Enumeration of trees: Formulas of Cayley and Kirchoff

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The Idea

Consider two sets

$$\{(T, a, z) : T \text{ is a tree on the vertex set } [n], \text{ and } a, z \in [n]\}$$

and

$$\{f : [n] \to [n]\}.$$

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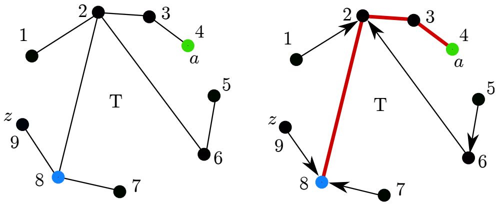
We are going to give a bijection between the two sets.

An example



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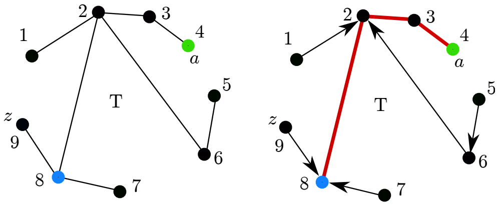

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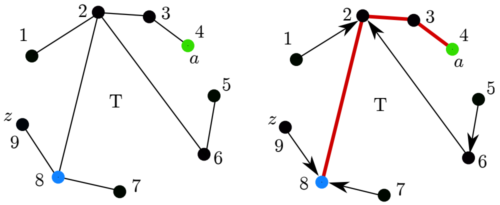

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The Joyal coding of the example is as follows:

(1) Take the unique a-z path in the tree (P). List the vertices of the path in two different ways. First, use the increasing order (in our example 2 3 4 8). Second, use the order as you walk from a to z (in our example 4 3 2 8).

An example (cont'd)

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Write down the two sequences in two lines and interpret it as a $\varphi : V(P) \to V(P)$ permutation.

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The complete Joyal code of our example is

i	1	2	3	4	5	6	7	8	9
$J_F(i)$	2	4	3	2	6	2	8	8	8

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The rest is left to the diligent audience: The Joyal coding of (T, a, z) is f. There are no other (T', a', z') triplet with f as its its Joyal code.

Break

Directed graphs

Definition

Directed graph is a (V, E, I, O) quadruple, where V is a finite vertex set and E is a finite edge set. $I, O \subset E \times V$ are two incidence relations with the property that for each edge there is exactly one edge incident to it.

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- From the directed graph $\overrightarrow{G} = (V, E, I, O)$ we can define the graph $G = (V, E, I \cup O)$. We say that we remove/erase the orientation of $\overrightarrow{G}$ and obtain G. Or $\overrightarrow{G}$ is an orientation of G.

Vertex-edge incidence matrix of directed graphs

Let $\overrightarrow{G}$ be a loopless directed graph.
$Inc_{\overrightarrow{G}} = (a_{ij}) \in \mathbb{R}^{n\times m} \equiv \mathbb{R}^{V\times E}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i I e_j, \\ -1, & \text{if } v_i O e_j, \\ 0, & \text{otherwise.} \end{cases}$$

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Note that the sum of the rows of $Inc_{\overrightarrow{G}}$ is $\overrightarrow{0} \in \mathbb{R}^E$. Hence the rank of $Inc_{\overrightarrow{G}}$ is at most $|V| - 1$

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$Inc_{\overrightarrow{G}}^{-r}[F]$ is a submatrix of $Inc_{\overrightarrow{G}}$, that we obtain by deleting the row of r and all the columns outside F.

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Lemma

Let G be a graph, $\overrightarrow{G}$ an arbitrary orientation of it, and r an arbitrary root. Let F be an edge set of size $|V|-1$ (i.e. $Inc_{\overrightarrow{G}}^{-r}[F]$ is an $(n-1)\times(n-1)$ matrix).

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Then the following properties are equivalent:

(1) $Inc^{-r}_{\overrightarrow{G}}[F]$ has full rank $(n-1)$, i.e. its rows are independent, also its columns are independent,

(2) F doesn't contain an edge set of a cycle,

(3) F is an edge set of spanning tree,

(4) $\det Inc^{-r}_{\overrightarrow{G}}[F]\in\{\pm1\}$.

$(1)\Rightarrow(2)$

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Second, consider the general case. Note that the corresponding matrix can be obtained from the matirx of the first case, by multiplying some of the columns by -1.

(2)⇒(3)

A pure graph theoretical statement. See in recitation session.

(3)⇒(4)

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Note that the matrix, where the row and column order follows the indices is a triangular matrix with ± 1's in the diagonal. The claim follows by linear algebra.

(4)⇒(1)

$(4) \Rightarrow (1)$

An obvious statement from linear algebra.

Binet-Cauchy formula

Theorem (Binet-Cauchy formula)

Let $A, B \in \mathbb{R}^{n\times m}$. Then

$$\det(A \cdot B^T)_{n\times n} = \sum_{\substack{F\subset\{1,\ldots,m\}=\text{'columns'}\\ |F|=n}} \det A[F] \cdot \det B[F].$$

The Theorem of Kirchoff

Corollary (Theorem of Kirchoff)

Let (G, r) be an arbitrary rooted graph, and let $\overrightarrow{G}$ be and arbitrary orientation of G. Then the number of spanning tree of G is

$$\det[Inc^{-r}_{\overrightarrow{G}} \cdot (Inc^{-r}_{\overrightarrow{G}})^T].$$

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Apply the Binet-Cauchy formula for $A = B = Inc_{\overrightarrow{G}}^{-r}$:

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The Theorem is proved.

Another view of $Inc_{\vec{G}} \cdot Inc_{\vec{G}}^T$

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Observation

$$(Inc_{\vec{G}} Inc_{\vec{G}}^T)_{uv} = \begin{cases} -(\sharp \text{ of edges between } u \text{ and } v) & \text{if } u \neq v, \\ d(u) & \text{if } u = v. \end{cases}$$

Laplace matrix of a graph

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Observation

$$A^{-r}_{\vec{G}}(A^{-r}_{\vec{G}})^T = L^{-r}_G$$

The Theorem of Kirchoff, the second form

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The number of spanning trees of a graph G is

$$\det L_G^{-r} = \det(D_G^{-r} - A_G^{-r}).$$

Cayley's Theorem from Kirchff's Theorem

Apply Kirchoff's Theorem for K_n

$\sharp$ of spanning trees of $K_n =$

$$\det \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & -1 & \dots & -1 & -1 \\ -1 & -1 & n-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & n-1 & -1 \\ -1 & -1 & -1 & \dots & -1 & n-1 \end{bmatrix}_{(n-1)\times(n-1)}$$

This is the end!

Thank you for your attention!