Crossing number and its applications

Peter Hajnal

Bolyai Institute, SZTE, Szeged

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Introduction

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The crsooing number will be a non-negative integer that is 0 iff our graph is planar.

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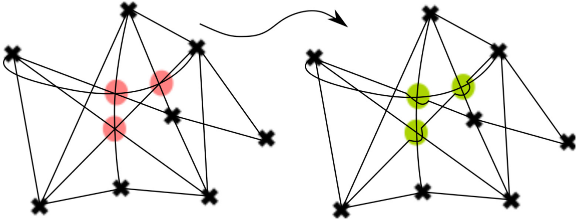

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Definition: Crossing number of a drawing

Let G be a graph, and λ be a regular drawing of it.

$$x(G,\lambda) = |\{P \in \mathbb{R}^2 - \lambda(V) : \ P \text{ is on more than one edge-curve}\}|.$$

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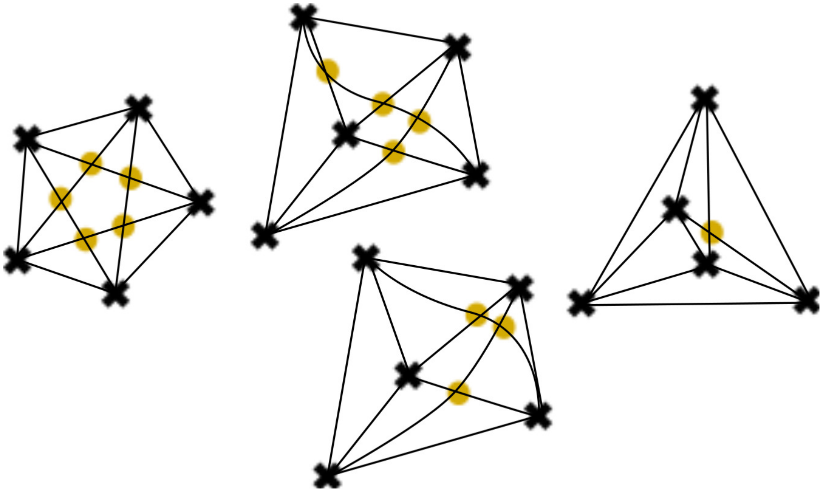
An alternative definition can be: Consider an arbitrary drawing. We count points P, that are an inner point of k edge-curves with multiplicity $\binom{k}{2}$.

Examples



Several drawings of $G = K_5$, and their crossing number: $x(K_5, \lambda) = 5$, $x(K_5, \lambda') = 4$, $x(K_5, \lambda''') = 3$, $x(K_5, \lambda'''') = 1$.

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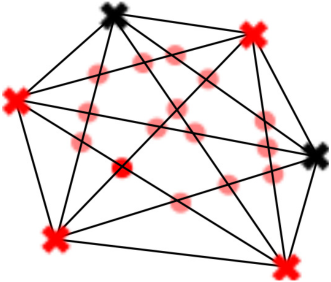

The case of K_6.

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Observartion

Let H be an arbitrary simple graph on n vertices, i.e. $H \subseteq K_n$. Then $x(H, \lambda|_H) \leq \binom{n}{4} = O(n^4)$, where λ is the drawing of the complete graph, we desribed above.

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Consider the drawing of G_0 around a vertex x. In a small enough neighborhood, the edge-curves meeting at x form a star shape. Between the branches there is „enough space" for the nice drawing of arbitrary number of loops.

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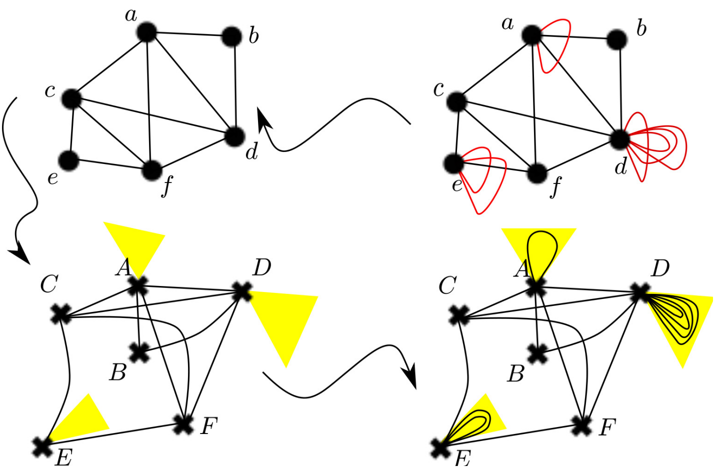

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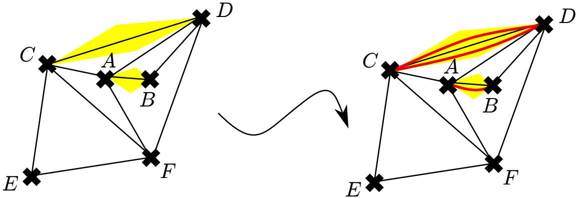
Observation

Given a nice drawing λ of G_0. We can extend it to $\widehat{\lambda}$, a nice drawing of G, i.e if $x(G_0, \lambda) = 0$ then $x(G, \widehat{\lambda}) = 0$.



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$x(K_5) = x(K_{3,3}) = 1.$

For any simple graph G on n vertices $x(G) = O(n^4)$.

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Although in both cases, the optimal drawings are conjectured, the conjecture is still a central open question.

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Indeed, from the neighbors of v we approach v. As soon as two edge-curves meet we redraw/switch, we avoid the crossing.

We obtain a drawing of the same graph. Meanwhile, the crossing number cannot increase.

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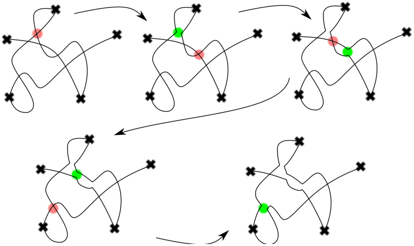

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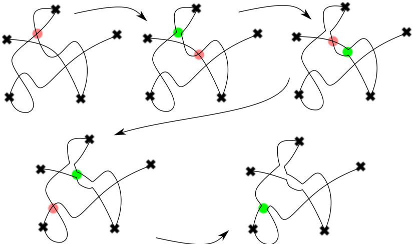

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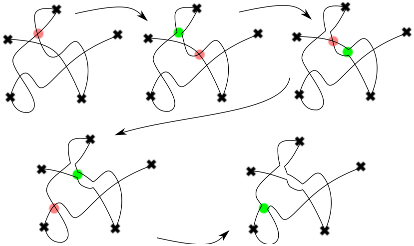

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Observation

For any drawing λ of a graph G one can find a V-nice drawing λ', that $x(G, \lambda') \leq x(G, \lambda)$.

Break

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Corollary

Let G be a simple graph and λ is its regular drawing. Then

$$x(G, \lambda) \geq |E| - 3|V|.$$

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Let R be a subgraph of G, that $V(G) = V(R)$ and $E(R)$ a maximal edge set with the property that $\lambda|_R$ is nice.

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We obtain, that

$$x(G, \lambda) \geq |E(G)| - 3|V|.$$

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The bound $|E| \geq 4|V|$ guarantees that G is not planar, i.e. $x(G) \geq 1$.

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$$x(K_n) \geq \frac{1}{64}\frac{\binom{n}{2}^3}{n^2} = \frac{1}{128}n^4 + O(n^3) = \Omega(n^4).$$

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The inequality holds for expected values too:

$$\mathbb{E}(x(\underline{R}, \lambda|_{\underline{R}})) \geq \mathbb{E}(|E(\underline{R})|) - 3\mathbb{E}(|V(\underline{R})|).$$

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From this:

$$p^4 x(G, \lambda) \geq p^2 |E(G)| - 3p|V(G)|.$$

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p will be positive, so we can divide the inequality by p^4:

$$x(G,\lambda) \geq \frac{|E(G)|}{p^2} - \frac{3|V(G)|}{p^3}.$$

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Let $p = \frac{4|V|}{|E|}$.

$$x(G,\lambda) \geq \frac{1}{16}\frac{|E|^3}{|V|^2} - \frac{3}{64}\frac{|E|^3}{|V|^2} = \frac{1}{64}\frac{|E|^3}{|V|^2}.$$

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The claim is proven.

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With more attention it can be improved, but the optimal value is not known.

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A geometric theorem

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Definition

Let $\mathcal{P} \subseteq \mathbb{R}^2$ a finite planar point set and $\mathcal{E}$ a finite set of lines on the plane.

$$I(\mathcal{P}, \mathcal{E}) = |\{(P, e) : P \in \mathcal{P},\ e \in \mathcal{E} \text{ and } P\, I\, e\}|,$$

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Theorem (Szemerédi—Trotter's theorem)

$$I(\mathcal{P}, \mathcal{E}) \leq 4(|\mathcal{P}||\mathcal{E}|)^{2/3} + 4|\mathcal{P}| + |\mathcal{E}|.$$

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$$\mathcal{O}(|P|^{2/3}|\mathcal{E}|^{2/3}+|\mathcal{P}|+|\mathcal{E}|)=\mathcal{O}(\max\{|\mathcal{P}|^{2/3}|\mathcal{E}|^{2/3},|\mathcal{P}|,|\mathcal{E}|\}).$$

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Note that for arbitrary p and e positive integers one can give a set of points $\mathcal{P}$ of size p and a set of lines $\mathcal{E}$ of size e such that number of incidences between them is at least a thousandth of the upper estimate.

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That is, the magnitude of the upper estimate is optimal.

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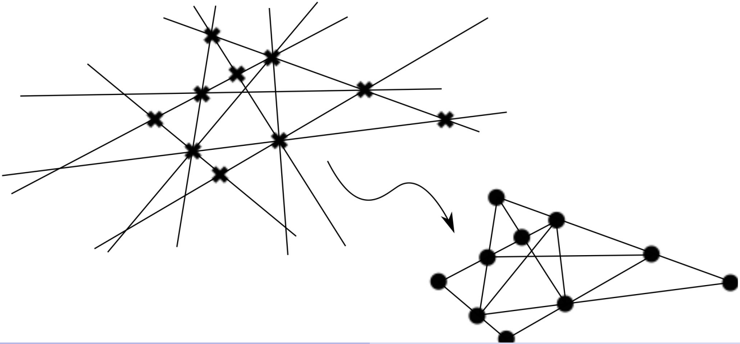

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From geometry it is obvious that $x(G) \leq x(G, \lambda) \leq \binom{|\mathcal{E}|}{2} \leq |\mathcal{E}|^2$.

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2nd C:ase $|E| \geq 4|V|$. Then the Crossing Lemma is applicable:

$$|\mathcal{E}|^2 \geq \binom{|\mathcal{E}|}{2} \geq x(G,p) \geq \frac{1}{64}\frac{(I(\mathcal{P},\mathcal{E}) - |\mathcal{E}|)^3}{|\mathcal{P}|^2}.$$

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2nd C:ase $|E| \geq 4|V|$. Then the Crossing Lemma is applicable:

$$|\mathcal{E}|^2 \geq \binom{|\mathcal{E}|}{2} \geq x(G,p) \geq \frac{1}{64}\frac{(I(\mathcal{P},\mathcal{E}) - |\mathcal{E}|)^3}{|\mathcal{P}|^2}.$$

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In both cases the theorem is proven.

Break

Basic problems

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Let $A, B \subset \mathbb{R}$ be finite set of numbers.
$A + B = \{a + b : \ a \in A \text{ és } b \in B\}$ and
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Question: How big and how small can be $|A + A|$ and $|A \cdot A|$, assuming $|A| = n$?

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We will give a lower bound on $|A+A|$. We can assume that the elements A are $a_1 < a_2 < \ldots < a_n$.

- Using $a_1 + a_1 < a_1 + a_2 < \ldots < a_1 + a_n < a_2 + a_n < \ldots < a_n + a_n$ we have at least $2n-1$ different values in $A+A$.
- If A contains n consequtive elements of an arithmetic progression, then $|A+A| = 2n-1$.

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Easy to give a linear lower bound on $|A \cdot A|$.

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Conjecture (Erdős—Szemeredi)

For every positive ε

$$\min_{A\subseteq\mathbb{R}, |A|=n} \max\{|A+A|, |A\cdot A|\} = \Omega(n^{2-\varepsilon}).$$

The conjecture is still open today.

Theorem of György Elekes

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Theorem (György Elekes)

For large enough n

$$\min_{A\subseteq\mathbb{R},|A|=n} \max\{|A+A|, |A\cdot A|\} \geq \frac{1}{10}n^{5/4},$$

i.e. for any n element set of numbers A we have $\max\{|A+A|, |A\cdot A|\} = \Omega(n^{5/4})$.

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$$\mathcal{E}_A = \{e_{a,a'} :\ y = \frac{1}{a} \cdot x + a',\ a, a' \in A\}.$$

Elekes' proof II

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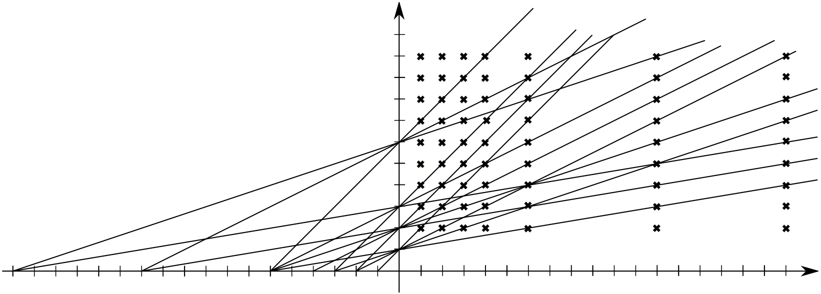

The points and lines in the case of $A = \{1, 2, 3, 6\}$

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- The equation of the line $e_{a,a'}$ is $\frac{1}{a} \cdot y - \frac{1}{a \cdot a'} \cdot x = 1$. It can be seen that the intersections with the axes and (a, a') determine each other. Hence $|\mathcal{E}_A| = |A|^2$.

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- The line $e_{a,a'}$ contains $(a \cdot a_1, a_1 + a')$, $(a \cdot a_2, a_2 + a'), \ldots$, where $A = \{a_1, a_2, \ldots\}$. We obtain that $I(\mathcal{P}_A, \mathcal{E}_A) \geq |A||\mathcal{E}_A| = |A|^3$.

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$$0{,}15\cdot n^{5/4} \le \sqrt{|A\cdot A||A+A|} \le \max\{|A\cdot A|, |A+A|\}.$$

This is the end!

Thank you for your attention!