Vertex coloring of graphs

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Reminder

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In this unit we always assume that G denotes a SIMPLE graph.

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The chromatic number of G is

$$\chi(G) = \min\{k : G \text{ has a proper } k\text{-coloring}\}.$$

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A coloring is proper iff for any color the set of vertices with the given color form an independent set.

So a proper coloring of G can be interpreted as a partition of the vertex set into independent vertex sets.

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The Observation is obvious since any proper coloring must color the vertices of an arbitrary clique with different colors.

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Exhibiting a 4-clique is a transparent method. Unfortunately the method is NOT COMPLETE.

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(Op2) Identifying two non-adjacent vertices: Let (x, x') two non-adjacent vertices in G. Let $N(x)$ be denote the set of neighbors of x. The operation substitute the two vertices x and x' with one new vertex $[xx']$ with neighborhood $N(x) \cup N(x')$. $\widetilde{G}$ denotes the graph we obtained.

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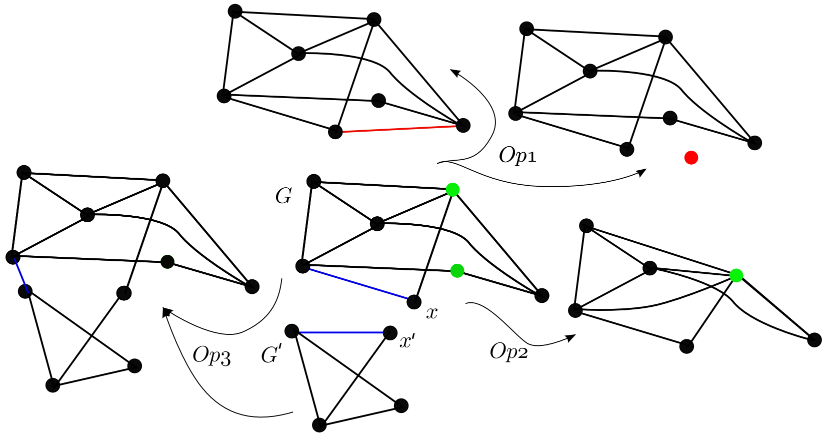
(Op3) Hajós operation: Let $e \in E(G), e' \in E(G')$, $\overrightarrow{e} = xy$, $\overrightarrow{e'} = x'y'$ be two edges. We produce the new graph H as follows $H = \text{Hajós}_{\overrightarrow{e}, \overrightarrow{e'}}(G, G')$, where
$V(H) = (V(G) - \{x\}) \dot\cup (V(G') - \{x'\}) \dot\cup \{[xx']\}$,
$E(H) = (E(G) - \{e\}) \dot\cup (E(G') - \{e'\}) \dot\cup \{yy'\}$, and incidence is the natural one.

Example



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For G^+ the Lemma is obvious. For $\widetilde{G}$ and Hajós(G, G') the Lemma is straight forward.

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Definition

The graph G is Hajós constructible from K_{k+1}'s iff there exists a sequence of graphs $G_1, G_2, \ldots G_l$ such that for each G_i is a K_{k+1}, or can be constructed from previous elements of our sequence using one of our operations.

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Definition

The graph G is Hajós constructible from K_{k+1}'s iff there exists a sequence of graphs $G_1, G_2, \ldots G_l$ such that for each G_i is a K_{k+1}, or can be constructed from previous elements of our sequence using one of our operations.

Corollary

If G is Hajós constructible, then G is non-k-colorable.

Example: 5-wheel is non-3-colorable

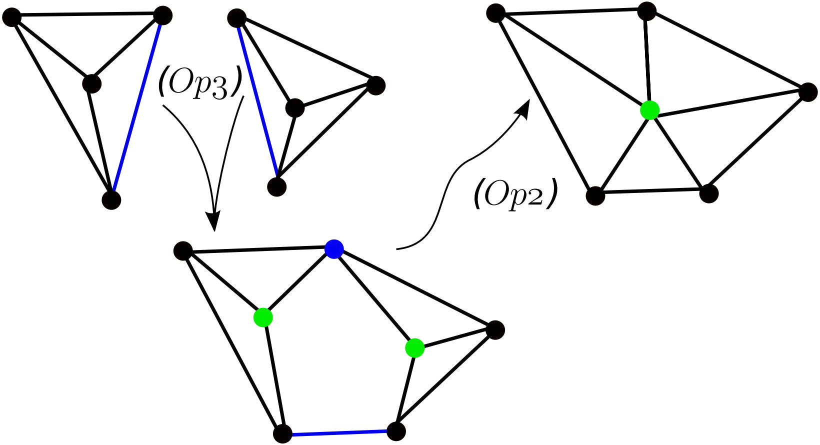

Figure: First we apply (Op3) for G_1 and G_2 (two K_4's'), G_3 is the result of the Hajós operation. We apply (Op2) on it and we obtain G_4. The result is a 5-wheel, a non-3-colorable graph.

The main Theorem

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We will prove the other direction by contradiction.

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- The saturation process preserves non-k-colorability. The key property of G^{satur} is that if we add any edge to it we must obtain a Hajós constructible graph.

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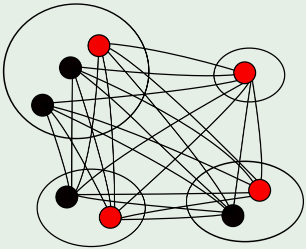

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The only way that can happen is that there are $x, y, z \in V(G^{satur})$ three different vertices, that $xy, xz \notin E(G^{satur})$, but $yz \in E(G^{satur})$.

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The saturation property of G^{satur} ensures the $G^{satur} + xy$ and $G^{satur} + xz$ are both are Hajós constructible.

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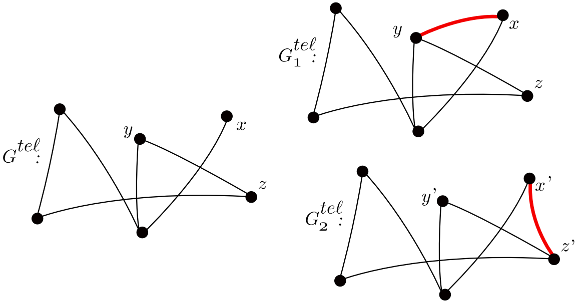

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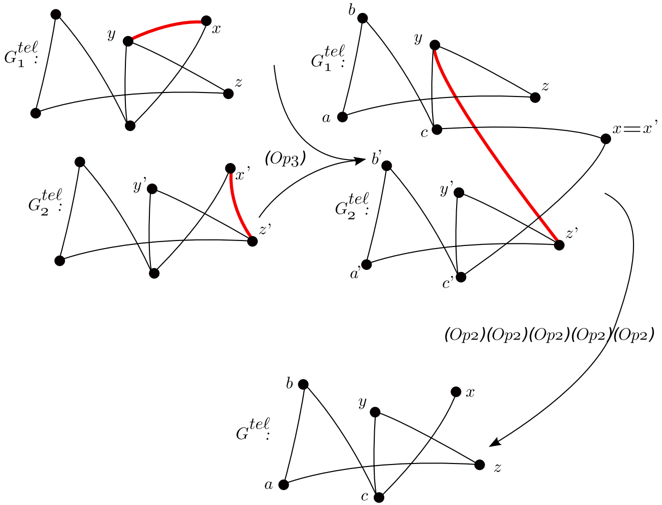

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(2) If $r \leq k$: G^{satur} is k-colorable, a contradiction.

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Theorem (BSc)

There exists a sequence of graphs $\{G_n\}$ such that $\omega(G_n) = 2$ (in other words G is triangle-free), furthermore $\chi(G_n) \to \infty$, assuming $n \to \infty$.

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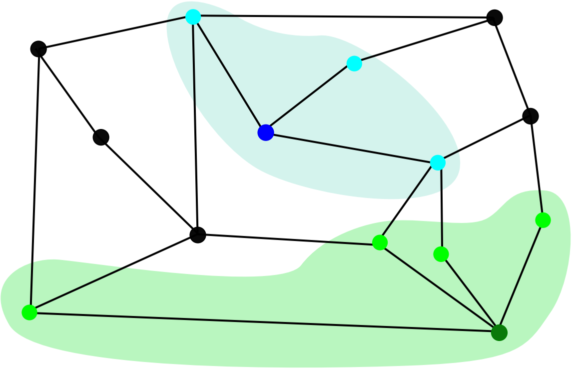

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In spite of the local simplicity, the global coloring problem can be hard.

We try to extend this example.

The extension: A ball

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Definition: A ball in a graph

Let G be an arbitrary graph, $o \in V(G)$, and $r \in \mathbb{N}^+$.

$$B(o, r) = G \mid_{\{v \in V : dist(o,v) \leq r\}},$$

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Definition

The girth of a graph G is

$$g(G) = \min\{\ell : \text{ there is cycle of length } \ell \text{ in } G\}.$$

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The proof will be a characteristic example for the probabilistic method.

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"If $\alpha(G)$ is small, then $\chi(G)$ is great". Or "assuming that color classes can't be small, we got to use many colors".

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In other words the event $\mathcal{A}_t$ is the complement of $\cup_{R\subseteq V, |R|=t}\mathcal{F}_R$.

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Using the obvious fact $\mathbb{P}(\cup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mathbb{P}(E_i)$ we obtain

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We can simplify the bound by the rude upper bound $\binom{n}{t} < n^t$ and the not so rude upper bound $1 - p < e^{-p}$ (p is positive and close to 0)

$$\mathbb{P}(\mathcal{A}_t) \geq 1 - n^t e^{-p\binom{t}{2}} = 1 - e^{t\log n} e^{-p\frac{t(t-1)}{2}} = 1 - e^{t\log n - p\frac{t(t-1)}{2}}.$$

The choice of t

We assume that $n/2t \geq \tau$ and the lower bound on the probability $\mathcal{A}_t$ is at least $2/3$.

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We have

$$\mathbb{E}\left(\xi_{\leq\gamma}\right) = \mathbb{E}\left(\sum_{\text{length of } C \leq \gamma} \xi_C\right) = \sum_{l=3}^{\gamma}\left(\sum_{\text{length of } C = l} \mathbb{E}\left(\xi_C\right)\right).$$

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Using the inequality

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we can give an upper bound on $\mathbb{E}(\xi_{\leq\gamma})$:

$$\mathbb{E}(\xi_{\leq\gamma}) \leq \sum_{\ell=3}^{\gamma} \frac{n^\ell}{6} p^\ell = \sum_{\ell=3}^{\gamma} \frac{n^\ell p^\ell}{6} \overset{(!)}{\leq} \sum_{\ell=3}^{\gamma} \frac{(np)^\gamma}{6} \leq \gamma\frac{(np)^\gamma}{6}.$$

Further promises

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Our choice of parameters are such that $\gamma(np)^{\gamma}/6 \leq n/6$ **is true**.

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Based on the promises and Markov's inequality we obtain

$$\mathbb{P}\left(\xi_{\leq\gamma} > \frac{n}{2}\right) < \mathbb{P}\left(\xi_{\leq\gamma} > 3\mathbb{E}\xi_{\leq\gamma}\right) < \frac{1}{3},$$

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After these we have

$$\mathbb{P}\left(\mathcal{A}_t \wedge \left(\xi_{\leq\gamma} \leq \frac{n}{2}\right)\right) > 0$$

since the two events connected by $\wedge$ have probability at least $\frac{2}{3}$.

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- Any independent set in G_0 has site at most t. Hence $\chi(G_0) \geq |V(G_0)|/t \geq \tau$.

G_0 proves the theorem.

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We have to fix the value p, t such a way that:

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and $n \geq 2t\tau$.

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A possible choice: Let p be satisfying $\gamma(np)^\gamma = n$. So the third promise "defines" p.

In this case $np = (n/\gamma)^{1/\gamma}$, i.e. the second assumption is automatically true (n is large enough).

$$p = \frac{1}{n} \cdot (n/\gamma)^{1/\gamma} = c(\gamma)n^{-(1-1/\gamma)},$$

where $c(\gamma)$ is a constant only depending on γ.

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$n \geq 2t\tau$ is obviously true.

This is the end!

Thank you for your attention!