Combinatorial matching algorithms

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2023 fall

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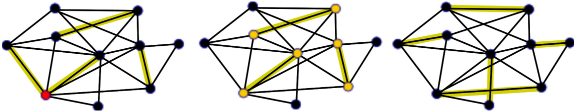

A graph with an edges set (yellow edges), which is not a matching (the red vertex is covered by two edges). In the middle one can see a matching, which is not a perfect matching (the set of yellow vertices in the set of matched vertices). On the left there is perfect matching.

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• $2\nu(G)$ is the maximum size of a vertex set, that can be matched. $|V(G)| - 2\nu(G)$ is the minimum number of unmatched vertices.

• There are several natural algorithmic questions about matchings in graphs:

Algorithmic matching problems: Given a graph G.

(1) Find an optimal matching M.

(2) Determine the value $\nu(G)$.

(3) Decide whether G has a perfect matching or not.

(4) Find a "large" matching M.

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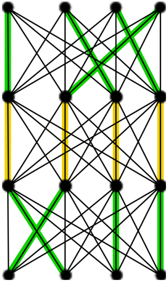
- We know that int the case of halting the output can't be augmented by extensions.

Example



Our graph has four levels of common size (let n be the size of the levels, in our example $n = 4$). Between two adjacent levels all possible edges are resent and there are no further edges. It is possible that the greedy algorithm first chooses the yellow edges, matching the two middle levels. Then it halts. The green edges form a perfect matching.

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Theorem

Let $\nu_{\text{greedy}}(G)$ denote the size of the output of the greedy algorithm. Then

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The second inequality is obvious since our algorithm computes a matching.

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Analysis: the proof

- Let M_{greedy} denote the output matching of the greedy algorithm.
- $L = V(M_{\text{greedy}})$ is the set of matched vertices.
- It is obvious that L is a covering vertex set, and $|L| = 2\nu_{\text{greedy}}(G)$.
- The size of L gives an upper bound on the size of an arbitrary matching, hence $\nu(G) \leq |L| = 2\nu_{\text{greedy}}(G)$.

Break

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• When halting the actual solution can't be augmented in certain way.

• We apply this scheme for finding optimum matching. Now on we are always given a graph G and a matching M.

The notion of augmenting path

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Definition

Let G be a graph and M a matching in it, $P : v_0, e_1, v_1, \ldots, e_k, v_k$ is a path of G. P is an augmenting path for M if v_0 and v_k unmatched, k is odd, furthermore the edges of M and $E(G) - M$ are alternating along P.

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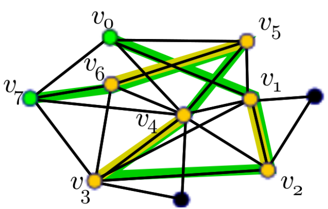

The yellow edges form a matching. The green path is an augmenting path for M. The right figure exhibits the augmented matching.

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Let P be an augmenting path for M. Then

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Let P be an augmenting path for M. Then

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- Note that the size of M' is larger than the size of M. The $M \leftarrow M'$ step is called augmentation of M.

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(Halting) We output the actual matching.
// In the case of halting we know that there is no augmenting path for M.

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- The major questions are:

(1) What one can say about the size of the output?

(2) How to find augmenting path for given matching M?

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Theorem of Berge (1957)

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„Corollary"

Furthermore we have a new problem: For input (G, M) we search for an augmenting path for matching M in the graph G.

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P is called a partial augmenting path, if $P: v_0, e_1, v_1, \ldots, e_k, v_k$ is a path, and $v_0 \notin V(M)$, furthermore $e_{2i} \in M$ $(i = 1, 2, \ldots)$.

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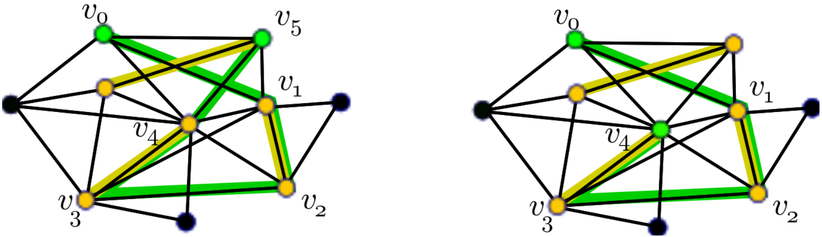

A partial augmenting path of length 5 (v_5 is matched). On the right hand side we have a partial augmenting path of length 4 (it has even length).

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(Halting) If we exit the above while loop without success then we say that the search is unsuccessful.
// In this case all neighbors of the vertices of O are labeled.

Search/alternating forest

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• Every vertex v of the search forest has a component, and the component has a root r. The unique rv path in the forest is the

A picture of a search forest

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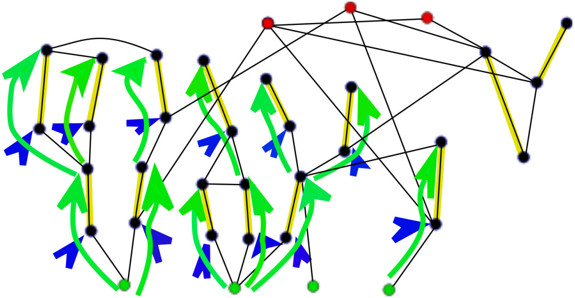

The yellow edges form our matching. The green vertices are the roots. The red vertices are unmatched, unlabeled (at the beginning) vertices.

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An important example

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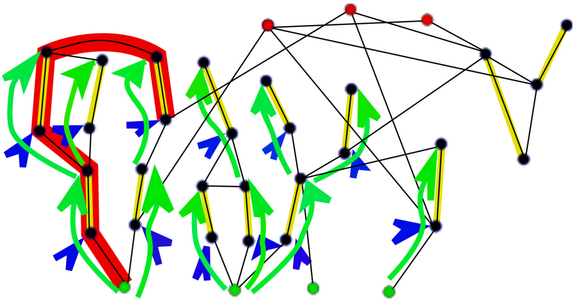

The red path is a partial augmenting path, that reaches its endvertex in a wrong phase.

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Let G be a bipartite graph with color classes L and U. Let M be a matching in G. Let $R = A \cap \overline{V}(M)$. $R = A \cap \overline{V}(M)$.

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Theorem (Harold Kuhn, Dénes Kőnig—Jenő Egerváry/Hungarian method)

Let G be a bipartite graph with color classes L and U. Let M be a matching in G. Let $R = A \cap \overline{V}(M)$. $R = A \cap \overline{V}(M)$.

If the greedy search for augmenting path is unsuccessful, then the matching is optimal (i.e. there is no augmenting path).

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First proof: Notations

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$\delta_L(P) = |L| - |P|$, i.e. the number of unmatched lower vertices.

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Let S be an arbitrary set of lower vertices ($S \subset L$). Let M be a matching. Then we have

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- The observation has mathematical content if $\epsilon(S) > 0$.

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Definition

S, a set of lower vertices is called Kőnig obstruction iff $\epsilon(S) > 0$.

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In the above case one says S lower-Kőnig-set, that proves the optimality of P.

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- So O is a lower-Kőnig-set that proves M optimality.

A mathematical corollary of the first proof

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Kőnig's theorem

$$\min\{\delta_L(P) : \ P \text{ is a matching}\} = \max\{\epsilon(S) : S \subset L\}.$$

Second proof: Initial notation

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Reminder

$C \subset V$ is a covering set of any edge has at least one endvertex in C.

Second proof: The main observation

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Let C be a covering set and M be a matching. If $|C| = |P|$, then P is an optimal matching in G, and C is an optimal covering set of G.

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In the case of sets C, P as above, one say that C proves that P is optimal. Or one can say that P proves that L is optimal.

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- Let U_{labeled} be the set of upper vertices of labeled edges of M. Let $L_{\text{unlabeled}}$ be the set of lower vertices of the unlabeled edges of M.

- Let $C = U_{\text{labeled}} \cup L_{\text{unlabeled}}$. Note that $|C| = |M|$.

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- The observation gives us that, C proves the optimality of M.

Mathematical proof of the second proof

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Kőnig's theorem

For a bipartite graph G

$$\nu(G) = \tau(G).$$

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Final remarks

- The Hungarian method gives an efficient algorithm to find an optimal matching in a bipartite graph.
- We will see that this algorithm can be extended to general graphs.
- As a byproduct we obtain an efficient algorithm that finds an optimal covering set in a given BIPARTITE graph.
- No one can extend this algorithm to the general case.

Break

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• Strange start. Success is impossible, there are no room to complete an augmenting path by the greedy approach.

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If $r = r'$, then (Case Edmonds) $\star$ // We will discuss it later.

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$\widetilde{M} = M \cap E(\widetilde{G})$.
$\widetilde{F}$: We have the "same" set of roots and we take all double outgrowth operations outside C_e.

Example

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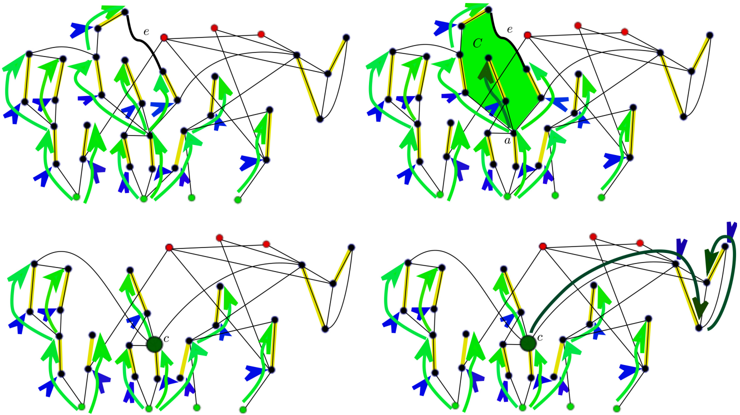

On the left the picture shows the search and the edge e. In the middle we can see the cycle, the vertex a_e. On the right we see the extension of labeling after the contraction.

Lemma

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(i) $\widetilde{M}$ is a matching in $\widetilde{G}$,

(ii) $\widetilde{F}$ is a search forest for $\widetilde{M}$,

(iii) c is an outer vertex in $\widetilde{F}$.

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(Contraction) We "build" the graph $\widetilde{G}$.
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(Iteration) We go back to the step (Label extension).

Where are we?

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• A "generic" run of our algorithm look like sequence of contractions

$$(G, M) = (G_0, M_0) \to (G_1, M_1) \to \ldots \to (G_\ell, M_\ell)$$

and a final successful or unsuccessful search.

What's next?

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A missing Theorem

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A missing Theorem

If the search in G_ℓ is unsuccessful, then the original matching M is optimal.

The end of the Edmonds' algorithm

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(Successful search in G_ℓ) We apply the Lemma ℓ times to construct an augmenting path for the original M.
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(Unsuccessful search in G_ℓ) We output: "There is no augmenting path for M".

The missing Theorem: Definitions

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We call this parameter of R the Berge-Tutte parameter.

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- They can be matched in G. They can be matched only to vertices in R.
- If the surplus, defined above β is positive, then it gives a lower bound on the number of unmatched vertices.
- In general, for arbitrary $R \subset V(G)$, and matching P

$$\beta(R) \leq \delta(P).$$

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Observation

For arbitrary $R \subset V(G)$, and arbitrary matching P we have

$$\beta(R) = \delta(P),$$

then P is an optimal matching. One says that R, P is a Berge pair, and R is a Berge proof for the optimality of P.

The missing Theorem: Reminder, notations

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- Note that there is a vertex in O_{i+1}, that is not present in O_i: the vertex representing the contracting cycle. On the other side the vertices of I_{i+1} are present in G_i.

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The Theorem gives us that M_i is optimal in G_i. The case $i = 0$ proves the missing Theorem and completes the proof of correctness of Edmonds' algorithm.

We need to prove a sequence of claims. We do induction and follow the order $i = \ell, \ell - 1, \ell - 2, \ldots, 2, 1, 0$.

The Theorem: The start of the induction

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- We discuss the case of $i = \ell$.

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- Specially $c_1(G_\ell - I_\ell) \geq |O_\ell|$, furthermore

$$\beta(I_\ell) = c_1(G_\ell - I_\ell) - |I_\ell| \geq |O_\ell| - |I_\ell| = \delta(M_\ell).$$

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- The number of vertices in the new component has the same parity as its ancestor. Specially $c_1(G_i - I_\ell) = c_1(G_{i+1} - I_\ell)$.

The Theorem: The induction step (cont'd)

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- To complete the proof we need to note that $\delta(M_i) = |V(G_i)| - 2|M_i|$ doesn't change during the algorithm.

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The above two theorems have two directions. They are not symmetric. One direction is easy, straight forward. The other direction is the essence of the mathematical content.

Petersen's Theorem

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Petersen's Theorem (1891)

If G is 3-regular 2-edge.connected graph then it has a perfect matching.

This is the end!

Thank you for your attention!