Higher order connectivity of graphs

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Flows: Reminder

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Theorem (the main theorem of flows)

Let $\mathcal{H}$ be a network and f be a flow in it. Then the following are equivalent:

(i) f is a maximum value flow in the network $\mathcal{H}$.

(ii) There is no augmenting path with respect to f in the network $\mathcal{H}$.

(iii) There exists a source/sink cut in $\mathcal{H}$ with capacity equal to the value of f.

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Let $\mathcal{H} : (\overrightarrow{G}, c, s, t)$ be a network. Then

$$\max\{\text{val}(f) : f \text{ is a flow in } \mathcal{H}\} = \\ \min\{c(\mathcal{V}) : \ \mathcal{V} \text{ is a source/sink cut in } \mathcal{H}\}.$$

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Another consequence of the fundamental theorem is the Ford-Fulkerson algorithm.

Integral Flow Theorem

If every edge in network $\mathcal{H}$ has an integer capacity ($c : E(\overrightarrow{G}) \to \mathbb{Z}$), then there exists an optimal flow in which every edge carries an integer amount of material.

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Easy observation/exercise

Let $\overrightarrow{G}$ be an arbitrary directed graph with two distinguished nodes s and t. Let $\mathcal{H}_{\overrightarrow{G}}$ be the following network: $(\overrightarrow{G}, c \equiv 1, s, t)$.

(i)

$$\max\{\mathrm{val}(f) : f \text{ is a flow in } \mathcal{H}_{\overrightarrow{G}}\} =$$
$$\max\{k : P_1, P_2, \ldots, P_k \text{ are edge-disjoint } \overrightarrow{st}\text{-paths in } \overrightarrow{G}\}$$

(ii)

$$\min\{c(\mathcal{V}) : \mathcal{V} \text{ is a source/sink cut in } \mathcal{H}_{\overrightarrow{G}}\} =$$
$$\min\{|S| : S \subset E(G) \text{ is a source}\rightarrow\text{sink separating edge set}\}.$$

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(ii)

$$\max\{k : \; P_1, P_2, \dots, P_k \text{ internally node-disjoint } \overrightarrow{st} \text{ paths in } \overrightarrow{G}\} = \min\{|U| : \; U \subset V(\overrightarrow{G}) - \{s, t\} \text{ is a source}\rightarrow\text{sink separating node set}$$

Menger's Theorems for Undirected Graphs

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Let G be an arbitrary undirected graph with two distinguished nodes s and t. Then

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$$\max\{k : P_1, P_2, \dots, P_k \text{ edge-disjoint } st \text{ paths in } G\} = \min\{|S| : \ S \subset E(G) \text{ is a source/sink separating edge set}\}.$$

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There is no suitable separating set U, and the paths P_i may be the same one-edge path (without internal nodes).

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There is no suitable separating set U, and the paths P_i may be the same one-edge path (without internal nodes).

That is, the optimum of both optimization problems is ∞. In this case, it is worthwhile to assume the absence of edges between s and t.

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For every set $F \subseteq E(G)$ with $|F| < k$, the graph $G - F$ is connected.

The condition must hold even for $F = \emptyset$, i.e., our base graph must be connected. Connectivity should be preserved when any proper but not *large* set of edges is removed.

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For every set $U \subseteq V(G)$ with $|U| < k$, the graph $G - U$ is connected, and $|V| > k$.

The technical condition for the vertex count serves to ensure that the graph is sufficiently large: after removing the *not too large* vertex set mentioned in the definition, at least two vertices should remain.

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Example

Among graphs with $k+1$ vertices, only the complete graph is k-connected.

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The horizontal connections are obvious from the definitions. The vertical arrows represent a somewhat more challenging relationship. The starred equivalence is only partially true. In 1-vertex-connectedness, the condition of having at least two vertices is essential; this is not a requirement for connectivity. The other vertical implications follow from the lemma below.

Lemma

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Let e be any edge of graph G and v be any vertex. Let $k \geq 2$.

(a) If G is k-edge-connected, then $G - e$ is $(k-1)$-edge-connected.

(b) If G is k-vertex-connected, then $G - v$ is $(k-1)$-vertex-connected.

(c) If G is k-edge-connected, then $G - v$ can have any number of components.

(d) If G is k-vertex-connected, then $G - e$ is $(k-1)$-vertex-connected.

Characterization of Higher Connectivity

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(i) A graph G is k-edge-connected if and only if, for any two of its vertices, there exist k pairwise edge-disjoint paths between them.

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(ii) A graph G is k-vertex-connected if and only if, for any two of its vertices, there exist k paths, whose internal vertices form pairwise disjoint sets (Path system is *vertex-independent*), and $|V(G)| > k$.

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Applying the condition to x and y, the guaranteed path system between x and y is in our graph. Removing the edges/vertices must eliminate each of them. Due to the independence of the paths, this cannot happen.

Proof: Non-trivial Direction (i)

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Let G be a graph, and $x, y \in V$ be any two vertices, with k given.

Assume that G is k-edge-connected, and apply Menger's theorem.

$$k \leq \min\{|L| : \ L \subseteq E(G), G - L \text{ does not have an } xy \text{ path}\} =$$
$$= \max\{l : P_1, \ldots, P_l \text{ pairwise edge-disjoint } xy \text{ paths in } G\}$$

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Thus, there exist k pairwise edge-disjoint xy paths in G.

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If $p \geq k$, then the statement holds.

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We show that there exist $k-p$ vertex-independent xy paths in $G-P$.

Apply the undirected, vertex-independent version of Menger's theorem (x and y are not connected in $G-P$):

$$k-p \leq \min\{|U|:\ U \subseteq V(G)\setminus\{x,y\},\ G-P-U \text{ does not have an } xy$$

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Hence, there exist $k-p$ vertex-independent xy paths in $G-P$. Adding the elements of P as 1-length xy paths, we obtain k vertex-independent xy paths in G.

Connectivity Parameters

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Definition

The connectivity parameters of graph G:

$$\kappa_e(G) = \begin{cases} \max\{k : \ G \text{ is } k\text{-edge-connected}\}, & \text{if } G \text{ is connected} \\ 0, & \text{if } G \text{ is not connected} \end{cases}$$

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Observation

For every graph G, the following hold:

$$\kappa_e(G) = \min_{x,y \in E(G)} \max\{k : \ P_1, \ldots P_k \text{ pairwise edge-disjoint } xy \text{ paths in } G\}$$

$$= \min_{x,y \in E(G)} \min_{\mathcal{V} \ xy \text{ cut}} |E(\mathcal{V})| = \min_{\mathcal{V} \text{ cut}} |E(\mathcal{V})|,$$

where $\mathcal{V} = \{S, T\}$, $S \cup T = V(G)$, $S \cap T = \emptyset$, $S, T \neq \emptyset$.

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Calculating $\max_{\mathcal{V} \text{ cut}} |E(\mathcal{V})|$ is *hard*, an $\mathcal{NP}$-complete problem.

Break

Minimal k-edge-connected graphs

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If G is minimal k-edge-connected, then it has no loops.

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If G is minimal k-edge-connected, then it has no loops.

If G is k-edge-connected and has at least two vertices, then every vertex has degree at least k.

Boundary of a Vertex Set: Definition

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Notation

The boundary of $U \subseteq V(G)$:

$$\partial U = \{xy \in E(G) : \ x \in U \text{ and } y \notin U, \text{ or } x \notin U \text{ and } y \in U\}$$

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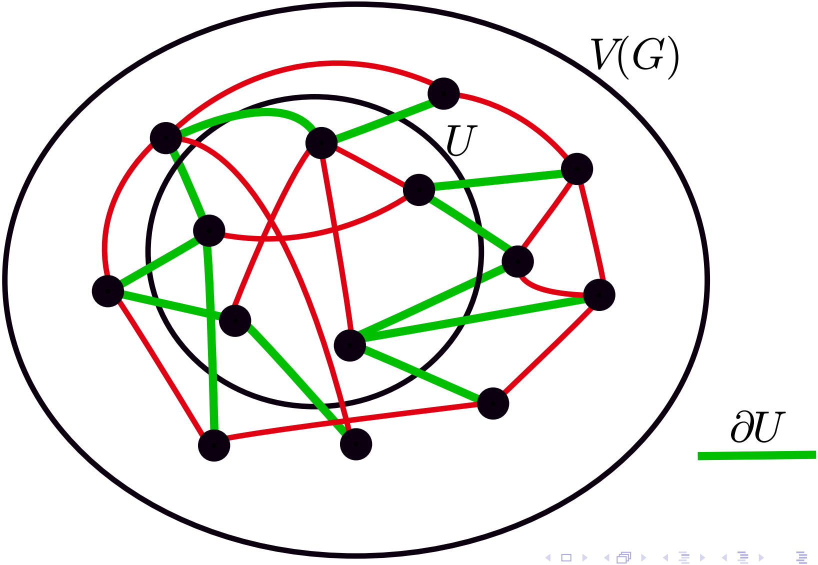

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If G has no loops, then for any $x \in V(G)$, $d(x) = |\partial\{x\}|$.

G is k-edge-connected if and only if the boundary of any proper, non-empty subset of $V(G)$ contains at least k edges.

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k a positive integer, G a minimal k-edge-connected graph. A set $P \subseteq V(G)$ is called a precise set if its boundary contains exactly k edges.

The statement (i) of the theorem is equivalent to the existence of a singleton precise set in G.

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If, for any $e = xy \in E(G)$, $G - e$ is not k-edge-connected, then there exists a separating set $C \subset V(G)$ such that $|\partial_{G-e} C| < k$. In this case, C is a precise set in G and separates x and y.

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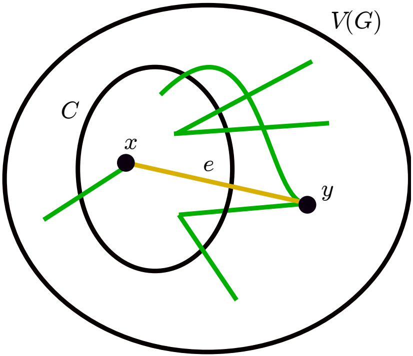

Proof of Mader's Theorem (i): Cases

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Let M be a minimal precise set in G, i.e., a precise set such that none of its proper subsets is precise. We claim that M is a singleton set.

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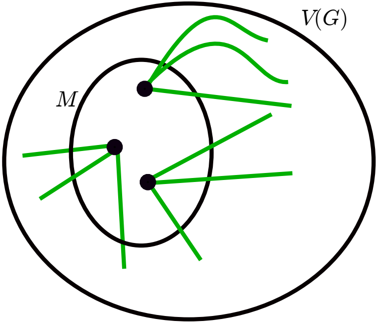
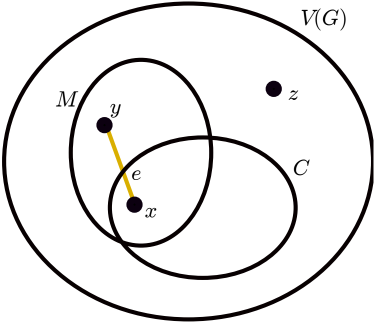


$1.eset$



$2.eset$

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Since every vertex in G has degree at least k, M can only be a singleton set.

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Due to the observations, there exists a precise set $C \subseteq V(G)$ that separates x and y. Without loss of generality, we can assume $z \notin C$; if z was an element of C, we could replace C with $\overline{C}$.

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There are eight cases. In all cases the right hand side counts e at least as many times as the left hand side.

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(ii) Let P be a precise set in G. In this case, $\overline{P}$ is also precise. P and $\overline{P}$ each have a minimal precise subset for containment, let these be M_1 and M_2. These are two different singleton precise sets in G.

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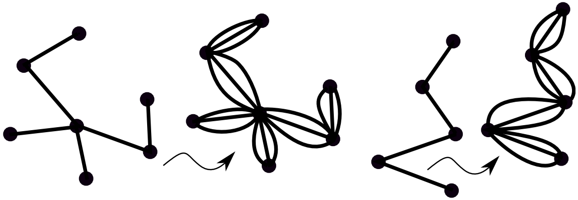

Break

Lovász's Lifting Lemma

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Let G be a graph, $u \in V(G)$, $G_0 = G - u$, $k \geq 2$ an integer. Suppose that the number of edges between u and G_0 is even and positive, and u satisfies the following condition:

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Then, there exist two edges $e = ux$ and $f = uy$ incident to u such that the graph $\widetilde{G} = G - e - f + xy$ also satisfies condition (L).

Lovász's Lifting Lemma in Pictures

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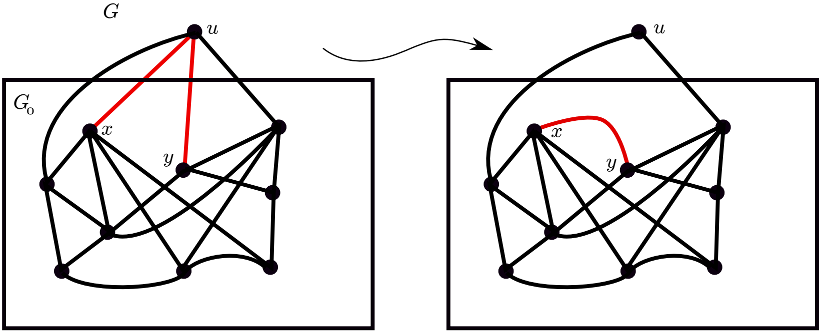

Figure: In the figure, the red edges are being exchanged. If an edge already exists between x and y, we add a new edge parallel to the existing xy edges.

G graph, k positive even integer, two operations

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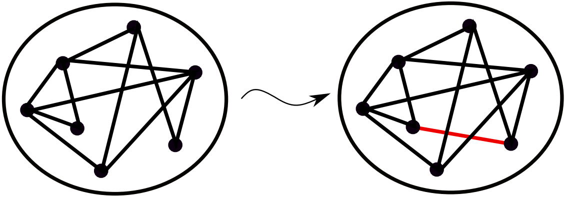
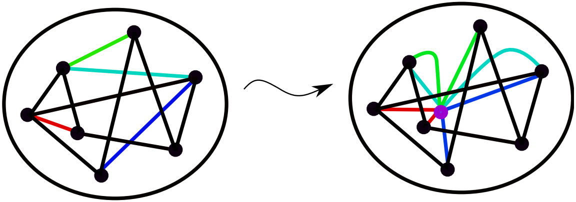
Edge addition: We add a new edge between two vertices of G: $G \to G^+$.





Contraction of $k/2$ edges: We remove $k/2$ edges from G, replace their endpoints with new vertices, and then identify the $k/2$ new vertices: $G \to \widetilde{G}$.

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$$G_0 \to G_1 \to \ldots \to G_l = G,$$

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Then G is k-edge-connected.

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Theorem

If k is a positive even number, and G is a k-edge-connected graph, then G can be built from G_0 (see above) using the previous two operations.

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From now on: G is minimally k-edge-connected, $|V(G)| \geq 2$.

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If we contract the edges in $E(H) \setminus E(G)$ to a single vertex u, we obtain the graph G.

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Then, for any edge $e = ux$, there exists an edge $f = uy$ such that the graph $\widetilde{G} = G - e - f + xy$ also satisfies property (L).

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If C_f separates x and y, then $|\partial_{\widetilde{G}} C_f| = |\partial_G C_f| \geq k$, which is a contradiction.

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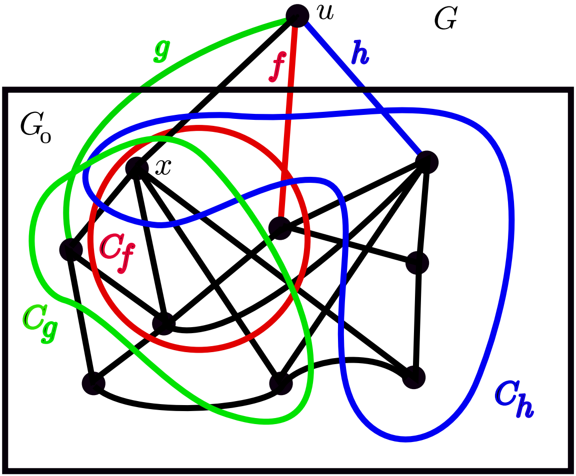
Thus, at most half of the edges incident to u can go to the counterexample set C_f.

Iteration

Repeat the procedure for other edges.



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Let $\mathcal{C}_0$ be the obtained system. Due to (1), it cannot be the case that $\mathcal{C}_0$ consists of only two counterexample sets:

System of Counterexample Sets

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Let $\mathcal{C}_0$ be the obtained system. Due to (1), it cannot be the case that $\mathcal{C}_0$ consists of only two counterexample sets: Otherwise, at most half of the edges incident to u could extend to both sets in a way that the edge ux is included in both, and the two sets together still cover the neighborhood of u. This is clearly impossible.

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For any graph H and sets $A, B, C \subseteq V(H)$, the following inequality holds:

$$|\partial(A \cap B \cap C)| + |\partial(A \cap \overline{B} \cap \overline{C})| + |\partial(\overline{A} \cap B \cap \overline{C})| + |\partial(\overline{A} \cap \overline{B} \cap C)| \leq |\partial A| + |\partial B| + |\partial C|$$

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The proof of the lemma (like proving submodular inequalities) involves simple calculations. We need to check for each edge how many times it contributes to the left and right sides. Each edge contributes at least as much to the right side as to the left side.

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Each of the four terms in the starting four-term sum involves the intersection of three sets, all of which are non-empty (the first has x as an element, the others are empty due to the minimality of $\mathcal{C}_0$). Thus, due to property (L), each term is at least k. Summing up, we have $4k \leq 3k + 1$, i.e., after sorting, $k \leq 1$.

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This is a contradiction because we assumed $k \geq 2$. Thus, one of the edges uy satisfies the lemma.

This is the End!

Thank you for your attention!