Degree sequences

Peter Hajnal, MSc – Discrete Mathematics

Bolyai Institute, University of Szeged, Hungary

Fall 2023

Introduction

Introduction

This lecture is the continuation of the BSC course, called Combinatorics.

Introduction

This lecture is the continuation of the BSC course, called Combinatorics.

Let us recall some important concepts of graph theory.

Introduction

This lecture is the continuation of the BSC course, called Combinatorics.

Let us recall some important concepts of graph theory.

Definition: Graph

Let $G = (V, E, I)$, where V and E are arbitrary finite sets, $I \subseteq V \times E$ incidence relation. The set V is called vertex set, E is called edge set. If $(v, e) \in I$ we say that the vertex v is an endvertex of edge e. G is called graph iff any edge has two endvertices. We mention that the two endvertices of e might coincide.

Introduction (cont'd)

Introduction (cont'd)

Definition: Simple graph

Edges with two identical endvertices are called *loops*. If e_1 and e_2 are such that they incident to the same pair of vertices, then we call them *parallel edges*. Graphs that do not contain loops and parallel edges are called *simple graphs*.

Introduction (cont'd)

Definition: Simple graph

Edges with two identical endvertices are called *loops*. If e_1 and e_2 are such that they incident to the same pair of vertices, then we call them *parallel edges*. Graphs that do not contain loops and parallel edges are called *simple graphs*.

Definition: Degree of a vertex

The degree of a vertex v is the number of edges incident to v, where the loops incident to v are counted twice.

Degree sequences

Degree sequences

Definition: Degree sequence

The sequence of natural numbers $d_1, \ldots, d_n$ is called *degree sequence* if it is the sequence of degrees of a graph G, and it is. a non-increasing sequence.

Degree sequences

Definition: Degree sequence

The sequence of natural numbers $d_1, \ldots, d_n$ is called *degree sequence* if it is the sequence of degrees of a graph G, and it is. a non-increasing sequence.
Especially $n = |V|$, $d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq d_n$ is satisfied.

Degree sequences

Definition: Degree sequence

The sequence of natural numbers $d_1, \ldots, d_n$ is called *degree sequence* if it is the sequence of degrees of a graph G, and it is. a non-increasing sequence.
Especially $n = |V|$, $d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq d_n$ is satisfied.

- Alternatively, we also write $d_1 = d_{max}$, and $d_n = d_{min}$.

Degree sequences

Definition: Degree sequence

The sequence of natural numbers $d_1, \dots, d_n$ is called *degree sequence* if it is the sequence of degrees of a graph G, and it is. a non-increasing sequence.
Especially $n = |V|$, $d_1 \geq d_2 \geq \dots \geq d_{n-1} \geq d_n$ is satisfied.

- Alternatively, we also write $d_1 = d_{max}$, and $d_n = d_{min}$.
- Note that from the degree sequence of a graph G we can calculate the number of edges:

$$2|E| = \sum_{i=1}^{n} d_i.$$

Degree sequences

Definition: Degree sequence

The sequence of natural numbers $d_1, \ldots, d_n$ is called *degree sequence* if it is the sequence of degrees of a graph G, and it is. a non-increasing sequence.
Especially $n = |V|$, $d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq d_n$ is satisfied.

- Alternatively, we also write $d_1 = d_{max}$, and $d_n = d_{min}$.
- Note that from the degree sequence of a graph G we can calculate the number of edges:

$$2|E| = \sum_{i=1}^{n} d_i.$$

- In other words

$$d_{\text{average}} = \sum_{i=1}^{n} d_i/n = 2|E|/n.$$

The basic problem

The basic problem

The basic problem

Given a non-increasing sequence of natural numbers $\langle d_i \rangle_{i=1}^n$. Is there a graph G that the sequence of degrees is the initial sequence?

The basic problem

The basic problem

Given a non-increasing sequence of natural numbers $\langle d_i \rangle_{i=1}^n$. Is there a graph G that the sequence of degrees is the initial sequence?

- If so, then in that case we say that the sequence is realizable, and G is a realizing graph.

The basic problem

The basic problem

Given a non-increasing sequence of natural numbers $\langle d_i \rangle_{i=1}^n$. Is there a graph G that the sequence of degrees is the initial sequence?

- If so, then in that case we say that the sequence is realizable, and G is a realizing graph.
- Of course, the elements of a realizable sequence are always natural numbers.

A remark

A remark

If there is no assumption on G then the basic problem can be answered easily.

A remark

If there is no assumption on G then the basic problem can be answered easily.

Claim

The sequence of natural numbers $\langle d_i \rangle_{i=1}^n$ can be realized if and only if (iff) $\sum_{i=1}^n d_i$ is even.

A remark

If there is no assumption on G then the basic problem can be answered easily.

Claim

The sequence of natural numbers $\langle d_i \rangle_{i=1}^{n}$ can be realized if and only if (iff) $\sum_{i=1}^{n} d_i$ is even.

The simple proof (see recitation session) makes use of the possibility of loops.

Further problems

Further problems

- A natural question is: "Can we realize a given sequence without loops?".

Further problems

- A natural question is: "Can we realize a given sequence without loops?".

- Or in general: When a given sequence of natural numbers can be realized with a graph with some special property?

Further problems

- A natural question is: "Can we realize a given sequence without loops?".

- Or in general: When a given sequence of natural numbers can be realized with a graph with some special property?

- In the following, we will consider such questions.

Further problems

- A natural question is: "Can we realize a given sequence without loops?".

- Or in general: When a given sequence of natural numbers can be realized with a graph with some special property?

- In the following, we will consider such questions.

- If the answer is yes, then we can extend our problem:

Further problems

• A natural question is: "Can we realize a given sequence without loops?".

• Or in general: When a given sequence of natural numbers can be realized with a graph with some special property?

• In the following, we will consider such questions.

• If the answer is yes, then we can extend our problem:

(1) We can ask for a realizing graph.

Further problems

• A natural question is: "Can we realize a given sequence without loops?".

• Or in general: When a given sequence of natural numbers can be realized with a graph with some special property?

• In the following, we will consider such questions.

• If the answer is yes, then we can extend our problem:

(1) We can ask for a realizing graph.

(2) We can ask for the list of all realizing graphs.

Realization with loopless graphs

Realization with loopless graphs

Theorem

$\langle d_i \rangle_{i=1}^n$ descending sequence of natural numbers is realizable with a graph without loops if and only if

- $\sum_{i=1}^n d_i$ is even, and
- $d_1 = d_{\max} \leq d_2 + d_3 \cdots + d_n$.

Realization with loopless graphs, the proof

Realization with loopless graphs, the proof

- First, suppose that $\langle d_i\rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.

Realization with loopless graphs, the proof

• First, suppose that $\langle d_i \rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.

• To see that Condition *2.* is necessary consider the realizing graph for the sequence. Let v_i be the vertex with degree d_i. How many edge are between $\{v_i\}$ and $V \setminus \{v_i\}$?

Realization with loopless graphs, the proof

- First, suppose that $\langle d_i \rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.
- To see that Condition *2.* is necessary consider the realizing graph for the sequence. Let v_i be the vertex with degree d_i. How many edge are between $\{v_i\}$ and $V \setminus \{v_i\}$?
- We can't have loops, so the answer is d_i.

Realization with loopless graphs, the proof

- First, suppose that $\langle d_i \rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.
- To see that Condition *2.* is necessary consider the realizing graph for the sequence. Let v_i be the vertex with degree d_i. How many edge are between $\{v_i\}$ and $V \setminus \{v_i\}$?
- We can't have loops, so the answer is d_i.
- On the other hand, it is obvious the result of the enumeration is at most $d_1 + \cdots + d_{i-1} + d_{i+1} + \ldots + d_n$.

Realization with loopless graphs, the proof

- First, suppose that $\langle d_i \rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.
- To see that Condition *2.* is necessary consider the realizing graph for the sequence. Let v_i be the vertex with degree d_i. How many edge are between $\{v_i\}$ and $V \setminus \{v_i\}$?
- We can't have loops, so the answer is d_i.
- On the other hand, it is obvious the result of the enumeration is at most $d_1 + \cdots + d_{i-1} + d_{i+1} + \ldots + d_n$.
- The two answers must be consistent. So we got n conditions: each element of our degree sequence is not more than the sum of the other degrees.

Realization with loopless graphs, the proof

- First, suppose that $\langle d_i \rangle_{i=1}^n$ is realizable without loops. Then we know that the condition *1.* is satisfied.
- To see that Condition *2.* is necessary consider the realizing graph for the sequence. Let v_i be the vertex with degree d_i. How many edge are between $\{v_i\}$ and $V \setminus \{v_i\}$?
- We can't have loops, so the answer is d_i.
- On the other hand, it is obvious the result of the enumeration is at most $d_1 + \dots + d_{i-1} + d_{i+1} + \ldots + d_n$.
- The two answers must be consistent. So we got n conditions: each element of our degree sequence is not more than the sum of the other degrees.
- Of these, only one inequality is not obvious. This is exactly the condition *2.*

Realization without loops, proof (cont'd)

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.
- If $n = 1$, then $d_1 = 0$ from assumption *2*. The realization is obvious.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.
- If $n = 1$, then $d_1 = 0$ from assumption *2*. The realization is obvious.
- If $n = 2$, then from Condition *2* we have that $d_1 = d_2$. So d_1 parallel edges between the two vertices realizes the sequence.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.
- If $n = 1$, then $d_1 = 0$ from assumption *2*. The realization is obvious.
- If $n = 2$, then from Condition *2* we have that $d_1 = d_2$. So d_1 parallel edges between the two vertices realizes the sequence.
- We assume that that $n \geq 3$ and $d_1 \geq 2$ (also $d_2 \geq 1$).

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.
- If $n = 1$, then $d_1 = 0$ from assumption *2*. The realization is obvious.
- If $n = 2$, then from Condition *2* we have that $d_1 = d_2$. So d_1 parallel edges between the two vertices realizes the sequence.
- We assume that that $n \geq 3$ and $d_1 \geq 2$ (also $d_2 \geq 1$).
- In the case $\sum_{i=1}^{n} d_i = 0$ the sequence can be realized by n node graph with no edge.

Realization without loops, proof (cont'd)

- The other direction is proven with mathematical induction according to $\sum_{i=1}^{n} d_i$. First we look at some easy cases.
- If $d_1 \leq 1$, then the statement is obvious.
- If $n = 1$, then $d_1 = 0$ from assumption *2*. The realization is obvious.
- If $n = 2$, then from Condition *2* we have that $d_1 = d_2$. So d_1 parallel edges between the two vertices realizes the sequence.
- We assume that that $n \geq 3$ and $d_1 \geq 2$ (also $d_2 \geq 1$).
- In the case $\sum_{i=1}^{n} d_i = 0$ the sequence can be realized by n node graph with no edge.
- Be $m := \sum_{i=1}^{n} d_i$. Suppose that in the case $\sum_{i=1}^{n} d_i < m$ our two conditions guarantee the realization without loops.

Realization without loops, the proof (cont'd)

Realization without loops, the proof (cont'd)

- Consider the sequence

$$d_1 - 1, d_2 - 1, d_3, \ldots, d_n.$$

Realization without loops, the proof (cont'd)

- Consider the sequence

$$d_1 - 1, d_2 - 1, d_3, \dots, d_n.$$

- This is not necessarily non-increasing. Its maximum element is $d_1 - 1$ or d_3 (when $d_1 = d_2 = d_3$). In both cases, our two conditions are satisfied.

Realization without loops, the proof (cont'd)

- Consider the sequence

$$d_1 - 1, d_2 - 1, d_3, \ldots, d_n.$$

- This is not necessarily non-increasing. Its maximum element is $d_1 - 1$ or d_3 (when $d_1 = d_2 = d_3$). In both cases, our two conditions are satisfied.

- The induction hypothesis can be applied: the new series is realizable with a loopless graph.

Realization without loops, the proof (cont'd)

- Consider the sequence

$$d_1 - 1, d_2 - 1, d_3, \ldots, d_n.$$

- This is not necessarily non-increasing. Its maximum element is $d_1 - 1$ or d_3 (when $d_1 = d_2 = d_3$). In both cases, our two conditions are satisfied.

- The induction hypothesis can be applied: the new series is realizable with a loopless graph.

- Its two different vertices have degree $d_1 - 1$, and $d_2 - 1$. Connecting them with an extra edge we get a graph that gives the realization of the original sequence.

Realization without loops, final remark

From the induction proof one can easily construct an algorithm that — from a given sequence, satisfying the two conditions — constructs a realizing graph that doesn't contain a loop.

Break

Realization with simple graphs

Realization with simple graphs

Lemma

If $\langle d_i \rangle_{i=1}^n$ is decreasing a sequence of natural numbers can be realized with a simple graph, then there is a realizing simple graph whose vertices are $v_1, \dots, v_n$, where $d_i = d(v_i)$, and the neighbors of the vertex v_1 are exactly the vertices $v_2, v_3, \dots, v_{d_1+1}$. (Note that $d_1 \leq n-1$ is guaranteed by the realizing graph of the original sequence).

A consequence of the Lemma

A consequence of the Lemma

Theorem, V. Havel (1955) and S. Hakimi (1962)

$\langle d_i \rangle_{i=1}^n$ can be realized with a simple graph if and only if the sequence

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is realizable too.

A consequence of the Lemma

Theorem, V. Havel (1955) and S. Hakimi (1962)

$\langle d_i\rangle_{i=1}^n$ can be realized with a simple graph if and only if the sequence

$$d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n$$

is realizable too.

- One direction is obvious.

A consequence of the Lemma

Theorem, V. Havel (1955) and S. Hakimi (1962)

$\langle d_i \rangle_{i=1}^n$ can be realized with a simple graph if and only if the sequence

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is realizable too.

- One direction is obvious.

- The other direction is an easy consequence of the former lemma? We take the graph realizing $\langle d_i \rangle_{i=1}^n$, in which v_1 is connected to the vertices with the smallest index. Then we delete the vertex v_1 to obtain a graph that realizes the required degree sequence.

The proof of the Lemma

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.
- Assume that in G the neighborhood of v_1 is not $\{v_2, \ldots, v_{d_1+1}\}$.

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.

- Assume that in G the neighborhood of v_1 is not $\{v_2, \ldots, v_{d_1+1}\}$. This means that there exists $i < j$ such that v_1 is connected to v_j, but not connected to v_i.

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.

- Assume that in G the neighborhood of v_1 is not $\{v_2, \dots, v_{d_1+1}\}$. This means that there exists $i < j$ such that v_1 is connected to v_j, but not connected to v_i.

- The neighborhood of v_j consists of v_1 and $d_j - 1$ further vertices. v_i has d_i neighbors (v_1 is not among them).

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.

- Assume that in G the neighborhood of v_1 is not $\{v_2, \ldots, v_{d_1+1}\}$. This means that there exists $i < j$ such that v_1 is connected to v_j, but not connected to v_i.

- The neighborhood of v_j consists of v_1 and $d_j - 1$ further vertices. v_i has d_i neighbors (v_1 is not among them).

- Due to the descending order, we have $d_j - 1 \leq d_i - 1 < d_i$. This implies that there is a vertex $x \neq v_1$ which is adjacent to v_j but not connected to v_i.

The proof of the Lemma

- Let G be a realizing graph for which $d(v_i) = d_i$ and the index sum of v_1's neighbors is minimal. We claim that this graph proves the lemma.

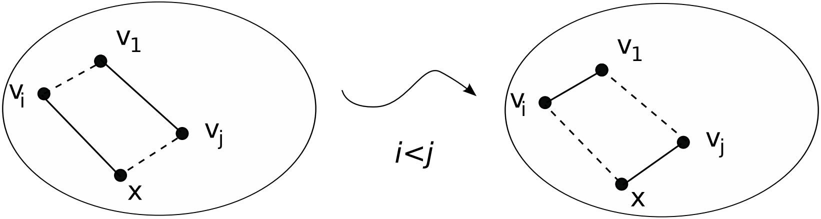
- Assume that in G the neighborhood of v_1 is not $\{v_2, \ldots, v_{d_1+1}\}$. This means that there exists $i < j$ such that v_1 is connected to v_j, but not connected to v_i.

- The neighborhood of v_j consists of v_1 and $d_j - 1$ further vertices. v_i has d_i neighbors (v_1 is not among them).

- Due to the descending order, we have $d_j - 1 \leq d_i - 1 < d_i$. This implies that there is a vertex $x \neq v_1$ which is adjacent to v_j but not connected to v_i.

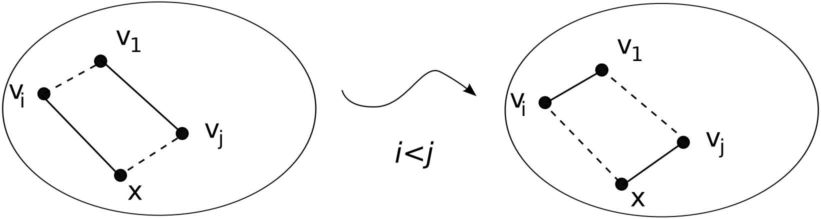
- Let $\widetilde{G}$ be the graph that we obtain from G by deleting edges $v_j v_1$ and $v_i x$ and adding edges $v_j x$ and $v_i v_1$.

The proof of the Lemma (cont'd)



Dashed lines indicate LACK of edges. This is the information guarantees that we get a simple graph after the "switch".

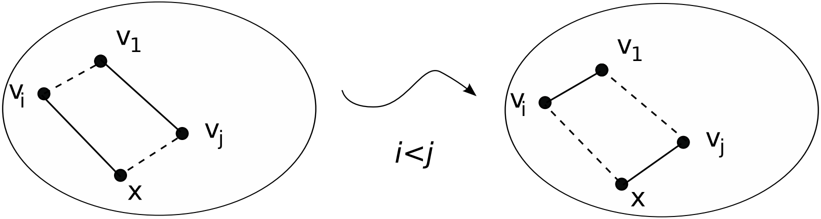
The proof of the Lemma (cont'd)



Dashed lines indicate LACK of edges. This is the information guarantees that we get a simple graph after the "switch".

- Thus, the graph remained simple and its degree sequence did not change, however in $\widetilde{G}$ the index sum of the neighbors of v_1 is decreased.

The proof of the Lemma (cont'd)



Dashed lines indicate LACK of edges. This is the information guarantees that we get a simple graph after the "switch".

- Thus, the graph remained simple and its degree sequence did not change, however in $\widetilde{G}$ the index sum of the neighbors of v_1 is decreased.
- This contradicts the choice of G.

One final remark

One final remark

- Notice that using the Lemma one can get a recursive algorithm (Havel-Hakimi algorithm) to decide whether there is a realization by simple graphs.

One final remark

- Notice that using the Lemma one can get a recursive algorithm (Havel-Hakimi algorithm) to decide whether there is a realization by simple graphs.

- In the case of affirmative answer the algorithm construct a realizing simple graph.

One final remark

- Notice that using the Lemma one can get a recursive algorithm (Havel-Hakimi algorithm) to decide whether there is a realization by simple graphs.

- In the case of affirmative answer the algorithm construct a realizing simple graph.

- Listing all realizing simple graphs is a non-trivial problem.

Break

Trees

Trees

Trees

Tree, branching operation.

Trees

Trees

Tree, branching operation.

Consequence (Basic theorem on the number of edges of a tree)

For each tree we have

$$|E| = |V| - 1.$$

Trees

Trees

Tree, branching operation.

Consequence (Basic theorem on the number of edges of a tree)

For each tree we have

$$|E| = |V| - 1.$$

In other words: Any tree on n vertices has $n - 1$ edges.

Trees

Trees

Tree, branching operation.

Consequence (Basic theorem on the number of edges of a tree)

For each tree we have

$$|E| = |V| - 1.$$

In other words: Any tree on n vertices has $n-1$ edges.

A straight forward claim

If a tree has at least two vertices, then it can't have an isolated node.

Sufficient and necessary conditions

Sufficient and necessary conditions

Theorem

Assume that $n \geq 2$. The sequence $\langle d_i \rangle_{i=1}^n$ can be realized by a tree if and only if $\sum_{i=1}^n d_i = 2n - 2$ and $d_{min} > 0$.

The proof

The proof

- We saw that the conditions are necessary.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.
- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.

- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.

- Assume that $n \geq 3$, and we have the claim for shorter sequences than n.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.

- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.

- Assume that $n \geq 3$, and we have the claim for shorter sequences than n.

- A simple arithmetic gives $1 < d_{\text{average}} = \frac{\sum_{i=1}^{n} d_i}{n} < 2$. Hence $d_1 = d_{max} \geq d_{\text{average}} \geq d_{min} = d_n$.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.

- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.

- Assume that $n \geq 3$, and we have the claim for shorter sequences than n.

- A simple arithmetic gives $1 < d_{\text{average}} = \frac{\sum_{i=1}^{n} d_i}{n} < 2$. Hence $d_1 = d_{max} \geq d_{\text{average}} \geq d_{min} = d_n$.

- We have $d_n = 1$, and $d_1 \geq 2$.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.

- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.

- Assume that $n \geq 3$, and we have the claim for shorter sequences than n.

- A simple arithmetic gives $1 < d_{\text{average}} = \frac{\sum_{i=1}^{n} d_i}{n} < 2$. Hence $d_1 = d_{max} \geq d_{\text{average}} \geq d_{min} = d_n$.

- We have $d_n = 1$, and $d_1 \geq 2$.

- The $d_1 - 1, d_2, \ldots, d_{n-1}$ sequence also satisfies the conditions. By the induction hypothesis it can be realized by a tree.

The proof

- We saw that the conditions are necessary.
- We use induction to prove that the conditions are sufficient.
- If $n = 2$ then the conditions imply that $d_1 = d_2 = 1$. The realization by tree is straight forward and unique.
- Assume that $n \geq 3$, and we have the claim for shorter sequences than n.
- A simple arithmetic gives $1 < d_{\text{average}} = \frac{\sum_{i=1}^{n} d_i}{n} < 2$. Hence $d_1 = d_{max} \geq d_{\text{average}} \geq d_{min} = d_n$.
- We have $d_n = 1$, and $d_1 \geq 2$.
- The $d_1 - 1, d_2, \ldots, d_{n-1}$ sequence also satisfies the conditions. By the induction hypothesis it can be realized by a tree.
- From this realizing tree we can obtain a new one by branching, that realizes the original sequence,

Realizations by trees: an algorithm

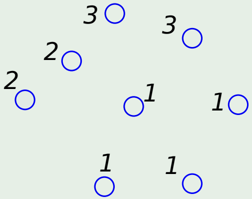
Realizations by trees: an algorithm

- Again based on the proof one can design an algorithm that from an input sequence, satisfying the conditions, constructs a realizing tree.

Realizations by trees: an algorithm

- Again based on the proof one can design an algorithm that from an input sequence, satisfying the conditions, constructs a realizing tree.

Example



Realizing by trees: Finding all realizations

Realizing by trees: Finding all realizations

- To list all realizing trees we choose an arbitrary vertex of degree 1. Let us refer to our choice as vertex u.

Realizing by trees: Finding all realizations

• To list all realizing trees we choose an arbitrary vertex of degree 1. Let us refer to our choice as vertex u.

• The possible neighbors of it (v) are the vertices with degree at least 2 ($|V| \geq 3$).

Realizing by trees: Finding all realizations

• To list all realizing trees we choose an arbitrary vertex of degree 1. Let us refer to our choice as vertex u.

• The possible neighbors of it (v) are the vertices with degree at least 2 ($|V| \geq 3$).

• We run through all possible v's and connect u and v. The unique neighbor of u is known. We have to realize a transformed sequence on $V - \{u\}$ (u disappears and the degree of v will be reduced by 1).

Realizing by trees: Finding all realizations

• To list all realizing trees we choose an arbitrary vertex of degree 1. Let us refer to our choice as vertex u.

• The possible neighbors of it (v) are the vertices with degree at least 2 ($|V| \geq 3$).

• We run through all possible v's and connect u and v. The unique neighbor of u is known. We have to realize a transformed sequence on $V - \{u\}$ (u disappears and the degree of v will be reduced by 1).

• We have smaller realization problems. If we can list all realizations in each cases then we can generate the required complete list.

Realizing by trees: Finding all realizations

- To list all realizing trees we choose an arbitrary vertex of degree 1. Let us refer to our choice as vertex u.
- The possible neighbors of it (v) are the vertices with degree at least 2 ($|V| \geq 3$).
- We run through all possible v's and connect u and v. The unique neighbor of u is known. We have to realize a transformed sequence on $V - \{u\}$ (u disappears and the degree of v will be reduced by 1).
- We have smaller realization problems. If we can list all realizations in each cases then we can generate the required complete list. We iterate the ideas if the number of vertices is at least 3. In the case of $|V| = 2$ we know the complete list without using any idea.

Complete list of tree realizing a given sequence: The algorithm

Algorithm

Complete list of tree realizing a given sequence: The algorithm

Algorithm

INPUT: A sequence $\langle d_i \rangle_{i=1}^n$ that satisfies $\sum_{i=1}^n d_i = 2n-2$ and $d_{min} > 0$.

Complete list of tree realizing a given sequence: The algorithm

Algorithm

INPUT: A sequence $\langle d_i \rangle_{i=1}^n$ that satisfies $\sum_{i=1}^n d_i = 2n-2$ and $d_{min} > 0$.

- If $n = 1, 2$, we know the complete list.

Complete list of tree realizing a given sequence: The algorithm

Algorithm

INPUT: A sequence $\langle d_i \rangle_{i=1}^n$ that satisfies $\sum_{i=1}^n d_i = 2n-2$ and $d_{min} > 0$.

- If $n = 1, 2$, we know the complete list.

- In the case of $n > 2$ we choose an arbitrary vertex of degree 1. We call it $u \in V$.

Complete list of tree realizing a given sequence: The algorithm

Algorithm

INPUT: A sequence $\langle d_i \rangle_{i=1}^n$ that satisfies $\sum_{i=1}^n d_i = 2n-2$ and $d_{min} > 0$.

• If $n = 1, 2$, we know the complete list.

• In the case of $n > 2$ we choose an arbitrary vertex of degree 1. We call it $u \in V$.

• For each vertex $v \in V$, that has degree at least 2 ($d(v) \geq 2$) take the degree sequence in the vertex set $V - \{u\}$

$$d'(x) = \begin{cases} d(x), & x \neq v, \\ d(v) - 1, & x = v. \end{cases}$$

Complete list of tree realizing a given sequence: The algorithm (cont'd)

Algorithm (cont'd)

Complete list of tree realizing a given sequence: The algorithm (cont'd)

Algorithm (cont'd)

- We generate the complete list of realizing trees for d' by our algorithm.

Complete list of tree realizing a given sequence: The algorithm (cont'd)

Algorithm (cont'd)

• We generate the complete list of realizing trees for d' by our algorithm.

• For each v, for all trees on the corresponding list we add the vertex u by a branching from v.

Complete list of tree realizing a given sequence: The algorithm (cont'd)

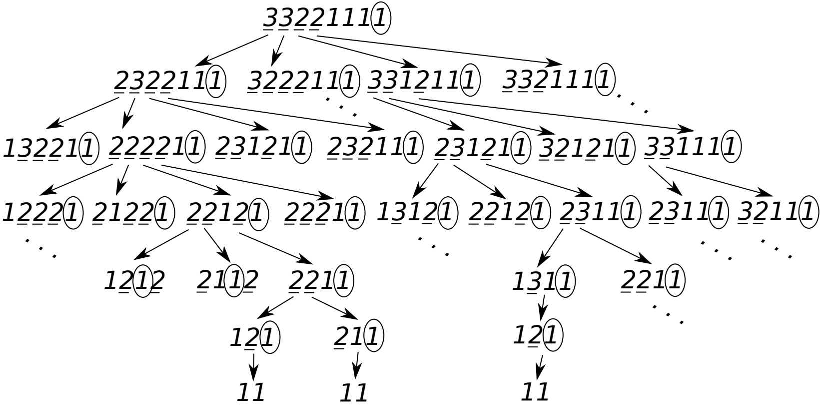
Algorithm (cont'd)

• We generate the complete list of realizing trees for d' by our algorithm.

• For each v, for all trees on the corresponding list we add the vertex u by a branching from v. We add the trees, we obtain this way, to the output list.

All realizations by tree, Example

All realizations by tree, Example



The chosen degree 1 is always the last one in the sequence, we circled them. The degrees of the possible neighbor are underlined.

All realizations by tree, numbers

All realizations by tree, numbers

- Easy to enumerate that in the case of the previous example there are 100 realizing trees.

All realizations by tree, numbers

- Easy to enumerate that in the case of the previous example there are 100 realizing trees.

- Note that V is given, its element can be distinguished.

All realizations by tree, numbers

- Easy to enumerate that in the case of the previous example there are 100 realizing trees.

- Note that V is given, its element can be distinguished.

- There are only 5 isomorphism classes of trees realizing the degree sequence of the example.

All realizations by tree, enumeration

All realizations by tree, enumeration

The above recursive algorithm can be easily transformed to an induction proof of a

All realizations by tree, enumeration

The above recursive algorithm can be easily transformed to an induction proof of a

Theorem

Let $\langle d_i \rangle_{i=1}^n$ be a sequence realizable by tree. The number of trees on the vertex set $\{v_1, v_2, \ldots, v_n\}$, that satisfies $d(v_i) = d_i$ $(i = 1, \ldots, n)$:

$$(n-2)! \prod_{i=1}^{n} \frac{1}{(d_i - 1)!} = (n-2)! \prod_{i=1}^{n} \frac{d_i}{d_i!}.$$

All realizations by tree, enumeration

The above recursive algorithm can be easily transformed to an induction proof of a

Theorem

Let $\langle d_i\rangle_{i=1}^n$ be a sequence realizable by tree. The number of trees on the vertex set $\{v_1, v_2, \ldots, v_n\}$, that satisfies $d(v_i) = d_i$ $(i = 1, \ldots, n)$:

$$(n-2)!\prod_{i=1}^n \frac{1}{(d_i-1)!} = (n-2)!\prod_{i=1}^n \frac{d_i}{d_i!}.$$

The adventage of the second form: we can deal with sequence of natural numbers, i.e. $d_i = 0$ is allowed $(\langle d_i\rangle_{i=1}^n \in \mathbb{N}^n\ (n \geq 2))$.

All realizations by tree, enumeration: The proof

All realizations by tree, enumeration: The proof

- If we have a 0 in the sequence, then our formula gives 0, as required.

All realizations by tree, enumeration: The proof

- If we have a 0 in the sequence, then our formula gives 0, as required.
- Assume that $1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$, and the vertex set of the realizing trees is $\{v_i\}_{i=1}^n$, where $d(v_i) = d_i$.

All realizations by tree, enumeration: The proof

- If we have a 0 in the sequence, then our formula gives 0, as required.
- Assume that $1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$, and the vertex set of the realizing trees is $\{v_i\}_{i=1}^n$, where $d(v_i) = d_i$.
- We know

$$\frac{1}{n}\sum_{i=1}^{n} d_i = 2\frac{n-1}{n} < 2,$$

hence we have that $d_n = d(v_n) = 1$.

All realizations by tree, enumeration: The proof

- If we have a 0 in the sequence, then our formula gives 0, as required.
- Assume that $1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$, and the vertex set of the realizing trees is $\{v_i\}_{i=1}^n$, where $d(v_i) = d_i$.
- We know

$$\frac{1}{n}\sum_{i=1}^{n} d_i = 2\frac{n-1}{n} < 2,$$

hence we have that $d_n = d(v_n) = 1$.

- The set of the realizing trees can be classified according to the only neighbor of v_n. We have $n-1$ classes! When the only neighbor of v_n is v_i then the size of the corresponding class is the number of realizations of $d_1, d_2, d_3, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{n-1}$.

All realizations by tree, enumeration: The proof

- If we have a 0 in the sequence, then our formula gives 0, as required.
- Assume that $1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$, and the vertex set of the realizing trees is $\{v_i\}_{i=1}^n$, where $d(v_i) = d_i$.
- We know

$$\frac{1}{n}\sum_{i=1}^{n} d_i = 2\frac{n-1}{n} < 2,$$

hence we have that $d_n = d(v_n) = 1$.

- The set of the realizing trees can be classified according to the only neighbor of v_n. We have $n-1$ classes! When the only neighbor of v_n is v_i then the size of the corresponding class is the number of realizations of $d_1, d_2, d_3, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{n-1}$.
- From the induction hypothesis we know contribution of this class to the final result of the enumeration.

The number of realizations by trees: The formal induction

The number of realizations by trees: The formal induction

If $n = 2$, then the claim is true. Using the induction hypothesis we can enumerate the trees realizing the given sequence as a sum of $n - 1$ numbers:

The number of realizations by trees: The formal induction

If $n = 2$, then the claim is true. Using the induction hypothesis we can enumerate the trees realizing the given sequence as a sum of $n-1$ numbers:

$$\sum_{j=1}^{n-1}(n-3)!\left(\prod_{i=1}^{j-1}\frac{d_i}{d_i!}\right)\cdot\frac{d_j-1}{(d_j-1)!}\cdot\left(\prod_{i=j+1}^{n-1}\frac{d_i}{d_i!}\right)=$$

$$(n-3)!\left(\prod_{i=1}^{n-1}\frac{d_i}{d_i!}\right)\sum_{j=1}^{n-1}(d_j-1)=$$

$$(n-3)!\left(\prod_{i=1}^{n}\frac{d_i}{d_i!}\right)\sum_{j=1}^{n}(d_j-1)=$$

$$(n-3)!\left(\prod_{i=1}^{n}\frac{d_i}{d_i!}\right)\left(\sum_{j=1}^{n}d_j-n\right)=(n-2)!\prod_{i=1}^{n}\frac{d_i}{d_i!}.$$

The number of trees on a given vertex set, Theorem of Cayley

The number of trees on a given vertex set, Theorem of Cayley

Theorem (Cayley)

On the vertex set $\{v_1, v_2, \ldots, v_n\}$ there are n^{n-2} tree.

The number of trees on a given vertex set, Theorem of Cayley

Theorem (Cayley)

On the vertex set $\{v_1, v_2, \ldots, v_n\}$ there are n^{n-2} tree.

Theorem (Cayley)

The complete graph on n vertices (K_n) ha n^{n-2} spanning tree.

The proof of the Theorem of Cayley

The proof of the Theorem of Cayley

Classify the trees on a given vertex set according to their degree sequences. In each of the disjoint classes we know how many trees are. The summation of the sizes give the number of all trees:

The proof of the Theorem of Cayley

Classify the trees on a given vertex set according to their degree sequences. In each of the disjoint classes we know how many trees are. The summation of the sizes give the number of all trees:

$$\sum_{\substack{d_1,d_2,\ldots,d_n\in\mathbb{N}\setminus\{0\}\\ d_1+d_2+\cdots+d_n=2(n-1)}} (n-2)!\prod_{i=1}^{n}\frac{1}{(d_i-1)!}=\sum_{d_1^-+d_2^-+\cdots+d_n^-=n-2}\frac{(n-2)!}{\prod_{i=1}^{n} d_i^-!},$$

where $d_i^-=d_i-1$.

The proof of the Theorem of Cayley

Classify the trees on a given vertex set according to their degree sequences. In each of the disjoint classes we know how many trees are. The summation of the sizes give the number of all trees:

$$\sum_{\substack{d_1,d_2,\dots,d_n\in\mathbb{N}\setminus\{0\}\\ d_1+d_2+\dots+d_n=2(n-1)}} (n-2)!\prod_{i=1}^{n}\frac{1}{(d_i-1)!} = \sum_{d_1^-+d_2^-+\dots+d_n^-=n-2}\frac{(n-2)!}{\prod_{i=1}^n d_i^-!},$$

where $d_i^- = d_i - 1$.

Note that the right hand side of the equation can be rewritten by the multinomial theorem

$$\sum_{d_1^-+d_2^-+\dots+d_n^-=n-2}\frac{(n-2)!}{\prod_{i=1}^n d_i^-!} =$$

$$\sum_{d_1^-+d_2^-+\dots+d_n^-=n-2}\frac{(n-2)!}{\prod_{i=1}^n d_i^-!}1^{d_1^-}1^{d_2^-}\cdots 1^{d_n^-} = (1+\dots+1)^{n-2} = n^{n-2}.$$

This is the end!

Thank you for your attention!