Sets systems and their fundamental extremal problems

Peter Hajnal

Bolyai Institute, SZTE, Szeged

2023 Fall

Sperner System

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The main question in this topic is: What is the largest possible size of a Sperner system over V?

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The maximum size of Sperner systems over V is $\binom{n}{\lfloor n/2 \rfloor}$.

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Let $\mathcal{H}$ be a family of subsets of V with n elements. The f-vector of $\mathcal{H}$ is the $(f_0, f_1, \ldots, f_n)$ vector, where f_i component indicates how many i-element sets are in $\mathcal{H}$.

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(LYM Inequality)

Let S be a Sperner system over V. Then, for the f-vector f,

$$\sum_{i=0}^{|V|} \frac{f_i}{\binom{n}{i}} \leq 1.$$

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The lemma is named after Lubell, Yamamoto, and Meshalkin, who independently proved it. It is also often associated with Béla Bollobás, who proved a related statement using a similar method.

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If we count, for each $E \in S$, all valid π orderings, then we get $\sum_{E \in S} |E|! \cdot (n - |E|)!$ such pairs.

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Now, let π be an arbitrary ordering. Since the inclusion relation is a total order on the initial segments of $[n]$, if $\pi(E_1), \pi(E_2)$ are initial segments of $[n]$, then either $E_1 \subset E_2$ or $E_2 \subset E_1$. So, for any ordering π, there can be at most one E such that $\pi(E)$ is an initial segment of $[n]$.

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$$1 \geq \sum_{i=0}^{|V|} \frac{f_i}{\binom{n}{i}} \geq \sum_{i=0}^{|V|} \frac{f_i}{\binom{n}{[n/2]}} = \frac{|\mathcal{S}|}{\binom{n}{[n/2]}}.$$

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The antichains over $(\mathcal{P}(V), \subset)$ precisely correspond to Sperner systems over V.

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Corollary

$$\max_{A \text{ antichain}} (|A|) \leq \min_{L_1, L_2, \ldots, L_k \text{ is a chain cover}} k$$

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$\mathcal{L} \subset \mathcal{P}(V)$, $\mathcal{L} : L_1 \subset L_2 \subset \ldots \subset L_t$ is a symmetric chain if there exists an i such that $|L_1| = i, |L_2| = i + 1, \ldots, |L_t| = |V| - i$.

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Lemma

$(\mathcal{P}(V), \subset)$ has a cover with disjoint symmetric chains.

Due to symmetry, each chain must contain a set of size $\lfloor n/2 \rfloor$. Therefore, the number of chains used must be $\binom{n}{\lfloor n/2 \rfloor}$.

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For $|V| = 1, 2, 3$, the statement is trivially true. For the induction step, let $|V| > 1$. Then, consider $V = V_0 \dot\cup \{u\}$, where $\mathcal{P}(V_0)$ already has a covering.

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$\mathcal{P}(V) = \mathcal{P}(V_0) \dot\cup \{R \subset V : u \in R\}$. Let $\mathcal{P}(V_0) = \mathcal{L}_1 \dot\cup \mathcal{L}_2 \dot\cup \ldots \dot\cup \mathcal{L}_k$ be the covering from the induction hypothesis. Now, we construct chains from the chains in $\mathcal{L}_t$ as follows:

$$\mathcal{L}_t' : L_1 \cup \{u\} \subset L_2 \cup \{u\} \subset \ldots \subset L_{j-1} \cup \{u\},$$

and

$$\mathcal{L}_t'' : L_1 \subset L_2 \subset \ldots \subset L_j \subset L_j \cup \{u\}.$$

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It can be observed that these chains are symmetric to the base set $V_0 \cup \{u\}$ and pairwise disjoint. Thus, they prove the lemma, and consequently, Sperner's theorem.

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The resolution of this apparent contradiction lies in the fact that $\mathcal{L}'_i$ can be empty.

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In both cases, we only need to prove that the optimum of the maximization problem is larger than the optimum of the

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Then, the statement follows directly from Kőnig's theorem.

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Let $R = \{p^+ : p \in L^+\} \dot\cup \{p^- : p \in L^-\}$ be a covering set of size $|P| - M$.

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For $|R|$ to be minimal, the optimal choice for $P^\pm(R)$ is $\emptyset$.

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Observation

$$\max_{L \text{ chain}} (|L|) = \omega(G_P),$$

$$\min_{A_1, A_2, \dots, A_k \text{ antichain cover}} k = \chi(G_P),$$

$$\max_{A \text{ antichain}} (|A|) = \alpha(G_P) = \omega(\overline{G_P}),$$

$$\min_{L_1, L_2, \dots, L_k \text{ chain cover}} k = \chi(\overline{G_P}).$$

Graph Theory

The connections established in the previous theorem lead to the following graph theoretical concept:

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A graph G is perfect if for every induced subgraph F obtained by deleting some vertices (i.e., a vertex set), we have

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The connections established in the previous theorem lead to the following graph theoretical concept:

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A graph G is perfect if for every induced subgraph F obtained by deleting some vertices (i.e., a vertex set), we have

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Intersecting Set Systems

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The basic extremal question is about the maximum number of edges in an intersecting set system over an n-element vertex set.

Intersecting Set Systems: Examples

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Let $x \in V$. $\mathcal{H}$ consists of all sets containing x. $\mathcal{H}$ is obviously intersecting, and $|\mathcal{H}| = 2^{|V|-1} = 2^{n-1}$.

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Example

If the base set has an even number of elements, and we include only one of the complementary pairs of exactly $|V|/2$-sized sets in $\mathcal{H}$, along with more than $|V|/2$-sized sets.

A High School Problem

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Indeed, we can divide the 2^n subsets of V into 2^{n-1} complementary pairs, and each pair can contribute to at most one intersecting set system.

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Theorem (Erdős—Ko—Rado Theorem)

Let $k \leq n/2$. Let $\mathcal{H}$ be a k-uniform intersecting set system over an n-element V. Then

$$|\mathcal{H}| \leq \binom{n-1}{k-1}$$

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Our estimate is the best possible, achieved by considering all k-sets containing a fixed element.

Circularly Ordered Sets, Arcs

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How many k-length arcs can be selected to form an intersecting system?

Solution

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k arcs can be chosen (for example, $[a_1, a_k], [a_2, a_{k+1}], \ldots, [a_k, a_{2k-1}]$), no more.

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There are $2(k-1)$ arcs, forming $k-1$ complementary pairs: typical pairs consist of arcs ending in a_j and starting in a_{j+1}. (Here we use $2k \leq n$.)

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Thus, there cannot be more than $1 + (k-1)$ arcs.

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First, given E, consider how many ways π can be chosen such that the corresponding pair is counted. It is easy to see that $\pi(E)$ is a k-length arc, and there are n possibilities. Once fixed, there are $k! \cdot (n-k)!$ good bijections.

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For all pairs,

$$\sum_{E \in \mathcal{H}} n \cdot k!(n-k)! = |\mathcal{H}| n \cdot k!(n-k)!$$

is obtained.

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Comparing the two results yields the theorem.

Break

Sunflowers

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Definition

$H_1, \dots, H_s$ form a *s-petal sunflower* (or Δ-system) if for every $i \neq j$ $(i, j \in \{1, \dots, s\})$, $H_i \cap H_j = \bigcap_{k=1}^{s} H_k$. The set $T = \bigcap_{k=1}^{s} H_k$ is called the *plate* of the sunflower.

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For example, a collection of s pairwise disjoint set systems forms an s-petal sunflower.

The fundamental question in the topic of sunflowers is: given a k-uniform set system with no s-petal sunflower, what is the maximum number of edges it can have?

Erdős—Rado Theorem

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Let $\mathcal{H}$ be a k-uniform set system that does not contain an s-petal sunflower. Then

$$|\mathcal{H}| \leq (s-1)^k k!.$$

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We prove the theorem by complete induction on k, specifically in the form: If $\mathcal{H}$ is a k-uniform set system and $|\mathcal{H}| > (s-1)^k k!$, then $\mathcal{H}$ contains an s-petal sunflower.

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Assume that we have established the statement for $k-1$. To prove it for k, we will need the following lemma.

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(ii) there exists a vertex $v \in V$ such that v is incident with at least $\frac{|\mathcal{H}|}{(t-1)k}$ edges.

Proof: Deriving the Theorem from the Lemma

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$$\widetilde{\mathcal{H}} \geq \frac{|\mathcal{H}|}{k(t-1)} > \frac{(s-1)^k k!}{k(s-1)} = (s-1)^{k-1}(k-1)!$$

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By the induction hypothesis, $\widetilde{\mathcal{H}}$ contains an s-petal sunflower, denoted as $S_1, \dots, S_s$. Then $S_1 \cup \{v\}, \dots, S_s \cup \{v\}$ form an s-petal sunflower in $\mathcal{H}$.

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The theorem gives an upper bound of $(s-1)^k k!$ on the number of edges in a set system that does not contain an s-petal sunflower. This bound grows faster than exponential.

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Construction: No three-petal sunflower

Let $V = \{a_1, a_2, \ldots, a_k\} \dot\cup \{b_1, b_2, \ldots, b_k\}$. $\mathcal{H}$ contains edges such that each $\{a_i, b_i\}$ $(i = 1, 2, \ldots, k)$ pair intersects it in exactly one element. It is easy to see that $\mathcal{H}$ is a 2^k-edge k-uniform set system.

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On the other hand, it cannot be k-sized. There must exist an i such that the plate is disjoint from $\{a_i, b_i\}$.

How do the three petals intersect with the set $\{a_i, b_i\}$? They must intersect disjointly and each must have exactly one element in common. Thus, the imaginary sunflower cannot exist.

A recent breakthrough

A recent breakthrough

Rao, Alweiss—Lovett—Wu—Zhang 2019

Let $\mathcal{H}$ be a k-uniform set system that does not contain an s-petal sunflower. Then

$$|\mathcal{H}| \leq \mathcal{O}((s\log(sk))^k).$$

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Naturally, the fundamental question is: what is the maximum number of edges in a λ-intersecting set system?

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From now on, we assume $\lambda \geq 1$.

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$\lambda = 1$,

$V = \{\{v_1, \ldots, v_{n-1}\}, \{v_1, v_n\}, \{v_2, v_n\}, \{v_3, v_n\}, \ldots \{v_{n-1}, v_n\}\}$

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Example: $\lambda = 1$ and the Fano Plane

Seven points $V = \{P_1, P_2, P_3, P_4, P_5, P_7\}$, and set system $\mathcal{H} = \{\{P_1, P_2, P_3\}, \{P_3, P_4, P_5\}, \{P_1, P_5, P_6\}, \{P_1, P_4, P_7\}, \{P_3, P_6, P_7\}, \{P_2, P_5, P_7\}, \{P_2, P_4, P_6\}\}$.

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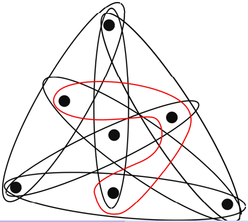

The Fundamental Theorem

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Theorem

Let $\lambda \geq 1$ and $\mathcal{F}$ be a λ-intersecting set system over a base set V. Then
$$|\mathcal{F}| \leq |V|.$$

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If there is an edge in $\mathcal{F}$ with cardinality less than λ, then no other edge in $\mathcal{F}$ is possible, and the statement is trivial.

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From now on, we assume that every edge has more than λ (at least $\lambda + 1$) elements.

Linear Algebra

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For an edge $F \in \mathcal{F}$, let χ_F be the characteristic vector of the set $F \subset V$ ($\chi_F \in \mathbb{R}^V \equiv \mathbb{R}^n$). We will show that the χ_F vectors ($F \in \mathcal{F}$) are linearly independent. This implies the theorem.

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If we set up the linear combination to get the 0 vector, then

$$\alpha_j = \frac{-\Lambda}{|A_j|-\lambda}.$$

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In combinatorics, this linear algebraic method has become particularly significant (for example, in proving the Fisher Inequality). It has become an important proof technique.

Break

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Let $\mathcal{H}$ be a set system over V, and A be a subset of V. Then define $Tr_A\mathcal{H} = \{E \cap A : E \in \mathcal{H}\}$ as the *trace* of $\mathcal{H}$ on A.

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It is clear that $Tr_A\mathcal{H} \subseteq \mathcal{P}(A)$. In the case where $Tr_A\mathcal{H} = \mathcal{P}(A)$, we say that A is *saturated*.

Vapnik–Chervonenkis Dimension

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(Vapnik–Chervonenkis)

Let $\mathcal{H}$ be a set system over $[n] = \{1, 2, \ldots, n\}$, and t be a positive integer such that the inequality $|\mathcal{H}| > 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{t-1}$ holds. Then $\dim_{VC}\mathcal{H} \geq t$. In other words, there exists a saturated set A of size t in $[n]$ for $\mathcal{H}$.

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However, in this system, there is no saturated set A of size t. To see this, note that for a set A to be saturated, $\mathcal{H}$ must contain an edge containing A.

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$$|\mathcal{H}| > \left[\binom{n-1}{0} + \ldots + \binom{n-1}{t-2}\right] + \left[1 + \binom{n-1}{1} + \ldots + \binom{n-1}{t-1}\right].$$

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Denote L_1 and L_2 as the two bracketed expressions.

Proof 1 (contd)

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Introduce the following notations:
$\mathcal{H}_1 = \{E \in \mathcal{H} : n \notin E, E \cup \{n\} \in \mathcal{H}\}$, $\mathcal{H}_2 = \mathcal{H} - \mathcal{H}_1$, and let $\widetilde{\mathcal{H}_2} = \{E \backslash \{n\} : E \in \mathcal{H}_2\}$.

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$$|\mathcal{H}_1| + |\mathcal{H}_2| = |\mathcal{H}| > L_1 + L_2$$

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Either *(i)* $|\mathcal{H}_1| > L_1$, or *(ii)* $|\mathcal{H}_2| = |\widetilde{\mathcal{H}_2}| > L_2$ holds.

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If *(i)* is true, then by the induction hypothesis, there exists a $t-1$-sized saturated set A with respect to $\mathcal{H}_1$. It is easy to see (since for $E \in \mathcal{H}_1$, both E and $E \cup \{n\}$ are edges in $\mathcal{H}$) that $A \cup \{n\}$ is saturated with respect to $\mathcal{H}$ (and has size t).

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If *(ii)* is true, then by the induction hypothesis, there exists a t-sized saturated set A with respect to $\widetilde{\mathcal{H}_2}$. This implies that for every $R \subseteq A$, there exists $E \in \widetilde{\mathcal{H}_2}$ such that $E \cap A = R$. However, for every $E \in \widetilde{\mathcal{H}_2}$, there uniquely exists $E_0 \in \mathcal{H}_2$, either E or $E \cup \{n\}$. In both cases, $E_0 \cap A = R$, which means A is saturated with respect to $\mathcal{H}$. ■

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Define the following S_i transformation: for $i \in V$, if $E \in \mathcal{H}$, then $S_iE = E\setminus\{i\}$ if $E\setminus\{i\} \notin \mathcal{H}$, and $S_iE = E$ otherwise. Let $S_i\mathcal{H} = \{S_iE : E \in \mathcal{H}\}$.

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It is not hard to see that if $\mathcal{H}$ is not down-closed, then there exists an i such that $S_i\mathcal{H} \neq \mathcal{H}$. (If it is not down-closed, there exist E and F such that $F \subset E$ and $E \in \mathcal{H}$ but $F \notin \mathcal{H}$. Then, any $i \in E \setminus F$ will do.)

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The third observation is stated as a lemma.

Lemma

$|Tr_A\mathcal{H}| \geq |Tr_A S_i\mathcal{H}|$ always holds.

From Lemma to Theorem

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For any $\mathcal{H} = \mathcal{H}_1$, there exist $i_1, i_2, \ldots$ such that $\mathcal{H}_k \neq \mathcal{H}_{k+1} = S_{i_k}\mathcal{H}_k$, iterating the S transformation until the set system stops changing.

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Let the last set system be $\mathcal{H}_s$. By what we have shown so far, $\mathcal{H}_s$ is down-closed, and the number of edges satisfies the condition of the theorem.

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Let the last set system be $\mathcal{H}_s$. By what we have shown so far, $\mathcal{H}_s$ is down-closed, and the number of edges satisfies the condition of the theorem.

Thus, there exists a t-sized edge A in $\mathcal{H}_s$. Then A is saturated with respect to $\mathcal{H}_s$, and its trace on A has $2^{|A|}$ elements. By the lemma, the trace of A with respect to $\mathcal{H}_1$ also has at least $2^{|A|}$ elements, which means A is saturated. The proof is complete.

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It suffices to show that every pair contributes at least as much to $Tr_A\mathcal{H}$ as to $Tr_A S_i\mathcal{H}$.

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The only potential issue is when R and $R \cup \{i\}$ are both in $Tr_A S_i\mathcal{H}$, but only one of them is in $Tr_A\mathcal{H}$. Clearly, the missing one must be $R \cup \{i\}$. However, if $R \cup \{i\}$ is not in $Tr_A\mathcal{H}$, then every edge E for which $E \cap A = R \cup \{i\}$ must satisfy $S_iE = E \setminus \{i\}$. This contradicts $R \cup \{i\} \in Tr_A S_i\mathcal{H}$, and thus, the lemma is proven.

This is the End!

Thank you for your attemtion!