ on the two sides we see colors } 1, 3 \text{ or } 2, 4\},$$

$$M_3 := \{e \in E(G) | \ e \text{ on the two sides we see colors } 1, 4 \text{ or } 2, 3\}.$$

Proof by picture



The end of the proof

We claim that M_1, M_2, M_3 are three perfect matchings.

First, they are disjoint.

Second, we claim that M_1, M_2, M_3 are matchings:

Finally $M_1 \cup M_2 \cup M_3 = E(G)$.

The edge coloring version of 4CT

The proven theorem is a result of XIXth century mathematics

The XXth century created computers and led to the proof of 4CT.

Theorem

If G is a 3-regular 2-edge-connected planar graph, then

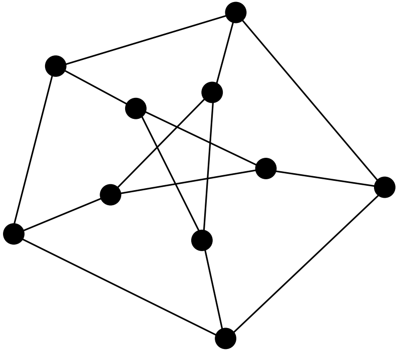
$$\chi_e(G) = 3.$$

The Petersen graph

In the new version of 4CT the assumption that G is a planar graph is a crucial condition.

If we do not assume planarity then there are counterexamples. The simplest one is given by Petersen

The Petersen graph is 3-regular, 2-edge-connected, non-planar, and $E(G)$ cannot be covered by three matchings.



Thank you for your attention!