Ramsey theory

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Another way of describing it: Color the edges of the graph green, and connect non-adjacent vertices with a red edge. In this way, we obtain a 2-edge-coloring of the complete graph on the vertex set $V(G)$, representing G. A homogeneous set is a set of vertices in which all edges have the same color, or in other words, it is *monochromatic*.

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Analysis of the Ramsey Algorithm

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Lemma

Let G be an arbitrary graph with n vertices. Let k be the number of vertices selected by the Ramsey Algorithm. Let ℓ be the size of the output homogeneous set. Then

$$k \geq \log_2 n, \qquad \ell \geq \frac{1}{2}\log_2 n.$$

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Corollary

Using the notations from the previous lemma, if $n = 4^e$, then $k \geq 2e$ and $\ell \geq e$.

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Corollary

In a graph with 4^k vertices, there always exists a homogeneous set of size k.

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Theorem (Campos—Griffiths—Morris—Sahasrabudhe (2023))

$$R(k) < 3.99999^k.$$

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The lack of knowledge is even more noticeable for $k = 10$. Currently, we only know that $798 \leq R(10) \leq 23,556$.

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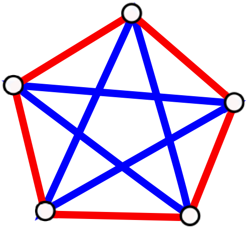

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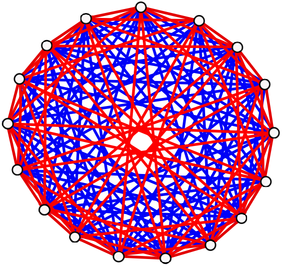

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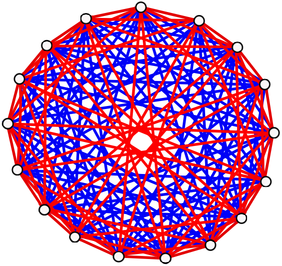

The set of vertices is $Z_{17} = \{0, 1, 2, 3, \ldots, 16\}$. An edge ij is red if and only if $i - j \in \{-8, -4, -2, -1, 1, 2, 4, 8\}$, where the arithmetic is done modulo 17 ($\mathbb{Z}_{17}$ arithmetic).

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In contrast to previous algorithms, this one does not *discard* vertices.

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Recursively call the algorithm on $G|_N$ and $G|_{\overline{N}}$ with $F(N)$ and $K(N)$ being the independent set and clique found in $G|_N$, and $F(\overline{N})$ and $K(\overline{N})$ being the independent set and clique found in $G|_{\overline{N}}$.

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We know that $|V| \geq \binom{k+\ell-2}{k-1}$ and $|N| + |\overline{N}| = |V| - 1$.

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Then

$$|N| + |\overline{N}| = |V| - 1 \geq \binom{k+\ell-2}{k-1} - 1$$
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Thus

$$|\overline{N}| > \binom{(k-1)+\ell-2}{(k-1)-1} - 1 \quad \text{or} \quad |N| > \binom{k+(\ell-1)-2}{k-1} - 1.$$

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This completes the justification of the statement.

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Simple Cases

(0) $R(k, \ell) = R(\ell, k)$,

(i) $R(1, \ell) = 1$,

(ii) $R(2, \ell) = \ell$.

Erdős—Szekeres Inequality

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Lemma: Erdős—Szekeres Inequality

$$R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1).$$

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◦ If the beings demand the determination of $R(6)$ to avoid violence, then the same support should be given to soldiers and weapon experts to solve the problem.

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As in the Erdős—Szekeres proof, we can break the symmetry of colors and thus introduce generalized asymmetric Ramsey numbers: $R_c(k_1, k_2, \ldots, k_c)$.

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The general case (3 colors instead of 2) can be handled by induction on the palette size, based on the above idea.

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The train of thought shown represents the inductive step.

Coloring r-sets with c Colors

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The two steps can be summarized. We can examine the c-coloring of r-element subsets. For a sufficiently large base set, a monochromatic set of size k is guaranteed in this case as well. The corresponding Ramsey numbers are denoted as $R_c^{(r)}(k)$ and $R_c^{(r)}(k_1, k_2, \ldots, k_c)$.

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The details of the development can be carried out by the interested student.

Break

Ramsey Theory

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Due to the many connections, a theory has developed around this philosophy. Various branches of mathematics provide partial results to this theory.

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Pál Erdős and György Szekeres

If $\mathcal{P}$ is a set of $R^{(4)}(5, k)$ points in general position, then we can select k points from them in such a way that they form the vertices of a convex polygon.

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Lemma

(i) If there are five points in general position in the plane, then we can choose four of them in such a way that they form a convex quadrilateral.

(ii) If there are k points in the plane such that any four of them form a convex quadrilateral, then the k points are in convex position.

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Lemma (ii) implies that this blue set forms a convex set of size k.

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In the proof, the elementary geometric statement was noticed by Klein Eszter, who then asked the question: Is it true that for a sufficiently large point set, we can always find a set of k points that forms the vertices of a convex k-gon in our point set?

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Erdős Pál named the problem "Happy End" because the question itself might have played a role in the later marriage of Szekeres György and Klein Eszter.

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Theorem, Suk 2017

$$2^{k-2}+1 \leq ESz(k) \leq 2^{k+o(k)}.$$

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Our first such theorem will be a lemma. This led to the investigation of the Fermat conjecture. According to this conjecture, the Diophantine equation $x^n + y^n = z^n$ has no non-trivial solutions for $n > 2$. (This conjecture was proven by Wiles in 1994.)

Schur's Theorem

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Let $n \in \mathbb{N}^+$. For sufficiently large prime p, the equation

$$x^n + y^n \equiv_p z^n$$

has non-trivial solutions, where $x \equiv_p y$ means $x \equiv y \mod p$, and a solution (x, y, z) is non-trivial if $x, y, z \not\equiv_p 0$.

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Schur's Lemma, 1916

Let ν be sufficiently large, and let $c \in \mathbb{N}_+$ be an arbitrary palette size. Take an arbitrary coloring $\varphi : \{1, 2, \ldots, \nu\} = [\nu] \to \{1, 2, \ldots, c\}$. Then the equation

$$x + y = z \text{ , where } x, y, z \in [\nu],$$

has a monochromatic solution.

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Then, the values $x = i - h$, $y = j - i$, $z = j - h$ form a suitable solution to the equation.

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Let p be a sufficiently large prime, and consider the multiplicative group of the p elements ($\mathbb{F}_p^*$). We define the subgroup

$$H = \{x^n | x \in \mathbb{F}_p^*\} = \{g^n, g^{2n}, \dots\}$$

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It is observed that the size of this subgroup, $|H|$, is at least $\frac{p-1}{n}$. Then, $\mathbb{F}_p^*$ decomposes into cosets according to H:

$$\mathbb{F}_p^* = m_1 H \dot\cup m_2 H \dot\cup \dots \dot\cup m_\ell H$$

The number of cosets ℓ is $\ell = \frac{|\mathbb{F}_p^*|}{|H|} = \frac{p-1}{|H|} \leq n$.

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Then, applying Schur's lemma with parameters $\nu = p-1$ and $c = n$, we find a suitable color/coset ($m_i H$) and suitable elements in this color/in this coset ($x, y, z \in m_i H$) such that $x + y = z$.

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Dividing by m_i $(m_i \neq 0)$, we get

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This gives us the sought non-trivial solutions.

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Definition: $Sch(c)$, the c-parameter Schur number

For any $c \in \mathbb{N}_+$, let $Sch(c)$ be the minimum ν such that, for any coloring of $[\nu]$, there exists a monochromatic $\{x, y, z\}$ satisfying $x + y = z$. In other words, $Sch(c)$ is the precise threshold in the *sufficiently large* ν from the lemma.

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Definition

The smallest n for which the above theorem holds is denoted as $W_c(k)$.

Break

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Example: Gomoku/Five in a row

The board is an infinite plane grid. Winning configurations consist of five adjacent positions either horizontally, vertically, or diagonally.

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This convention is natural. For example, in the original Tic-Tac-Toe game, positions can be identified as $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$.

Generalized Tic-Tac-Toe Winning Positions

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Let $e \in \{*, 1, \ldots, k\}^d \setminus \{1, 2, \ldots, k\}^d$, and associate with it a line $\mathcal{L}_e = \{P_1, P_2, \ldots, P_k\}$, where $P_i = P_i(e)$ denotes the position obtained by replacing the asterisks in e with i.

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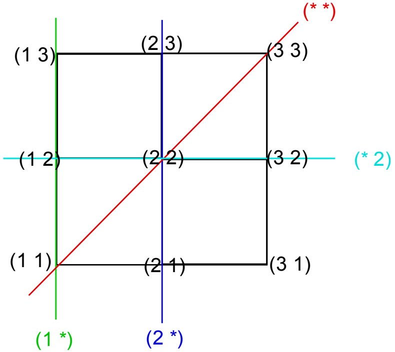
The U_k^d board has $(k+1)^d - k^d$ lines in total.

Example

In the following figure, an example is shown for $k = 3$ and $d = 2$. On the $(1\ *)$ line, the positions are $(1, 1)$, $(1, 2)$, and $(1, 3)$. The $(2\ *)$ line contains the positions $(1, 1)$, $(2, 2)$, and $(3, 3)$. The other diagonal won't form a line. In this case, there are a total of 7 lines.



Generalized Lines, Subspaces

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Definition

In the U_k^d table, an e-dimensional subspace can be described by a vector $a \in \{*_1, *_2, \ldots, *_e, 1, 2, \ldots, k\}^d$ where each indexed asterisk appears at least once. The elements of the subspace $\mathcal{A}_a$ described by this vector are obtained by replacing the $*_i$'s with the same element from $\{1, 2, \ldots, k\}$ (independently for different indices).

Thus, an e-dimensional subspace occupies k^e positions. For $e = 1$, the 1-dimensional subspace corresponds to a line.

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This can also be interpreted as follows: on the table U_k^d, for sufficiently large dimension d, if c players share the positions/color them, then there cannot be a tie, i.e., one of the players contains a winning set of positions/line within their color class.

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Consider the set $\{0, 1, \ldots, n-1\}$ and express its elements in base k. If, during the conversion, we pad the digit sequences with leading zeros to make them of length d, we establish a bijection

$$\{0, 1, \ldots, n-1\} \longleftrightarrow \{0, 1, \ldots, k-1\}^d$$

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The coloring of van der Waerden's theorem corresponds to a Hales—Jewett-style coloring of our table, where the Hales—Jewett theorem guarantees the existence of a monochromatic line corresponding to a k-length arithmetic sequence.

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For the case $k = 2$, observe that the set of positions (expressed in a monotonically increasing sequence) $00\ldots000$, $00\ldots001$,$00\ldots011$, $\ldots$, $01\ldots111$, $11\ldots111$ forms a set (of length $d+1$) such that any two elements form a line. If $d \geq c$, then the pigeonhole principle guarantees two monochromatic elements, i.e., a monochromatic line.

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The proof proceeds as follows:

$$\text{HJ-Statement}(k) \Rightarrow \text{Statement}(k+\frac{1}{2}) \Rightarrow \text{HJ-Statement}(k).$$

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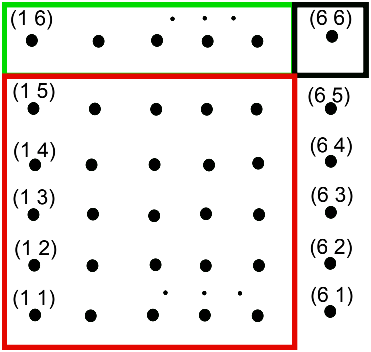
Note that the above definition requires that the order of our e asterisks is fixed.

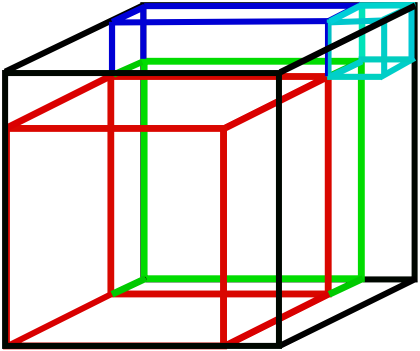
Example

$k = 6$ and $e = 2$. The black square corresponds to $S_6^2(2)$, as in this case everywhere a_1 to 6 must be 6. The green rectangle represents $S_6^2(1)$, and the red square corresponds to $S_6^2(0)$. The non-framed part does not satisfy the condition because it has 6 in the first position, but the next position is less than 6.



Example

In the following figure, $e = 3$ is illustrated. The red cube represents $S_k^3(0)$, the green box represents $S_6^2(1)$, the blue box represents $S_6^2(2)$, and the light blue cube represents $S_6^2(3)$.



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From Statement($k+\frac{1}{2}$) to HJ-Statement($k+1$)

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By the pigeonhole principle, there will be two that have the same color. The proof of Hales—Jewett statement comes from the fact that the union of any two $S^e_{k+1}(i)$ sets contains a line. (This is easily verified after studying examples.)

HJ-Statement(k) $\Rightarrow$ Statement($k + \frac{1}{2}$)

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For $e=1$, it follows easily: the positions of U_{k+1}^d contain the narrower U_k^d table, where our assumption guarantees a monochromatic line.

This line becomes part of the larger table ($*$ can now take the value of $k+1$). Thus, in the larger table, the appropriate line is an extension of the narrow but monochromatic line by one position. Monochromaticity may be lost, but we still obtain a nicely colored line/1-dimensional subspace.

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For a fixed beginning, the possible ends are identified with the positions of $U_{k+1}^{d''}$. In this, each end (with the fixed beginning) describes a colored position in the entire table. Thus, fixing corresponds to a colored $U_{k+1}^{d''}$ table.

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The nicely colored line is a subset of $S_{k+1}^{1}(0)$, i.e., every element (position's beginning) has the same super-color, i.e., the same colored $U_{k+1}^{d''}$ table belongs to it.

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That is, $U_{k+1}^{d'}$ has a super-coloring. In this, there is a nicely colored line (see the case $e=1$). The selection of the line: asterisk the first d' coordinates and fix.

The nicely colored line is a subset of $S_{k+1}^{1}(0)$, i.e., every element (position's beginning) has the same super-color, i.e., the same colored $U_{k+1}^{d''}$ table belongs to it.

d'' should be large enough so that there is a nicely colored e-dimensional subspace in it. Selecting this subspace: asterisk the last d'' coordinates.

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Break

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If a larger monochromatic set is our goal, then more edges can be specified while avoiding the large monochromatic set. The validity of Ramsey's theorem is of a structural nature. If the red edges avoid forming a large monochromatic set, then the complementary set (the blue edges) cannot have a similar structure.

Summary

Theorem	Structure to be Colored	Monochromatic Substructure to be Found	Maximum Size of Possible Color Class
Ramsey Theorem	Edges of a complete graph with n vertices	Edges of a complete graph with 3 vertices	$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, the n-vertex bipartite Turán graph
Ramsey Theorem	Edges of a complete graph with n vertices	Edges of a complete graph with k vertices	$T_{n,k-1}$, the n-vertex, $k-1$ part Turán graph
Schur	$[n]$	$\{x, y, x+y\}$	I. Example: odd numbers. II. Example: $[n]\setminus[\lfloor n/2\rfloor]$.
van der Waerden	$[n]$	AP of length k (non-constant)	???

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(Erdős Pál—Turán Pál, 1936)

$r_k(n) = o(n)$, if $k \geq 3$.

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(Szemerédi's Theorem, 1975)

For every $k \ge 3$, the conjecture holds. That is,

$$r_k(n) = o(n).$$

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- 1977 Fürstenberg. His proof uses ergodic theory.
- 2001 Gowers. His proof employs strong combinatorial number-theoretical results and Fourier techniques. The Fourier method was introduced by Roth, but its successful application required additional brilliant ideas.

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(Gowers' Estimate)

$$W_2(k) \leq 2^{2^{2^{2^{2^{k+9}}}}}.$$

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Green and Tao Theorem (Density Version)

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(Green—Tao Theorem, Density Version)

Let $P_n = \{2, 3, 5, 7, 11, p_6, \ldots, p_n\}$ be the set of the first n primes. Let ϵ be any (small) positive constant. If $A \subset \mathbb{N}$ satisfies $|A \cap P_n| \geq \epsilon n$ for infinitely many n, then A contains an arithmetic progression of length k for every positive integer k.

This is the end!

Thank you for your attention!