

Dynamic programming

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The basic idea

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- We assume that $P \in \mathcal{P}$. First, it seems that we are making our life harder. We introduces extra problems in addition to the initial one.
- We will have very easy problems in $\mathcal{P}$.
- We can order the elements of $\mathcal{P}$ a way, that solving the actual problem (following the order) will be always easy, based on the answers given so far.

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- Solving problems of $\mathcal{P}$ is very similar then proving a multitude of claims by induction.

Example: Fibonacci numbers

Definition: Fibonacci numbers

$$F_1 = F_2 = 1,$$

If $n > 2$, then

$$F_n = F_{n-1} + F_{n-2}.$$

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- The dynamical programming solution of a problem very often looks like just filling a sequence, or an array with numbers, and announcing the last number as the output.

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- This program is much slower than the one, based on dynamic programming.

Break



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Independent vertex set

$F \subset V(G)$ is an independent set iff there is no uv edge, with $u, v \in F$.

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For each edge, e there is an index/generation $i \in \mathbb{N}$, that e connects $x \in G_i$ and $y \in G_{i+1}$. In this case we say that x is the parent vertex, and y is the child vertex (according to e).

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- The depth of a rooted tree is the length of the longest root-leaf path. The depth is 0 if and only if the tree has only one vertex, that vertex is a root and a leaf at the same time.
- Many of the graph theoretical slang, introduced above, are originated in the language of family trees.

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- We have $|V(T)|$ problems in our collection of problems. One of them is the original problem (when x is the original root, r).

Roll up the problems

We order the problems $\mathcal{F}_x$ by the depth of (T_x, x) .

The scheme

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(Computation) We solve the the actual problem assuming that the previous problems are already solved.

(Output) Announce the answer for (T_r, r) as the output.

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 - (a) The independent sets not containing r_i , the root.
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We determine the largest size among the two types.

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To obtain an independent set of type (a) in T_{r_i} take an arbitrary independent set for each $T_{s_1}, T_{s_2}, \dots, T_{s_d}$ and take their union.

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- We can maximize the size of the independent set by taking a maximum size independent set from each of the T_{s_i} 's.

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- The answer for the actual problem is $1 + \sum_i \mu_i$.

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(Output) Print the solution for the rooted subtree (T_r, r) ($T_r = T$).

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For the depth 0 initial case we know an optimal independent set too. Using the above ideas we can compute not only the maximal size, but one of the optimal independent set too.

Break



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Finding the Largest Subset in Convex Position of a Finite Planar Point Set.

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Definition: Convex Position

A finite set of points $\mathcal{R} \subset \mathbb{R}^2$ is in convex position if it forms the vertex set of a convex polygon with $|\mathcal{R}|$ vertices.

Example

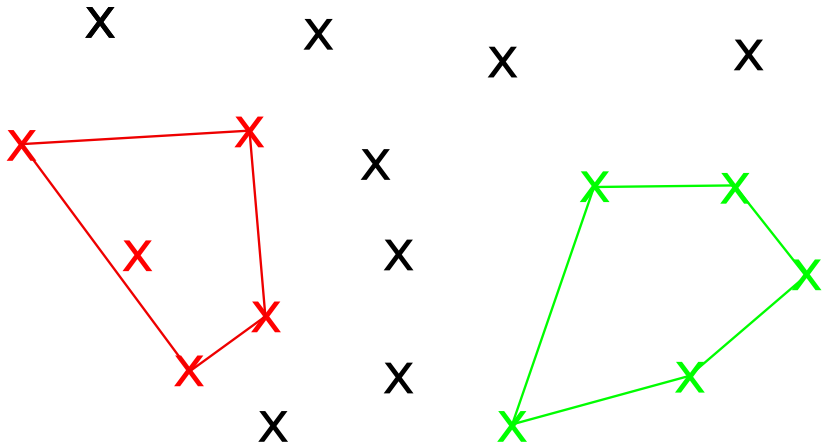


Figure: A finite set of points and a subset of five points (in red) which is not in convex position, along with another subset of five points (in green) which is in convex position.

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- This part of the algorithm can be solved with $\mathcal{O}(n \log n)$ comparisons.

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- The lower arc can be treated similarly. We refer to it as a *cup*.

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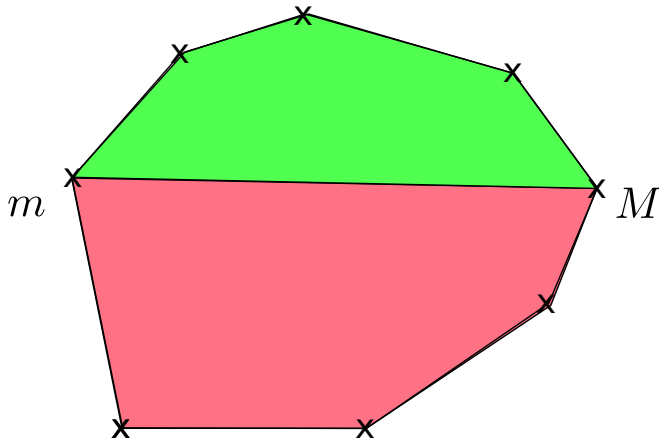


Figure: A point set in convex position with two highlighted vertices. The green part is the cap component of the the convex hull, and the red part is the cup component of the convex hull.

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- Naturally, the following ideas can also be applied to determine the largest cup sizes.
- Thus, the algorithm solving the original problem can be derived from the following steps.

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- The simplest/base subproblems are the $\mathcal{F}_{k,k,s}$ problems. In this case, the two points P_k and P_s form a cap, which is the optimal cap.

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- Let's guess the vertex that could be the third-to-last point of the cap (i.e., the point just before P_ℓ).
- This vertex is geometrically constrained within a triangle: It must lie to the left of the vertical line through P_k , above the line $P_k P_\ell$, and below the line $P_\ell P_s$. The vertex cannot lie on the boundary of the triangle, except for P_k , which may appear as the point directly before P_ℓ .

The Roll-up Step is continued

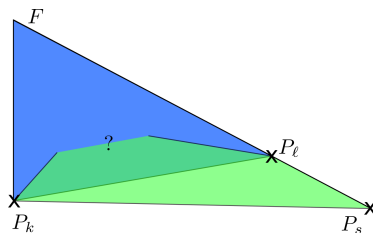


Figure: F is the intersection point of the vertical line passing through P_i and the line $P_\ell P_s$. The optimal cap must be found within the blue triangle $P_i P_\ell F$.

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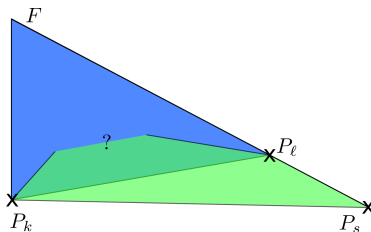


Figure: F is the intersection point of the vertical line passing through P_i and the line $P_\ell P_s$. The optimal cap must be found within the blue triangle $P_i P_\ell F$.

We take the points that lie within the triangle $P_k P_\ell F$ (these include P_k and potentially other interior points of the triangle). For each point $P_{\ell-}$, we retrieve the solution to $\mathcal{F}_{k,\ell-, \ell}$ from our previous work. By taking the maximum of these values and adding 1, we obtain the solution to $\mathcal{F}_{k,\ell, s}$.

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- It is crucial that we store the solutions to the already solved $\mathcal{O}(n^3)$ subproblems and retrieve the necessary information from them as needed.
- It is easy to modify the algorithm so that it not only provides the maximum size, but also outputs the largest set of points in convex position.

This is the end!

Thank you for your attention!