Enumeration of trees: Formulas of Cayley and Kirchoff

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2023 Fall

The Idea

Consider two sets

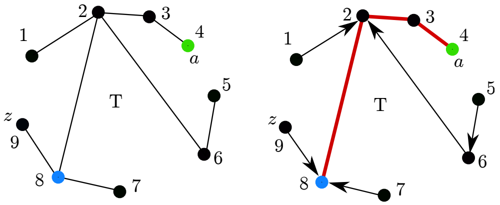
$$\{(T, a, z) : T \text{ is a tree on the vertex set } [n], \text{ and } a, z \in [n]\}$$

and

$$\{f : [n] \to [n]\}.$$

We are going to give a bijection between the two sets.

An example



$a = 4$ and $z = 8$

The Joyal coding of the example is as follows:

(1) Take the unique a-z path in the tree (P). List the vertices of the path in two different ways. First, use the increasing order (in our example 2 3 4 8). Second, use the order as you walk from a to z (in our example 4 3 2 8).

An example (cont'd)

Write down the two sequences in two lines and interpret it as a $\varphi : V(P) \to V(P)$ permutation.

$$\begin{pmatrix} 2 & 3 & 4 & 8 \\ 4 & 3 & 2 & 8 \end{pmatrix}.$$

(2) We extend the domain φ to $V(G)$ and define the Joyal code of our example: Take a vertex v from $V(G) \setminus V(P)$. It might have several neighbors. Exactly one is closer to P than v: v'. Define $f(v)$ as v' outside P, and as $\pi(v)$ on P.

The complete Joyal code of our example is

i	1	2	3	4	5	6	7	8	9
$J_F(i)$	2	4	3	2	6	2	8	8	8

Decoding

For decoding we are given an arbitrary $f : [n] \to [n]$. We must prove that there exists exactly one (T, a, z) triplet such that the above coding procedure assigns f to it.

First, take the directed graph of f. I.e. $\overrightarrow{G}_f$ the directed graph on $V = [n]$, such that it has n edges: $\overrightarrow{if(i)}$ $(i = 1, 2, \ldots, n)$. This a directed graph with the property that each node has out-degree 1.

Consider the sequence of sets:

$$V, f(V), f(f(V)), f(f(f(V))), \ldots$$

This sequence will 'converge' to a set U.

$f|_U$ will be a permutation of U. $\overrightarrow{G}|_U$ consists of directed cycles as components.

Decoding (cont'd)

Write $f|_U : U \to U$ using the 'two lines notation'. Use increasing order in the first line. $s_1, s_2, \ldots, s_\ell$ denotes the second line.

The second line 'lines up' the elements of U.

Now take the directed graph of f. Throw away the edges inside U. 'Forget' the orientation of the remaining edges. Inside U add the edges $s_{i-1}s_i$ $(i = 1, 2, \ldots, \ell - 1)$.

The $n - 1$ edges on n vertices form a tree T.

Let $a := s_1$, $z := s_\ell$.

The rest is left to the diligent audience: The Joyal coding of (T, a, z) is f. There are no other (T', a', z') triplet with f as its its Joyal code.

Break

Directed graphs

Definition

Directed graph is a (V, E, I, O) quadruple, where V is a finite vertex set and E is a finite edge set. $I, O \subset E \times V$ are two incidence relations with the property that for each edge there is exactly one edge incident to it.

- If uOe and vIe, then we say the e goes from u into v, and we write $e = \overrightarrow{uv}$.
- From the directed graph $\overrightarrow{G} = (V, E, I, O)$ we can define the graph $G = (V, E, I \cup O)$. We say that we remove/erase the orientation of $\overrightarrow{G}$ and obtain G. Or $\overrightarrow{G}$ is an orientation of G.

Vertex-edge incidence matrix of directed graphs

Let $\overrightarrow{G}$ be a loopless directed graph.
$Inc_{\overrightarrow{G}} = (a_{ij}) \in \mathbb{R}^{n\times m} \equiv \mathbb{R}^{V\times E}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i I e_j, \\ -1, & \text{if } v_i O e_j, \\ 0, & \text{otherwise.} \end{cases}$$

or

$$a_{ve} = \begin{cases} 1, & \text{if } vIe, \\ -1, & \text{if } vOe, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the sum of the rows of $Inc_{\overrightarrow{G}}$ is $\overrightarrow{0} \in \mathbb{R}^E$. Hence the rank of $Inc_{\overrightarrow{G}}$ is at most $|V| - 1$

Submatrices with $n-1$ rows

Definition

Given a non-oriented or directed graph G with a distinguished vertex $r \in V(G)$. The (G, r) pair is called *rooted graph*. r is called the root of G.

Notation

Let (G, r) be a rooted graph. Let $\overrightarrow{G}$ be an arbitrary orientation of G.

$Inc_{\overrightarrow{G}}^{-r}[F]$ is a submatrix of $Inc_{\overrightarrow{G}}$, that we obtain by deleting the row of r and all the columns outside F.

The main lemma: Submatrices of size $(n-1)\times(n-1)$

Lemma

Let G be a graph, $\overrightarrow{G}$ an arbitrary orientation of it, and r an arbitrary root. Let F be an edge set of size $|V|-1$ (i.e. $Inc_{\overrightarrow{G}}^{-r}[F]$ is an $(n-1)\times(n-1)$ matrix).

Then the following properties are equivalent:

(1) $Inc_{\overrightarrow{G}}^{-r}[F]$ has full rank $(n-1)$, i.e. its rows are independent, also its columns are independent,

(2) F doesn't contain an edge set of a cycle,

(3) F is an edge set of spanning tree,

(4) $\det Inc_{\overrightarrow{G}}^{-r}[F]\in\{\pm1\}$.

(1)⇒(2)

Proof by contradiction. Assume that $E(C) = \{e_1, \ldots, e_\ell\} \subset F$, where C is a cycle.

First, assume that in $\overrightarrow{G}$ the cycle C corresponds to a directed cycle.

Easy to see that the sum of the columns corresponding to the edges of C is the null-vector. The columns are not linearly independent. The claim is proven.

Second, consider the general case. Note that the corresponding matrix can be obtained from the matirx of the first case, by multiplying some of the columns by -1.

(2)⇒(3)

A pure graph theoretical statement. See in recitation session.

(3)⇒(4)

Assume that F is an edge set of a spanning tree T. We can think about T as a tree that is constructed from r by branching process.

Assume that in the ith step we extended our actual tree by the edge e_i and v_i $(i = 1, 2, \ldots, n-1)$. Note that $Fv = \{e_1, e_2, \ldots, e_{n-1}\}$, and $V(G) \setminus \{r\} = \{v_1, v_2, \ldots, v_{n-1}\}$.

Note that the matrix, where the row and column order follows the indices is a triangular matrix with ± 1's in the diagonal. The claim follows by linear algebra.

(4)$\Rightarrow$(1)

An obvious statement from linear algebra.

Binet-Cauchy formula

Theorem (Binet-Cauchy formula)

Let $A, B \in \mathbb{R}^{n\times m}$. Then

$$\det(A \cdot B^T)_{n\times n} = \sum_{\substack{F\subset\{1,\dots,m\}=\text{'columns'}\\ |F|=n}} \det A[F] \cdot \det B[F].$$

The Theorem of Kirchoff

Corollary (Theorem of Kirchoff)

Let (G, r) be an arbitrary rooted graph, and let $\overrightarrow{G}$ be and arbitrary orientation of G. Then the number of spanning tree of G is

$$\det[Inc_{\overrightarrow{G}}^{-r} \cdot (Inc_{\overrightarrow{G}}^{-r})^T].$$

The proof

Apply the Binet-Cauchy formula for $A = B = Inc^{-r}_{\overrightarrow{G}}$:

$$\det[Inc^{-r}_{\overrightarrow{G}} \cdot (Inc^{-r}_{\overrightarrow{G}})^T] = \sum_{\substack{F \subset E(G) \\ |F|=|V|-1}} Inc^{-r}_{\overrightarrow{G}}[F] \cdot Inc^{-r}_{\overrightarrow{G}}[F].$$

By our Lemma these determinants are 0, or $(-1)^2$, or $(+1)^2$. If we ignore the 0 terms then we obtain

$$\sum_{\substack{F \subset E(G) \\ F \text{is a spanning tree}}} 1.$$

The Theorem is proved.

Another view of $Inc_{\vec{G}} \cdot Inc_{\vec{G}}^T$

Observation

$$(Inc_{\vec{G}} Inc_{\vec{G}}^T)_{uv} = \begin{cases} -(\sharp \text{ of edges between } u \text{ and } v) & \text{if } u \neq v, \\ d(u) & \text{if } u = v. \end{cases}$$

Laplace matrix of a graph

Definitions

$D_G \in \mathbb{R}^{V \times V}$ is a diagonal matrix with the degrees on the diagonal:

$$(D_G)_{v,v} = d(v).$$

$A_G \in \mathbb{R}^{V \times V}$, the adjacency matrix of G, a loopless graph is a symmetric matrix:

$$(A_G)_{u,v} = \sharp \text{ of edges between } u \text{ and } v.$$

$L_G = D_G - A_G$ is the Laplace matrix of G.

Observation

$$A_{\vec{G}}^{-r}(A_{\vec{G}}^{-r})^T = L_G^{-r}$$

The Theorem of Kirchoff, the second form

The Theorem of Kirchoff, the second form

The number of spanning trees of a graph G is

$$\det L_G^{-r} = \det(D_G^{-r} - A_G^{-r}).$$

Cayley's Theorem from Kirchff's Theorem

Apply Kirchoff's Theorem for K_n

$\sharp$ of spanning trees of $K_n =$

$$\det \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & -1 & \dots & -1 & -1 \\ -1 & -1 & n-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & n-1 & -1 \\ -1 & -1 & -1 & \dots & -1 & n-1 \end{bmatrix}_{(n-1)\times(n-1)}$$

This is the end!

Thank you for your attention!