Extremal graph theory

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Fall 2023

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In this lecture we always assume that our graph is SIMPLE.

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$F \leftarrow F \cup \{x\}$ // We extend F with x.

$T \leftarrow T - \{x\} - N_T(x)$ // After choosing x only the non-neighboring vertices of T will survive the extension.

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Lemma

The size of the output of the greedy algorithm for finding large independent set is at least

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Hence we executed at least $|V(G)|/(D(G)+1)$ extension steps.

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The analysis of the improved algorithm

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Theorem (Turán's theorem, algorithmic form)

The size of the output of the improved greedy algorithm for finding large independent set is at least

$$\frac{|V(G)|}{\overline{d}(G)+1},$$

where $\overline{d}(G)$ denotes the average degree of G,
i.e. $\overline{d}(G)=\frac{\sum_{x\in V} d(x)}{|V|}=\frac{2|E|}{|V|}$.

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It is straight forward that $V(G) = H_1 \dot\cup H_2 \dot\cup \ldots \dot\cup H_\ell$, where ℓ is the number of extension steps, i.e. the size of the output.

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Let $\mathcal{E} = \mathcal{E}(H_1, H_2, \ldots, H_\ell)$ be that simple graph where two vertices is connected if and only if (iff) they belong to the same bite/cut.

Proof (1st Observation)

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Observation

For each vertex x we have $d_G(x) \geq d_{\mathcal{E}}(x)$, specially

$$|E(\mathcal{E}(H_1, \ldots, H_\ell))| \leq |E(G)|.$$

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(2) edges leading to $H_i \cup \ldots \cup H_\ell$ from x

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We have $d_{\mathcal{E}}^{\text{back}}(x) = 0 \leq d_G^{\text{back}}(x)$, furthermore

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Hence the claim $d_{\mathcal{E}}(x) \leq d_G(x)$ is obvious.

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We have

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Observation

$$\mu_{n,\ell} := \min\{|E(\mathcal{E}(H_1, \ldots, H_\ell))| : \sum_{i=1}^{\ell} |H_i| = n\} \geq \ell \binom{n/\ell}{2}.$$

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The proof of the algorithmic form of Turán's theorem: The claim is a simple consequence of the two Observations and simple arithmetics.

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The Jensen inequality is sharp if we work with real numbers. In our setting this is not true.

Balanced partitions

Definition

We consider a partition of a set of n elements into k parts. We say that the partition is balanced or its parts are almost "equal" iff one of (or all of) the following equivalent properties holds

(i) For each part O $|O| \in \left\{ \left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil \right\}$.

(ii) For any two parts O and O' $||O| - |O'|| \leq 1$.

(iii) $n - k\left\lfloor \frac{n}{k} \right\rfloor$ parts have size $\left\lceil \frac{n}{k} \right\rceil$ and $k - \left(n - k\left\lfloor \frac{n}{k} \right\rfloor\right)$ parts have size $\left\lfloor \frac{n}{k} \right\rfloor$.

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Definition: $\mathcal{E}_{n,k}$, the balanced equivalence graphs on n vertices with k components

Its vertex set is an n elements set, divided into k parts
$V = O_1 \dot\cup \ldots \dot\cup O_k$, of "almost equal" size.
Its edge set contains exactly those edges that are connecting two vertices from the same part.

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$$T_{n,k} = \overline{\mathcal{E}_{n,k}}$$

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Estimating $\mu_{n,\ell}$ using Jensen's inequality:

$$\mu_{n,\ell} \geq \ell \binom{n/\ell}{2} = 875.$$

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$$\mu_{n,\ell} = \min\{|E(\mathcal{E}(H_1,\ldots,H_\ell))| : \sum_{i=1}^{\ell} |H_i| = n\} = |E(\mathcal{E}_{n,\ell})|.$$

Example

$n = 700$, $\ell = 200$

Estimating $\mu_{n,\ell}$ using Jensen's inequality:

$$\mu_{n,\ell} \geq \ell\binom{n/\ell}{2} = 875.$$

Applying Lemma:

$$\mu_{n,\ell} = |E(\mathcal{E}_{n,\ell})| = 900.$$

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Easy computation give us that this modification

Reformulation of the Theorem

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If G is a simple graph on n vertices and $|E(G)| < |E(\mathcal{E}_{n,\ell})|$, then the improved greedy algorithm will have at least $\ell + 1$ extension steps. Specially we have $\alpha(G) \geq \ell + 1$.

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Theorem

Let G be a simple graph on n vertices. If $\alpha(G) < k$, then $|E(G)| \geq |E(\overline{T}_{n,k-1})|$.

The reformulated theorem, complementary form

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Theorem of Paul Turán

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then it must contain a clique of size k.

Turán's theorem, algorithmic version in complementary form

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If we run the improved greedy algorithm for $\overline{G}$ then we will find a "large" vertex set, that forms a clique in G. The algorithm can be formulated in terms of the original graph.

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The improved greedy algorithm for finding large clique in the case of $|E(G)| > |E(T_{n,k-1})|$ will find a clique of size at least k.

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2nd argument (coloring): $T_{n,k}$ has a legal k-coloring. Hence $T_{n,k}$ cannot contain a subgraph R with $\chi(R) \geq k+1$. Hence $T_{n,k}$ doesn't contain a clique of size $k+1$, furthermore it doesn't contain any subgraph that is nor k-colorable.

Break

Generalizations

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A special case of Turán's theorem consider a simple graph on n vertices doesn't have a clique of size 4 (or K_4 subgraph). The claim of the theorem is that G cannot have more than $|E(T_{n,3})|$ edges.

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Extremal graph theory has been born.

The function ext

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Definition

$$ext(n; T) = \max\{|E(G)| : \ G \text{ is a simple graph on } n \text{ vertices}, \ T \not\subseteq G\}.$$

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It will be very useful to introduce the following notation: $\mathcal{G}_n$ denotes the class of simple graphs on n vertices. So $G \in \mathcal{G}_n$ expresses that G is a simple graph on n vertices.

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Turán-type problems form only a small portion of extremal graph theory.

Extremal graph theory considers a special class of graphs and a graph parameter. We are interested what are the extremal values of this parameter if our graphs are chosen from the special class. Very often only the maximal, or only the minimal value has theoretical significance.

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• One can assume that the forbidden subgraph T doesn't have isolated vertex. (Why?)

Break

The forbidden graph with one edge

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Observation

Let I be the gaph containing two vertices and oone connecting edge. Then $ext(n; I) = 0$.

Forbidden graphs with two edges

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Proof: Easy.

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The upper bounds is easy if T doesn't have a cycle.

Theorem on forbidden trees

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Theorem

Let T be a forest (i.e. a graph without cycle; i.e. a graph with tree components). If $|E(T)| > 1$, then for suitable constants $\alpha_T, \beta_T > 0$

$$\alpha_T \cdot n \leq ext(n; T) \leq \beta_T \cdot n.$$

Hence the order of magnitude of $ext(n; T)$ is linear.

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Hence the order of magnitude of $ext(n; T)$ is linear.

The lower bound follows from the condition that the forbidden subgraph has at least two edges. This part of the claim is easy.

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For $G \in \mathcal{G}_n$ we have a subgraph R ($R \subseteq G$) with

$$\delta(R) \geq \frac{\overline{d}(G)}{2}.$$

The proof of the Lemma

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Until we find $x \in V(A)$: $d_A(x) < \frac{\overline{d}}{2}$ do

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The Lemma is equivalent to the fact the the algorithm doesn't „consume" G completely.

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$|E|$ is $n\frac{\overline{d}}{2}$. A contradiction.

The proof of the Theorem

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We proof by induction that T_i is a subgraph of R.

The proof on a Figure

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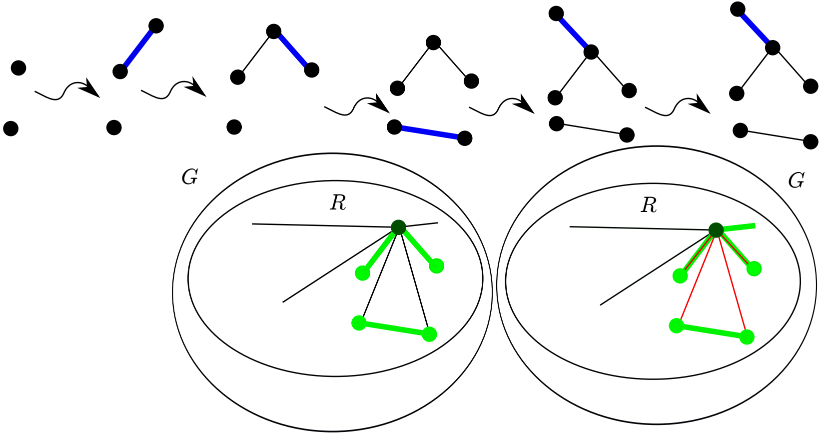

Break

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If T is such $\chi(T) = k \geq 2$, then $ext(n; T) = |E(T_{n,k-1})| + o(n^2)$.

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(i) Let T be a forbidden subgraph with chromatic number $k \geq 3$ (i.e. $k - 1$ — the number of parts of the corresponding — is at least 2). Then $ext(n; T) = |E(T_{n,k-1})| + o(n^2)$ (In this case $o(n^2)$ is the remainder term in the formula).

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(ii) Let T be a forbidden non-empty, bipartite graph, i.e. $\chi(T) = 2$. Then $ext(n; T) = o(n^2)$ (i.e. the formal remainder term is the main term now).

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$$|E(T_{n,k-1})| + \varepsilon \cdot n^2 = \frac{1}{2}\left(1 - \frac{1}{k-1}\right) n^2 + \varepsilon \cdot n^2.$$

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In this case G contains $K_{S,S,\dots,S} = K_{k\times S}$, assuming that n is large enough.

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Let us given $2 \geq k \in \mathbb{N}$. Let $\varepsilon > 0$ be an arbitrary small real number. Let S an arbitrary natural number. Let $G \in \mathcal{G}_n$ be a graph, with "many" edges:

$$|E(T_{n,k-1})| + \varepsilon \cdot n^2 = \frac{1}{2}\left(1 - \frac{1}{k-1}\right) n^2 + \varepsilon \cdot n^2.$$

In this case G contains $K_{S,S,\dots,S} = K_{k\times S}$, assuming that n is large enough. $K_{S,S,\dots,S} = K_{k\times S}$ is a complete k-partite graph, with parts of size S.

The proof: The first steps

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Lemma

For $G \in \mathcal{G}_n$ $|E(G)| \geq \delta\binom{n}{2}$ (the average degrre is at least $\delta(n-1)$). The G has a subgraph R, with all degrees at least $\delta(|V(R)|-1)$.

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The Lemma is equivalent to the fact that the algorithm do not delete all vertices.

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edges. Contradiction, since the number of edges in G is $\delta\binom{n}{2}$.

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Then we have a subgraph R that satisfies

$$(\delta - \varepsilon_0)(|V(R)| - 1)$$

and $|V(R)| \geq N$.

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If this is not the case then the number of deleted edges and the number of remainder edges is at most

$$(\delta - \varepsilon_0)(n-1) + (\delta - \varepsilon_0)(n-2) + \ldots + (\delta - \varepsilon_0)(N+1) + \binom{N}{2}$$
$$= (\delta - \varepsilon_0)\binom{n}{2} + (1 - \delta + \varepsilon_0)\binom{N}{2}.$$

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If n is "large", then this is a contradiction.

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Them for any $s \in \mathbb{N}$ a large enough G contains $K_{(k+1)\times s}$ subgraph.

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• The start of the induction is obvious: $K_{1\times s_1}$ is an empty graph on s_1 verices and n is large enough.

• For the induction step we need to prove that for any s we can find a large $S = S(s)$, that in a large enough G satisfying the assumption on the minimal degree if $K_{\ell\times S}$ is a subgraph, then a subgraph, $K_{(\ell+1)\times s}$ is guaranteed too.

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$x \in J/x$ is good iff x has at least s neighbors on each part of F.

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We take s good vertices with the same type. We also take the $\ell \cdot s$ vertices corresponding the common type. These $(\ell+1)s$ vertices span a $K_{(\ell+1)\times s}$ subgraph.

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So between F and $\overline{F}$ at most

$$|F|\cdot\left(\frac{1}{k}-\frac{\varepsilon}{2}\right)(n-1)=\ell S\cdot\left(\frac{1}{k}-\frac{\varepsilon}{2}\right)(n-1)\leq S\cdot\left(1-\frac{\varepsilon\ell}{2}\right)n$$

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So every vertex in R is not connected to at least $S - s$ vertices in F. The number of missing edges is at least

$$|R|(S - s) = (n - |F| - |J|)(S - s) = (n - \ell S - |J|)(S - s).$$

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Hence

$$\left(\frac{\varepsilon}{2}\ell S-s\right)n-\ell S(S-s)\leq (S-s)|J|,$$

$$\frac{\varepsilon\ell S-2s}{2(S-s)}\cdot n-\ell S\leq |J|.$$

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This guarantees the induction step.

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For example the case of C_8, C_{12}, C_{14}, ..., $K_{4,4}$, $K_{4,5}$, ..., or the cube graph (the order of magnitude of $ext(n; T)$) is not known.

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We need to prove a mathematical statement: If $G \in \mathcal{G}_n$ is C_4-free graph them it cannot have too many edges.

On the other hand, we need to construct one $G \in \mathcal{G}_n$ that is C_4-free, and has many edges.

The mathematical theorem

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Theorem

Let $G \in \mathcal{G}_n$ be a C_4-free graph. Them

$$|E(G)| \leq \frac{1}{4} \cdot n\sqrt{4n-3} + \frac{1}{4} \cdot n.$$

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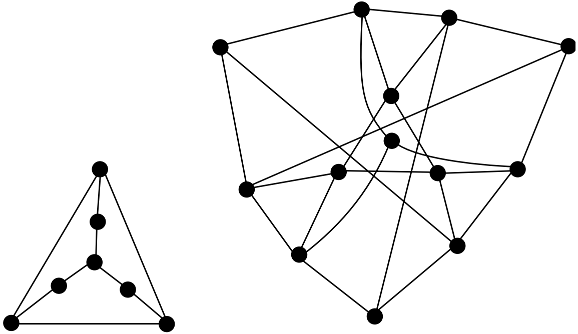

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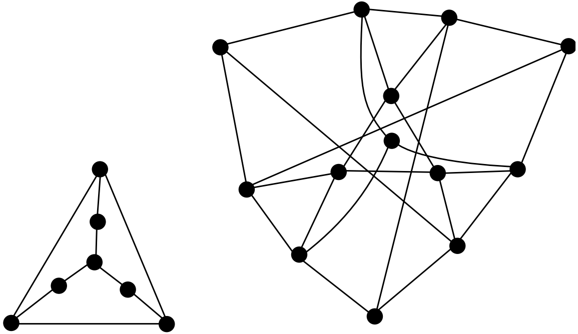

We just mention one important property for us: $|E| \sim \frac{1}{2}n^{3/2}$.

The summary of the results

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Theorem

$$ext(n, C_4) \sim \frac{1}{2} n^{\frac{3}{2}}.$$

This is the end!

Thank you for your attention!