Vertex coloring of graphs

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Reminder

Definition

A function $c : V(G) \to P$ is called vertex coloring of the graph G. $c(v)$ is called the color of vertex v.

The set P is called palette, its elements are colors. Examples are $P = \{\text{red}, \text{blue}\}$ or $P = \mathbb{N}_+$.

Definition

A coloring is called proper iff for each $e = uv \in E(G)$ one has $c(u) \neq c(v)$.

In this unit we always assume that G denotes a SIMPLE graph.

Reminder(con't)

Definition

c is a k-coloring of G, if the palette has size k.

Definition

The chromatic number of G is

$$\chi(G) = \min\{k : G \text{ has a proper } k\text{-coloring}\}.$$

Colorings and independent sets

Definition

$F \subset V(G)$ is an independent vertex set in the graph G if there is no edge $e = uv \in E(G)$ with $u, v \in F$.

Definition

$$\alpha(G) = \max \{|F| : F \text{ independent vertex set in } G\}.$$

A coloring is proper iff for any color the set of vertices with the given color form an independent set.

So a proper coloring of G can be interpreted as a partition of the vertex set into independent vertex sets.

Colorings and cliques

Definition

$K \subset V(G)$ vertex set of G is called clique iff any two vertices of K are adjacent.

Definition

$$\omega(G) = \max\{|K| : K \text{ is a clique in } G\}.$$

Observation

For any graph G one has

$$\chi(G) \geq \omega(G).$$

The Observation is obvious since any proper coloring must color the vertices of an arbitrary clique with different colors.

Break

Reminder

Triviality

G is non-1-colorable $\iff$ G has an edge.

Lemma

G is non-2-colorable $\iff$ G has an odd cycle.

Observation

G is non-3-colorable $\Longleftarrow$ G has a subgraph K_4 (K_4 is the complete graph on four vertices).

The reveres ($\Longrightarrow$) of the last claim is not true.

Exhibiting a 4-clique is a transparent method. Unfortunately the method is NOT COMPLETE.

Operations

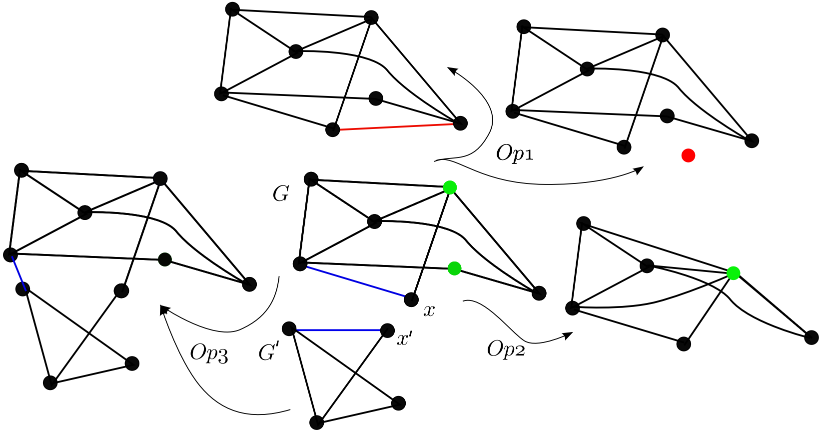
Definition

(Op1) Extension: Adding a new edge or vertex to our graph.

(Op2) Identifying two non-adjacent vertices: Let (x, x') two non-adjacent vertices in G. Let $N(x)$ be denote the set of neighbors of x. The operation substitute the two vertices x and x' with one new vertex $[xx']$ with neighborhood $N(x) \cup N(x')$. $\widetilde{G}$ denotes the graph we obtained.

(Op3) Hajós operation: Let $e \in E(G), e' \in E(G')$, $\overrightarrow{e} = xy$, $\overrightarrow{e'} = x'y'$ be two edges. We produce the new graph H as follows $H = \text{Hajós}_{\overrightarrow{e}, \overrightarrow{e'}}(G, G')$, where
$V(H) = (V(G) - \{x\}) \dot{\cup} (V(G') - \{x'\}) \dot{\cup} \{[xx']\}$,
$E(H) = (E(G) - \{e\}) \dot{\cup} (E(G') - \{e'\}) \dot{\cup} \{yy'\}$, and incidence is the natural one.

Example



Observation

Lemma

If G and G' are non-k-colorable, then neither G^+, nor $\widetilde{G}$, nor Hajós(G, G') are k-colorable.

Lemma

If G^+ or $\widetilde{G}$ is k-colorable, then G is k-colorable too. If Hajós(G, G') is k-colorable, then G OR G' is k-colorable too.

For G^+ the Lemma is obvious. For $\widetilde{G}$ and Hajós(G, G') the Lemma is straight forward.

The effective usage of the Observation

Definition

The graph G is Hajós constructible from K_{k+1}'s iff there exists a sequence of graphs $G_1, G_2, \dots G_l$ such that for each G_i is a K_{k+1}, or can be constructed from previous elements of our sequence using one of our operations.

Corollary

If G is Hajós constructible, then G is non-k-colorable.

Example: 5-wheel is non-3-colorable

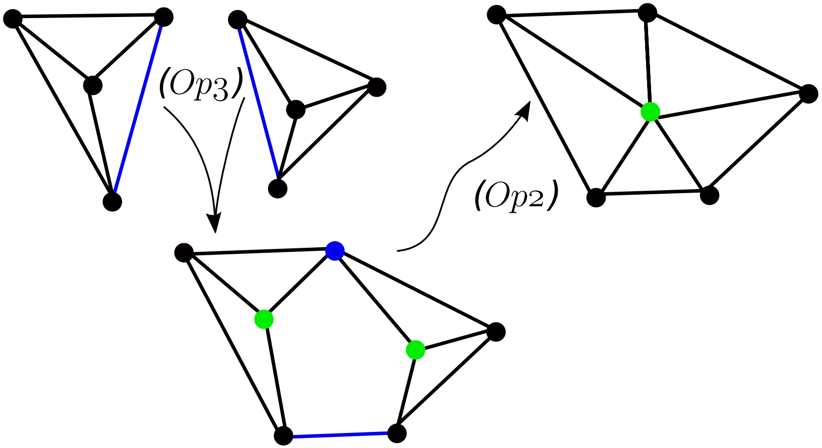


Figure: First we apply (Op3) for G_1 and G_2 (two K_4's'), G_3 is the result of the Hajós operation. We apply (Op2) on it and we obtain G_4. The result is a 5-wheel, a non-3-colorable graph.

The main Theorem

Theorem (György Hajós)

G is non-k-colorable iff G is Hajós constructible from K_{k+1}'s.

We already have proven one direction.

We will prove the other direction by contradiction.

The proof: The first steps

• Assume that there is a counterexample, i.e. there is a graph G that is non-k-colorable and non-Hajós-constructible.

• We can saturate G: i.e. we add edges till the "counterexample" property remains true. G^{satur} denotes the output of the saturation property.

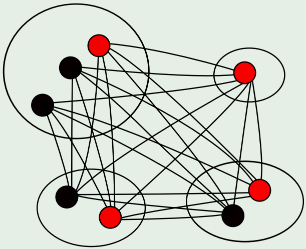
• The saturation process preserves non-k-colorability. The key property of G^{satur} is that if we add any edge to it we must obtain a Hajós constructible graph.

Multipartite graphs

Definition

G is a complete r-partite graph, iff $V(G)$ is partitioned into r parts and its edge set $E(G)$ contains all "cross edges" (uv edges where u and v are in different parts).

Example



Lemma

Definition: An alternative definition of the complete r-partite graphs

A graph G where „to be equal or non-adjacent" is an equivalence relation on $V(G)$, furthermore the number of equivalence classes is r.

Lemma

G^{satur} is a complete r-partite graph.

The proof of Lemma

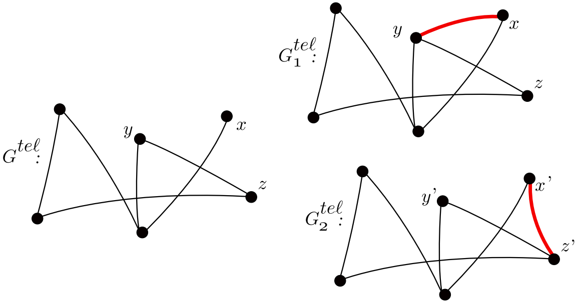
We prove by contradiction.

Assume that in G^{satur} „to be equal or non-adjacent" is not an equivalence relation.

The only way that can happen is that there are $x, y, z \in V(G^{satur})$ three different vertices, that $xy, xz \notin E(G^{satur})$, but $yz \in E(G^{satur})$.

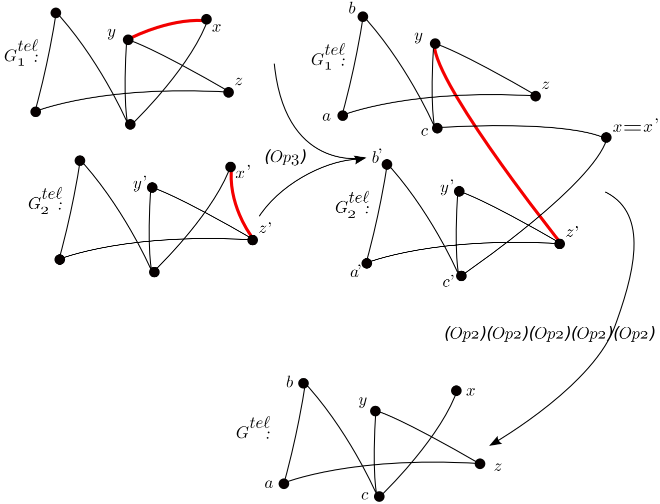
The saturation property of G^{satur} ensures the $G^{satur} + xy$ and $G^{satur} + xz$ are both are Hajós constructible.

The proof of Lemma (cont'd)



The proof of Lemma (cont'd)

$\text{Hajós}_{xy,x'z'}(G^{satur} + xy, G'^{satur} + x'z')$:



The end of the proof of Hajós' Theorem

Now we now that G^{satur} is a counter example and a complete r-partite graph.

We consider two cases.

(1) If $r \geq k+1$: G^{satur} has a K_{k+1} as a subgraph. This implies that G^{satur} is Hajós constructible, a contradiction.

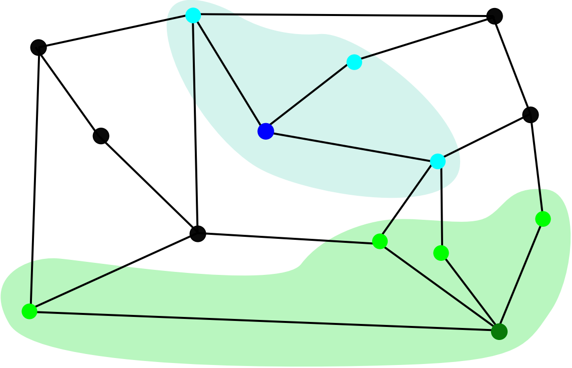
(2) If $r \leq k$: G^{satur} is k-colorable, a contradiction.

Break

Reminder

Theorem (BSc)

There exists a sequence of graphs $\{G_n\}$ such that $\omega(G_n) = 2$ (in other words G is triangle-free), furthermore $\chi(G_n) \to \infty$, assuming $n \to \infty$.



Local vs global

Take a triangle-free graph and think about it as a universe. We can be any vertex and see the vertex, its neighbors and all the edge between. The local information: We don't see any obstruction for coloring.

In spite of the local simplicity, the global coloring problem can be hard.

We try to extend this example.

The extension: A ball

Definition: A ball in a graph

Let G be an arbitrary graph, $o \in V(G)$, and $r \in \mathbb{N}^+$.

$$B(o, r) = G\mid_{\{v \in V : dist(o,v) \leq r\}},$$

where $dist(o, v)$ denotes the length of the shortest ov path/walk.

$B(o, 1)$ is the subgraph spanned by o and its neighbors.

The extension: The problem

Instead of $B(o,1)$'s we consider a local person in G with a farther horizon.

There are several ways to formalize the extension.

(1) For each o the ball $B(o,r)$ is bipartite. Equivalently, in G there is no cycle of odd length at most $2r+1$.

(2) For each o the ball $B(o,r)$ is a tree. Equivalently, in G there is no cycle of length at most $2r+1$.

Definition

The girth of a graph G is

$$g(G) = \min\left\{\ell : \text{ there is cycle of length } \ell \text{ in } G\right\}.$$

Theorem of Paul Erdős

Theorem (Paul Erdős)

For any $\gamma, \tau \in \mathbb{N}^+$ there is a graph G, with $g(G) \geq \gamma$ and $\chi(G) \geq \tau$.

Our proof won't be constructive.

The proof will be a characteristic example for the probabilistic method.

Random graphs

Let V be an n element vertex set. We will fix the value of n later. Till then we will say "n is large enough"

For any pair of vertices we connect then with an edge with probability p. (Hence with probability $1-p$ the chosen two vertices are not adjacent.) For different pairs the probabilistic decisions are independent.

The value of p $(0 < p < 1)$ will be given later, depending on n, τ, γ.

We just described a random model of graphs. This model is the so called Erdős-Rényi model for random graph. It is denoted by $\underline{G}_{n,p}$.

We have several unfixed parameters. During the proof we need to make some promises in order to be able to advance. At the end of the proof we will make our choices and check that our promises are fulfilled.

Independent sets instead of chromatic number

Observation

$$\chi(G) \cdot \alpha(G) \geq |V(G)| = n.$$

"If $\alpha(G)$ is small, then $\chi(G)$ is great". Or "assuming that color classes can't be small, we got to use many colors".

The event $\mathcal{A}_t$

Let $\mathcal{A}_t$ be the event that "$\alpha(G) < t$". Equivalently there are no independent set of size t.

Let $\mathcal{F}_R$ be the event, that R is an independent set in G. We have

$$\mathbb{P}(\mathcal{F}_R) = (1-p)^{\binom{|R|}{2}}.$$

Furthermore:

$$\mathbb{P}(\mathcal{A}_t) = 1 - \mathbb{P}\left(\bigcup_{\substack{R\subseteq V\\ |R|=t}} \mathcal{F}_R\right).$$

In other words the event $\mathcal{A}_t$ is the complement of $\cup_{R\subseteq V, |R|=t}\mathcal{F}_R$.

Bounding the probability of the event $\mathcal{A}_t$

Using the obvious fact $\mathbb{P}(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n \mathbb{P}(E_i)$ we obtain

$$\mathbb{P}(\mathcal{A}_t) \geq 1 - \binom{n}{t}(1-p)^{\binom{t}{2}}.$$

We can simplify the bound by the rude upper bound $\binom{n}{t} < n^t$ and the not so rude upper bound $1 - p < e^{-p}$ (p is positive and close to 0)

$$\mathbb{P}(\mathcal{A}_t) \geq 1 - n^t e^{-p\binom{t}{2}} = 1 - e^{t\log n} e^{-p\frac{t(t-1)}{2}} = 1 - e^{t\log n - p\frac{t(t-1)}{2}}.$$

The choice of t

We assume that $n/2t \geq \tau$ and the lower bound on the probability $\mathcal{A}_t$ is at least $2/3$.

Number of short cycle instead of girth

Let $\xi_{\leq\gamma}$ be the random variable that enumerates the cycles not longer than γ. We will be interested in the expected value of $\xi_{\leq\gamma}$.

We introduce the random variable

$$\xi_C = \begin{cases} 1, & \text{if } C \subseteq G_{n,p} \\ 0, & \text{otherwise} \end{cases},$$

where C is a possible cycle.

We have

$$\mathbb{E}\left(\xi_{\leq\gamma}\right) = \mathbb{E}\left(\sum_{\text{length of } C \leq \gamma} \xi_C\right) = \sum_{l=3}^{\gamma}\left(\sum_{\text{length of } C = l} \mathbb{E}\left(\xi_C\right)\right).$$

Bounding the expected value of the number of short cycles

If the length of C is ℓ, then $\mathbb{E}(\xi_C) = p^\ell$. How many possible cycles of length ℓ? The answer is $\binom{n}{\ell}\frac{(\ell-1)!}{2}$, since the vertex set of the cycle can be chosen $\binom{n}{l}$ ways, and chosen ℓ vertices can be ordered $\frac{(\ell-1)!}{2}$ ways.

Using the inequality

$$\binom{n}{\ell}\frac{(\ell-1)!}{2} = \frac{n(n-1)\dots(n-\ell+1)}{2\ell} \le \frac{n^\ell}{2\ell} \le \frac{n^\ell}{6},$$

we can give an upper bound on $\mathbb{E}(\xi_{\le\gamma})$:

$$\mathbb{E}(\xi_{\le\gamma}) \le \sum_{\ell=3}^{\gamma} \frac{n^\ell}{6}p^\ell = \sum_{\ell=3}^{\gamma} \frac{n^\ell p^\ell}{6} \overset{(!)}{\le} \sum_{\ell=3}^{\gamma} \frac{(np)^\gamma}{6} \le \gamma\frac{(np)^\gamma}{6}.$$

Further promises

We assume that $np \geq 1$.

Our choice of parameters are such that $\gamma(np)^\gamma/6 \leq n/6$ **is true**.

The final conclusion

Based on the promises and Markov's inequality we obtain

$$\mathbb{P}\left(\xi_{\leq\gamma} > \frac{n}{2}\right) < \mathbb{P}\left(\xi_{\leq\gamma} > 3\mathbb{E}\xi_{\leq\gamma}\right) < \frac{1}{3},$$

$$\mathbb{P}\left(\xi_{\leq\gamma} \leq \frac{n}{2}\right) > \frac{2}{3}.$$

After these we have

$$\mathbb{P}\left(\mathcal{A}_t \wedge \left(\xi_{\leq\gamma} \leq \frac{n}{2}\right)\right) > 0$$

since the two events connected by $\wedge$ have probability at least $\frac{2}{3}$.

The graph, we are looking for, exists!

We know that there is a graph G, on vertex set of size n and the following two properties:

- In G the number of cycles of length at most γ is smaller then $\frac{n}{2}$, hence after deleting suitable $n/2$ vertices we obtain a graph G_0 such that $g(G_0) \geq \gamma$.
- Any independent set in G_0 has site at most t. Hence $\chi(G_0) \geq |V(G_0)|/t \geq \tau$.

G_0 proves the theorem.

The consistency of our promises

We have to fix the value p, t such a way that:

$$0 < e^{t \log n - p\frac{t(t-1)}{2}} < \frac{1}{3}, \qquad np > 1, \qquad \gamma(np)^\gamma \leq n,$$

and $n \geq 2t\tau$.

A possible choice: Let p be satisfying $\gamma(np)^\gamma = n$. So the third promise "defines" p.

In this case $np = (n/\gamma)^{1/\gamma}$, i.e. the second assumption is automatically true (n is large enough).

$$p = \frac{1}{n} \cdot (n/\gamma)^{1/\gamma} = c(\gamma)n^{-(1-1/\gamma)},$$

where $c(\gamma)$ is a constant only depending on γ.

The consistency of our promises (cont'd)

The exponent in the first promise

$$t \log n - c(\gamma) n^{-\left(1-\frac{1}{\gamma}\right)} t(t-1)/2.$$

This should be negative with big absolute value.

This is satisfied if

$$t = 3(\log n) n^{1-\frac{1}{\gamma}}$$

(n is large enough).

$n \geq 2t\tau$ is obviously true.

This is the end!

Thank you for your attention!