Face coloring of plane graphs

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The 4-color-conjecture

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The puzzle was the so called 4-color problem.

Today it is a theorem: 4-color-theorem, or simply 4CT.

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(+) We assume that any two different edge-curves have finitely many common points. These are common endpoints or points where the two edge-curves transversally meet.

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Definition: Plane graphs

A plane graph is a pair (G, ρ), where G is a graph and ρ is a nice drawing it.

Faces

A nice drawing od a graph divide the plane into regions.

Definition: Faces of a nice drawing

We introduce a relation on the set of the point on the plane, that are not covered by edge-curves: $P \sim Q$, there is continuous curve connecting P and Q and not meeting any edge-curve.

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Theorem

Let G be a cycle-free graph and a nice drawing λ of it. In this case there is only one face

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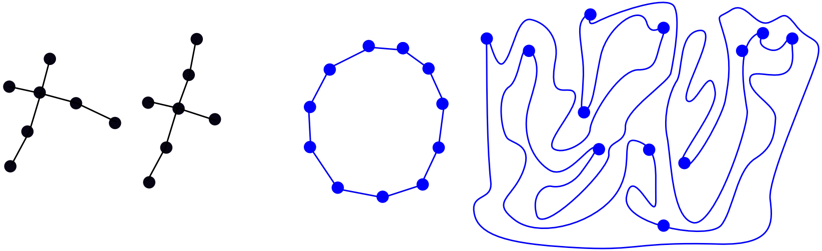
Without uniqueness the claim is the famous Jordan curve theorem.

We accept this Theorem, we don't prove it.

Example



On the left hand side the two drawings of the same tree are different (why?). On the right hand side we see two topologically equivalent drawings of the same cycle.

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The boundary of a face is the set of walks, we described above. The length of a boundary is the sum of the length of the walks, contained in the boundary.

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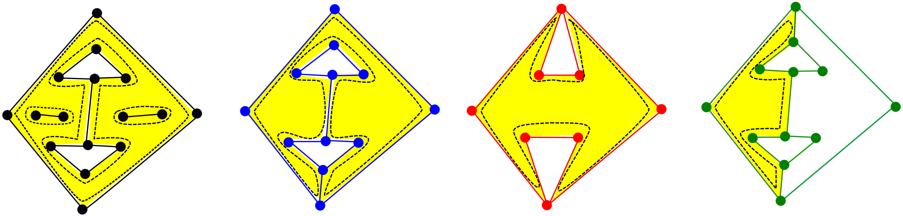
If G is connected and v is a cut-vertex, then it has a face, that its boundary travereses v more than once.
Moreover if G is connected and it has no cut-vertex, then the boundary of each face is a cycle.

Examples



The figure shows four drawings, each contains an emphasized yellow face.

Break

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Definition

Let (G^*, λ^*) be the plane graph we obtain after merging the two half-edges meeting at each border crossing to an edge-curve.

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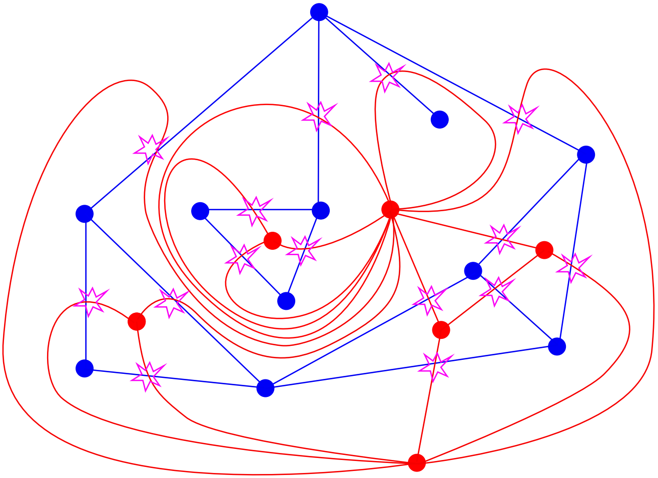

On the figure we see purple star that match the edges of the original graph and the edges of the dual graph.

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The length of the boundary of face τ is the same as the degree of the dual vertex τ^*.

Dictionary between G and G^*

ORIGINAL	DUAL
G plane graph	G^* plane graph
faces	vertices
edges	edges
two faces with common bordering edge	two adjacent vertices
face coloring	vertex coloring
proper face coloring (for any edge the two faces on the two sides of it get different colors)	proper vertex coloring
condition for proper face coloring: no edge	condition for proper vertex coloring: no loop

Dictionary between G and G^*

ORIGINAL	DUAL
vertices	faces
set of edges, that adjacent to a vertex	edges bounding a face
degree	length of the boundary
4-color-theorem ($4CT$): Faces of any 2-edge-connectred plane graph can be legally colored with 4 colors	4-color-theorem ($4CT$): Any looples planar graph can be legally vertex colored with 4 colors
We can assume: G 3-regular	We can assume: Each face is a triangle

4-color-theorem: Classical form

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4CT: face coloring version, 3-regular case

Let (G, λ) be a map, where G is 3-regular. G has a proper face coloring with 4 colors.

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4CT: triangulated vertex coloring version

Let (G, λ) be a loopless graph with a nice drawing If each face is a triangle, then $\chi(G) \leq 4$.

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We can find a proper 4-coloring of the faces of triangulated map.

Break

4CT as an edge coloring problem

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Theorem

The following two claims are equivalent:

(i) For any G, a 3-regular, 2-edge-connected planar graph $\chi_e(G) = 3$.

(ii) 4CT.

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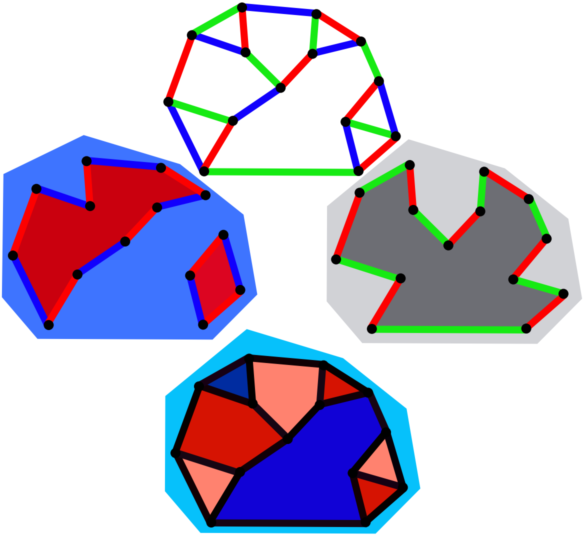
We can do the same for $M_1 + M_3$. The two colors can be chosen as dark/light.

Proof by picture



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This is a legal face coloring of the given plane graph.

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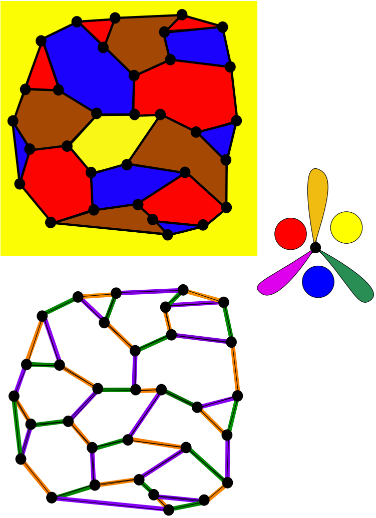

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First, they are disjoint.

Second, we claim that M_1, M_2, M_3 are matchings:

Finally $M_1 \cup M_2 \cup M_3 = E(G)$.

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Theorem

If G is a 3-regular 2-edge-connected planar graph, then

$$\chi_e(G) = 3.$$

The Petersen graph

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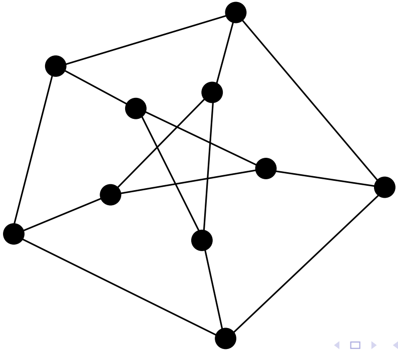

Thank you for your attention!