Circuits, $\mathcal{P}$- and $\mathcal{NP}$-complete problems

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Normalized Turing Machines

Reminder

For $L \in_T \mathcal{TIME}(t(n))/\mathcal{NTIME}(t(n))$, we always assume that $t(n)$ is a nice time function, i.e., there exists a Turing machine that solves L and run for $t(n)$ steps on each input of length n.
For an input $\omega \in \Sigma^n$, the run of T can be represented as

$$\kappa_0(\omega) \to \kappa_1 \to \kappa_2 \to \ldots \to \kappa_\ell,$$

where $\kappa_0(\omega)$ is the initial configuration corresponding to ω, κ_{i+1} is the successor of κ_i, and κ_ℓ is the first configuration where the state is ACCEPT or REJECT. We can assume $\ell = t(n)$.

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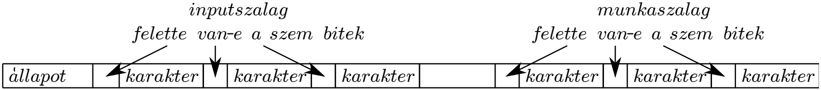

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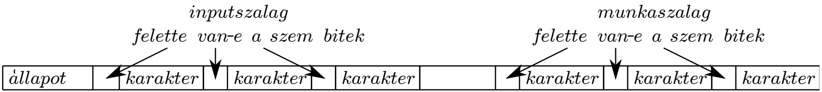

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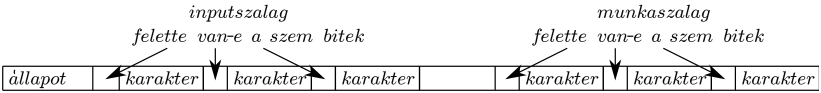

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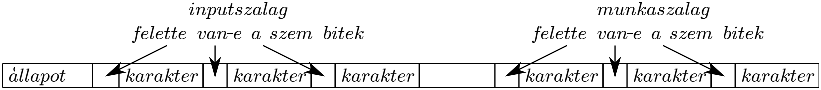

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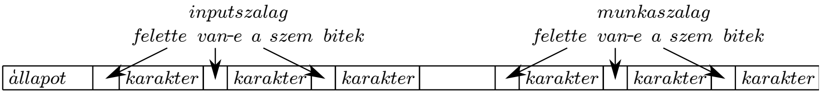

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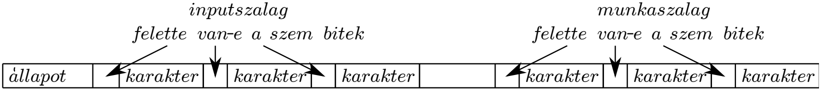

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The length of blocks encoding states and symbols depends on $|S|$, $|\Sigma|$, and $|\Gamma|$. In any case, a constant number of bits is sufficient (depending on the Turing machine).

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In the following, n and the encoding agreement are always fixed (accordingly, the lengths of the corresponding codes are always known).

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We show that it is possible to straightforwardly determine/calculate a small (polynomial-size) circuit that, given the code of a configuration, computes the code of the next configuration.

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A labeled directed graph $(\overrightarrow{G}, \ell)$ is called a circuit.

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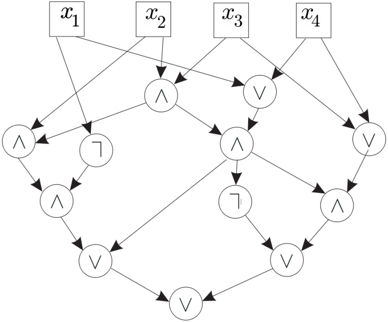

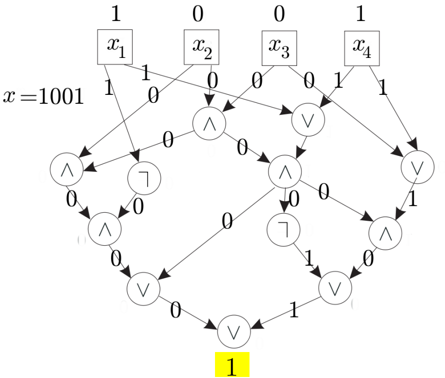

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Definition

Let $f_{\mathcal{C}}$ be the Boolean function computed/realized by the circuit $\mathcal{C}$ as described above.

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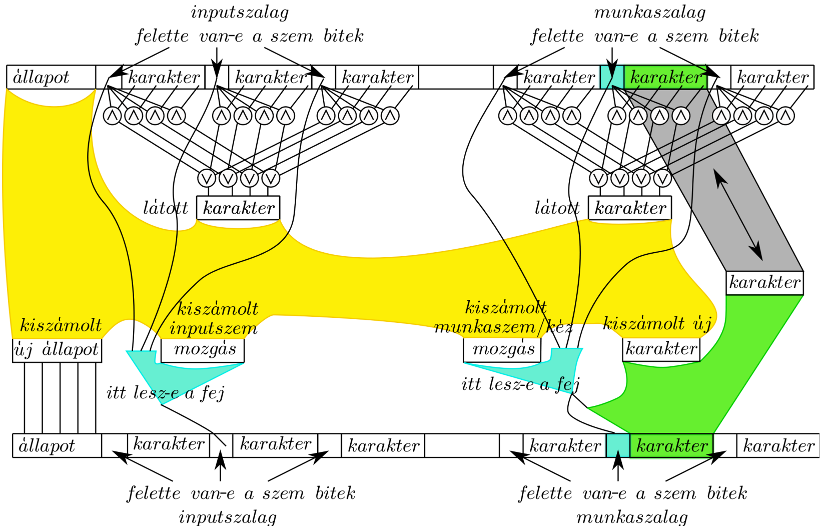
Our construction is simple but involves many details and agreements. Instead of providing a formal description, we illustrate the ideas through an example.

Mathematizing the Last Observation in a Picture



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For a cell, we perform the logical AND operation on the bit indicating whether the eye/hand is there and the bits encoding the content of the cell. The resulting bit sequence is either all 0 (if the eye/hand is not there, we ANDed with all 0s) or the code of the seen character (if the eye/hand is there).

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For the bit sequences obtained for the tape cells, we read the first, then the second, and so on, characters by performing the logical OR operation. We obtain the code of the seen character on the tape.

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In the yellow area, more complex calculations are performed: we compute constant many bits from constant many bits (depending on the Turing machine). The concrete implementation depends on the transition function. If we have no idea about the dependencies of individual bits, we can write down the obvious DNF formula based on the transitions. Even in this case, working with a constant number of gates allows us to accomplish our task.

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In the light blue area, we calculate one of the bits describing the position of the head. This depends on whether the head was there or stood over one of the neighboring cells, and also on the direction the transition prescribes. This part could be explicitly written down, but it is unnecessary based on our previous remark. This blue part is there for each head-position bit.

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In the green area, we calculate the new content of the tape. Each tape cell has such a green block (we only displayed one for simplicity). The new character depends on the new one, the old one, and whether the head is there. This part could also be easily implemented if we knew the number ℓ of bits used to encode the elements of Γ. In the green area, we calculate the function $f(\epsilon, k_0, k_1) = k_\epsilon$, which computes ℓ from $1 + 2\ell$ bits.

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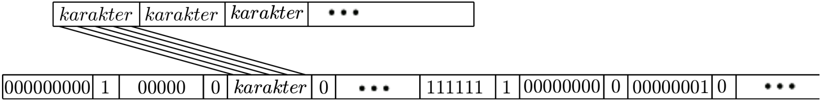

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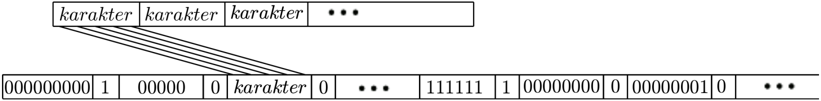

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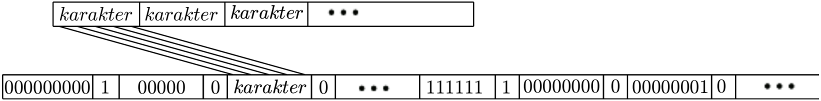

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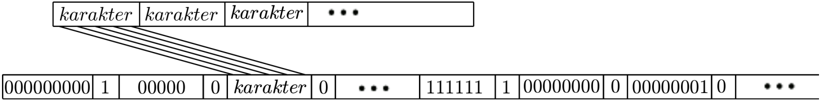

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On the tape, the code for $\triangleright$ is $0\ldots00$, and the code for the blank character is $0\ldots01$ (of length $\lceil \log_2 |\Gamma| \rceil$, in our example 8).

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Definition

Let

$$\text{CIRCUIT-EVAL} = \{\lceil \mathcal{C}, \omega \rceil : \mathcal{C}(\omega) = 1\},$$

i.e., the decision problem that, given a circuit $\mathcal{C}$ and a bit sequence ω, decides whether the circuit computes the 1 bit when given the bit values of $x_1, x_2, \ldots$ as ω (i.e., it evaluates $\mathcal{C}_n(\omega)$).

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Theorem

CIRCUIT-EVAL is $\mathcal{P}$-complete (with respect to $\mathcal{L}$-reductions).

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We can assume that T is such that after reaching the ACCEPT/REJECT state, it „holds" its state. Thus, answering the question $\omega \in L$? ($\omega \in \Sigma^n$) is equivalent to determining whether, during the run on ω, the configuration $\kappa_{t(n)}$ in state ACCEPT.

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Based on the above, for any $L \in_T \mathcal{P}$, given an arbitrary $\omega \in \Sigma^n$, we can construct a circuit $\mathcal{T}_{T,\omega}$ that encodes the input gates with ω (3rd observation), and some of its levels encode the elements of the configuration sequence of the Turing computation (2nd observation).

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The theoretical part of the reduction follows from the earlier observations. The construction/reduction is log-space.

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Consequence

(i) For any $L \in \mathcal{NP}$ language, $L \prec_{\mathcal{L}}$ CIRCUIT-SAT

(ii) CIRCUIT-SAT is $\mathcal{NP}$-complete

(iii) $\mathcal{P} = \mathcal{NP} \Leftrightarrow \text{CIRCUIT-SAT} \in \mathcal{P}$

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Therefore, constructing the code of $C(\lceil\omega\rceil, y_1, y_2, \ldots, y_{q(n)})$ (which can be done in $\mathcal{L}$) is a good reduction.

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The two steps together form a polynomial time algorithm solving the decision problem for L. From this, we obtain $\mathcal{NP} \subseteq \mathcal{P}$, so $\mathcal{P} = \mathcal{NP}$ follows.

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Let $V = \{x_1, x_2, \ldots, x_n\}$ be a set of variables. Let $L = V \dot{\cup} \overline{V}$ be the set of literals ($\overline{V}$ is the set of negated variables, i.e., $\{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$).

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For this set of clauses, we think of the clauses as connected by the $\wedge$ logical operator, i.e., the conjunction.

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That is, SAT is the problem where, given a CNF formula φ, we need to decide whether it is satisfiable.

The Cook–Levin Theorem

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SAT (satisfiability of CNF formulas) is $\mathcal{NP}$-complete.

From One $\mathcal{N}P$-Complete Problem to Another

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The above two steps are very characteristic once we have our first $\mathcal{N}P$-complete problem.

We don't need the full power of $\mathcal{N}P$ to formulate a problem C. It is enough to formulate a $\mathcal{N}P$-complete problem C' using C.

Plan for the Proof of CIRCUIT-SAT $\preceq$ SAT

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An equation system $\varphi_i(x_1, x_2, \ldots, x_n) = \psi_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, \ell$, is called a Boolean equation system if φ_i and ψ_i are Boolean formulas. BOOLEAN-EQUATION-SYSTEM-SAT is the language containing the encodings of solvable/satisfiable Boolean equation systems.

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$x_g = 1$ if g is the output gate of the circuit.

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Thus, generating the code of $x_g = 1$ (which can be done in $\mathcal{L}$) is a good reduction.

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The solvability of the equation system is equivalent to the satisfiability of the formula.

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suspected $\mathcal{P} \neq \mathcal{NP}$ holds, then there is no polynomial time algorithm for these problems. In other words, we consider them inherently difficult based on theoretical conjectures.

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Lemma

$(=3)$-SAT$\preceq_{\mathcal{P}}$ (≤ 3)-SAT, and (≤ 3)-SAT$\preceq_{\mathcal{P}}$ $(=3)$-SAT.

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Repeat this for every small clause. What we obtain is an equivalent 3-CNF.

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In the example above, the small clause had two literals. Our idea can be applied to clauses with fewer literals. The result: an equivalent formula with one larger clauses. We need to iterate our idea.

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3-SAT

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3-SAT is trivially in $\mathcal{NP}$ (it is a special case of SAT).

Reduction from SAT to 3-SAT

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Reminder: What Are We Claiming?

For a SAT $\rightarrow$ 3-SAT reduction, we need a function that can be computed in polynomial time, such that $\mathcal{C} \in$ SAT $\Leftrightarrow \mathcal{C}' \in$ 3-SAT.

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The assignment is as follows: for $C = \langle \ell_1, \dots, \ell_k \rangle$, introduce new variables $u_1, \dots, u_{k-1}$, and add the following clauses to $\mathcal{C}'$:

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Do this for every clause in $\mathcal{C}$. What we get is a 3-CNF.

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Repeat this for every clause in $\mathcal{C}$. What we obtain is a 3-CNF.

Proof, The Other Direction

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$\mathcal{C}'$ has no satisfying assignment where, for some $\mathcal{C}$ clause $C = \langle \ell_1, \ldots, \ell_k \rangle$, all literals $\ell_1 = \ldots = \ell_k = h$. Because the clause

$$\langle \overline{u}_1 \rangle, \langle u_1, \overline{u}_2 \rangle, \ldots, \langle u_{i-1}, \overline{u}_i \rangle, \ldots, \langle u_{k-2}, \overline{u}_{k-1} \rangle \langle u_{k-1} \rangle$$

is unsatisfiable.

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The following reduction chain is evident:

$$2\text{-}SAT \preceq_{\mathcal{P}} 3\text{-}SAT \preceq_{\mathcal{P}} 4\text{-}SAT \preceq_{\mathcal{P}} \ldots \preceq_{\mathcal{P}} k\text{-}SAT \preceq_{\mathcal{P}} \ldots \preceq_{\mathcal{P}} SAT.$$

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It is easy to see that 2-SAT $\in \mathcal{P}$. Moreover, 2-SAT $\in co\mathcal{NL}$.

NOT-ALL-TRUE-SAT

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Definition

An assignment makes a clause homogeneous if every literal in the clause receives the same truth value. In other words, a clause becomes non-homogeneous if it is satisfied (contains a true literal) but not all literals are true.

Let

$$\text{NOT-ALL-TRUE-SAT} = \{\lceil\varphi\rceil : \varphi \text{ is a CNF that is satisfiable but has no clause where all literals are true}\}$$

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Trivially, NOT-ALL-TRUE-SAT $\in \mathcal{NP}$.

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For an SAT $\rightarrow$ NOT-ALL-TRUE-SAT reduction, we need a function that can be computed in polynomial time, such that $\mathcal{C} \in$ SAT $\Leftrightarrow \mathcal{C}' \in$ NOT-ALL-TRUE-SAT.

Reduction from SAT to NOT-ALL-TRUE-SAT

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For an SAT $\rightarrow$ NOT-ALL-TRUE-SAT reduction, we need a function that can be computed in polynomial time, such that $\mathcal{C} \in$ SAT $\Leftrightarrow \mathcal{C}' \in$ NOT-ALL-TRUE-SAT.

The assignment is as follows: for $C = \langle \ell_1, \ldots, \ell_k \rangle$, introduce a new variables s, and add the following clauses to $\mathcal{C}'$:

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Do this for every clause in $\mathcal{C}$. What we get is a NOT-ALL-TRUE-SAT instance.

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To complete the proof is an easy exercise.

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The proof will be a reduction from NOT-ALL-TRUE-SAT to NOT-ALL-TRUE-3-SAT.

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Examine what the reduction constructs from C:

$\langle \ell_1, \overline{u}_1 \rangle, \langle u_1, \ell_2, \overline{u}_2 \rangle, \ldots, \langle u_{i-1}, \ell_i, \overline{u}_i \rangle, \ldots, \langle u_{k-2}, \ell_{k-1}, \overline{u}_{k-1} \rangle, \langle u_{k-1}, \ell_k \rangle.$

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Keep the values of the original variables; below, we describe how the new variables get their values.

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After ℓ_i, the negation of a new variable is introduced. Assigning a true value to the new variable makes the ℓ_i literal false and true in the *small* clause. Moving right, assigning true values to the subsequent new variables reaches the *small* clause for ℓ_j. Meanwhile, every clause receives both true and false values.

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In summary, we have shown that

$$R(\varphi) \in \text{NOT-ALL-TRUE-3-SAT}.$$

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Conversely, assume that $R(\varphi) \in$ NOT-ALL-TRUE-3-SAT. We have seen that an assignment satisfying every clause in $R(\varphi)$ (which is a non-all-true assignment) cannot result in the original clauses having all true values.

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We only need to rule out the possibility that $R(\varphi)$ is a non-all-true assignment to the original variables, restricting each clause of the original ones to have every literal true.

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This would imply that the

$$\langle \overline{u}_1 \rangle, \langle u_1, \overline{u}_2 \rangle, \ldots, \langle u_{i-1}, \overline{u}_i \rangle, \ldots, \langle u_{k-2}, \overline{u}_{k-1} \rangle \langle u_{k-1} \rangle$$

clauses all need to be false. This (as before) is impossible.

Break

Coloring Problems

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3-COLORABILITY $\in \mathcal{NP}$: the witness is a coloring, and it can be checked in polynomial time whether it is a valid coloring.

3-SAT $\preceq$ 3-COLORABILITY Reduction

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To each 3-CNF $\mathcal{C}$, we assign a graph $G_{\mathcal{C}}$, with vertices n, h, the variables of $\mathcal{C}$, their negations, and for each $C \in \mathcal{C}$ clause, the vertices C_1, C_2, C_3, C_4, C_5.

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The edges of $G_{\mathcal{C}}$ are as follows: nh, for each variable x_i $x_i\overline{x_i}$, nx_i and $n\overline{x_i}$, and for each $C = \langle z_1, z_2, z_3 \rangle$ clause C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_5C_1, C_1z_1, C_2z_2, C_3z_3, C_4h, C_5h.

3-SAT $\preceq$ 3-COLORABILITY Reduction (Visual)

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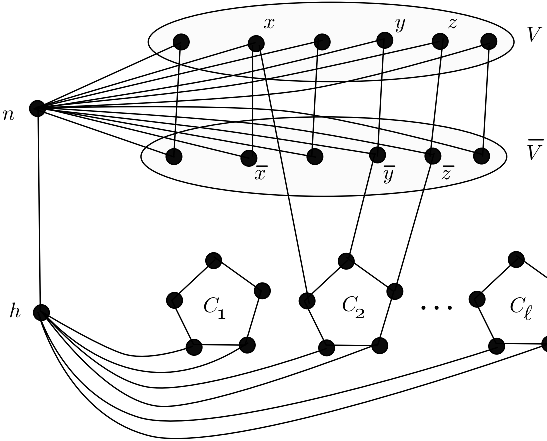

3-SAT $\preceq$ 3-COLORABILITY Reduction (Verbal)

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It is easy to verify that $G_{\mathcal{C}}$ can be determined in polynomial time, and it is 3-colorable if and only if $\mathcal{C}$ is satisfiable (using the observation that the 3-coloring of z_1, z_2, z_3, h can be extended to a valid coloring for the 5 vertices corresponding to the clause $C = \langle z_1, z_2, z_3 \rangle$, if the colors of the 4 vertices are distinct).

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COLORING PROBLEM is $\mathcal{NP}$-hard, as it generalizes 3-COLORABILITY.

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Proof I

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Reduction from SAT:
$\mathcal{C} = (C_1 = \langle z_{1,1}, \ldots, z_{1,r_1}\rangle, \ldots, C_k = \langle z_{k,1}, \ldots, z_{k,r_k}\rangle) \mapsto (G_{\mathcal{C}}, k)$
(where (i,j) indicates the j-th literal in the i-th clause),
$V(G_{\mathcal{C}}) = \{(i,j) : i \leq k, j \leq r_i\}$,
$E(G_{\mathcal{C}}) = \{(i,j),(i',j') : i = i' \text{ or } z_{i,j} = \overline{z}_{i',j'}\}$.

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An assignment is satisfying if, for each clause, we can choose a true literal (the edges ensure that the variable and its negation do not appear together, and at most one literal is chosen from each clause).

Proof II

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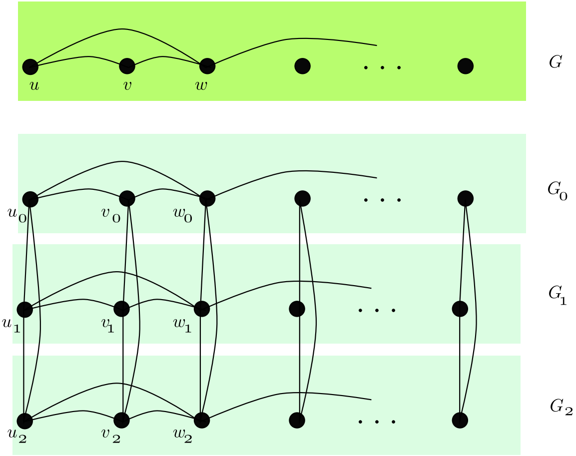
Reduction from the Coloring Problem: $G \mapsto (G', |V(G)|)$, where $V(G') = \{(v,i) : v \in V(G), i \in [3]\}$ (here, (v,i) represents that vertex v is assigned color i),
$E(G') = \{(v,i)(v',i') : v = v', i \neq i' \text{ or } vv' \in E(G), i = i'\}$ (i.e., edges are used to forbid, it is forbidden that a vertex receives more than one color, or connected vertices receive the same color).

Proof II in Figure



Proof II in Words

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It's easy to see that G' and $|V(G)|$ can also be determined in polynomial time, and there is an independent set of size $|V(G)|$ in G' if and only if G is 3-colorable.

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In contrast to the Coloring Problem, if k is not part of the input but a constant, then the resulting k-INDEPENDENT SET problem can be solved in polynomial time (every n-vertex graph has polynomially many k-element subsets if k is fixed).

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Equivalent to the INDEPENDENT SET problem.

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Expanding the Statement

We demonstrate that VERTEX-COVER can be reduced to HAMILTON. That is, given a graph G and $k \in \mathbb{N}$, we can effectively define a graph R such that R has a Hamiltonian cycle if and only if G can be covered by k vertices.

We illustrate the reduction with an example/figure. The general, formal description can be easily inferred from the figure, and we leave that to the interested reader.

Proof in Figure: G in red, $k = 2$, R in blue

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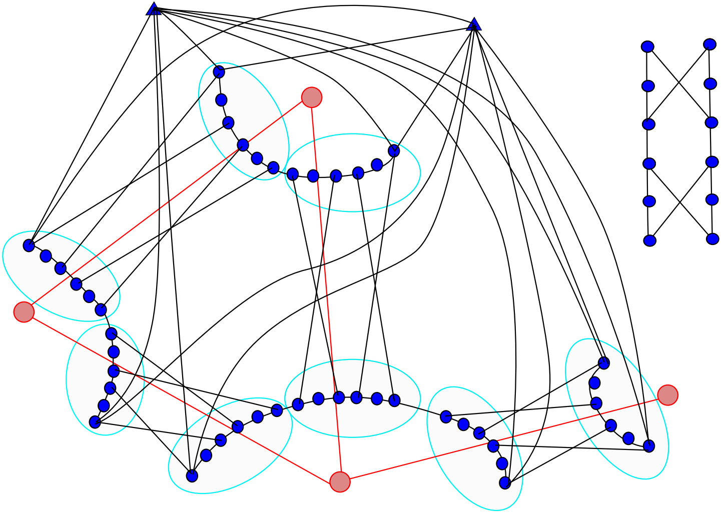

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It is easy to check that for each edge, two blocks corresponding to the edge can be crossed in two different ways, as illustrated on the right.

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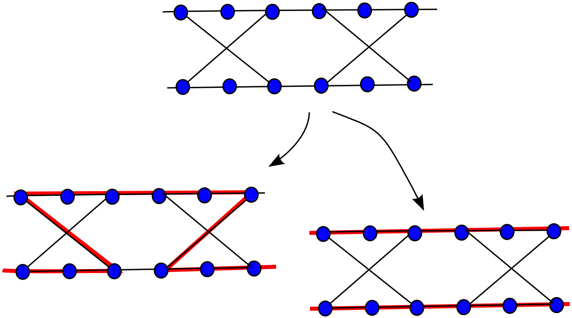

(Left) Crossing occurs on a single edge. We traverse one path of a vertex, but also traverse the other block of the edge. (Right) Crossing occurs in two separate parts, each on the path of a vertex.

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This completes the theoretical part of the reduction. The technical details of implementation (polynomial time) are omitted.

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MAX-CUT $\in \mathcal{NP}$: a witness is a red/blue coloring, the number of edges can be calculated in polynomial time, and it can be compared with k.

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MAX-CUT is $\mathcal{NP}$-hard: We reduce NOT-ALL-TRUE-3-SAT to it.

To every 3-CNF $\mathcal{C}$, we associate the graph $G_{\mathcal{C}}$, where vertices are variables and their negations (the literals).

For each clause $C \in \mathcal{C}$, we connect every three literals pairwise. (We refer to these edges as clause-edges.) Each clause corresponds to three clause-edges. If a literal appears in multiple clauses, multiple edges will be created in the graph constructed by the reduction.

For every variable x, we draw an edge between x and $\overline{x}$. (We refer to these edges as variable-edges.) This describes all the edges of the $G_{\mathcal{C}}$ graph.

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If a cut (I, H) achieves this upper bound, then every literal-edge is included, meaning each variable x and $\overline{x}$ falls into either I or H. Thus, the cut defines an evaluation of our variables.

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Furthermore, every clause has two out of three clause-edges in the cut, meaning the described evaluation satisfies every clause in $\mathcal{C}$ in a non-all-true manner.

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We also note that our reduction created a graph whose optimal cut was a balanced split. This implies that the MAX-BISECTION problem is also $\mathcal{NP}$-complete. Complementing this gives us that the MAX-BISECTION and MIN-BISECTION problems are (in polynomial time) equivalent. Specifically, the MIN-BISECTION problem is also $\mathcal{NP}$-complete.

Break

Set Systems

Set Systems

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For every $E \in \mathcal{E}$, there is a subset $\{v \in V : v \in E\}$ that is a subset of V.

Alternative Descriptions of Set Systems

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Observation

A set system $\mathcal{H}$ over V can be easily described by a bipartite graph B. The two color classes are V (upper points) and $\mathcal{H}$ (lower points). An element of the base set is connected to an element of $\mathcal{H}$ if and only if it belongs to the corresponding edge.

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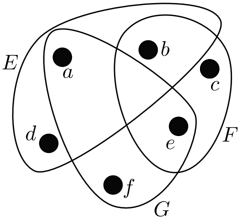

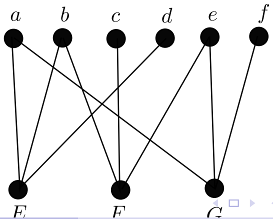

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

INDEPENDENT-NODES-IN-SET-SYSTEM

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The concept of an independent node set in graph theory can be extended to set systems in two ways:

I is independent if, for every $E \in \mathcal{H}$, $E \not\subseteq I$.

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Set Systems Harder Than Graphs

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Theorem

(i) INDEPENDENT-NODES $\preceq$ INDEPENDENT-NODES-IN-SET-SYSTEM,

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(ii) INDEPENDENT-NODES $\preceq$ INDEPENDENT*-NODES-IN-SET-SYSTEM.

Indeed, the graph-theoretical problem graph is a special case of set systems. The concept of independence in graph theory is a special case of both types of independence in set systems.

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Note: The problem INDEPENDENT-EDGES-IN-GRAPHS, alternatively MATCHING $= \{\lceil G, k \rceil : \nu(G) \geq k\}$, is easily solvable. According to Edmonds' algorithm, this problem is in $\mathcal{P}$. Hence, the case for graphs is manageable.

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Theorem

INDEPENDENT*-NODES-IN-SET-SYSTEM $\preceq$ INDEPENDENT-EDGES-IN-SET-SYSTEM

Set Systems: Duality

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Based on the bipartite graph representation, it is easy to describe independent* sets. For upper points in B, there exists a set I such that V is not covered, i.e., there is a triple $a \in A, f, f' \in I \subset F$ where a is connected to both f and f'.

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Definition

Let B be a bipartite graph describing a set system. By exchanging upper and lower roles, we obtain the dual graph B^*. Reading B^* as a set system and restoring it, we get a dual set system with $V^* = \mathcal{H}$ and $\mathcal{H}^* = V$.

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Thus, the initial transformation is the reduction proving the theorem.

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INDEPENDENT-EDGES-IN-SET-SYSTEM $\preceq$ TILING.

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In the second step, assume that $\mathcal{H}$ is a S-uniform set system. In this step, add $|V(H)| - kS$ new vertices to $V(H)$ (let $\widetilde{V}$ be the resulting set), and the elements of $\widetilde{\mathcal{H}}$ are the elements of $\mathcal{H}$ plus one set for each old-new vertex pair.

The Reduction in Pictures

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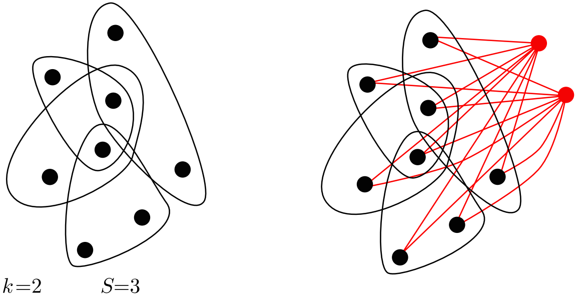

The second step of the reduction: $|V| - kS = 8 - 2 \cdot 3 = 2$. The two new vertices and the corresponding graph edges are shown in red on the right.

The Reduction in Words

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Observation

To tile $(\widetilde{V}, \widetilde{\mathcal{H}})$, we need to cover the $|V| - kS$ new vertices, which can only be done with $|V| - kS$ new vertex pairs. The remaining tiling edges can only be old edges, covering kS vertices. Thus, the tiling gives k independent edges in $\mathcal{H}$.

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Inverting the reasoning of the observation completes the theoretical part of the proof.

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Theorem

3-UNIFORM-SET-SYSTEM-PARTITION and PERFECT-TRIPLE are both $\mathcal{N}P$-complete.

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By this, for clause C, we have introduced two new vertices and three new triples.

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The resulting 3-uniform balanced triple set system is the outcome of the reduction.

Proof of 3-SAT $\preceq$ PERFECT-TRIPLE (continued)

In the constructed triple system, the number of *a* vertices and *b* vertices is the same. If the number of *c* vertices is more, add new *a* vertices and *b* vertices for balancing. If the number of *c* vertices is less, add new *c* vertices for balancing.

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Generalization of graph coloring problem.

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TILING $\preceq$ SET SYSTEM 2-COLORABILITY.

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Construction: $\widetilde{V} = \mathcal{H} \cup \{p, z\}$. For $\widetilde{H}$, for every intersecting pair E, F of $\mathcal{H}$ edges, add $Z_{E,F} = \{E, F, z\}$ as an edge. For every $v \in V$, add $R_v = \{E : v \in E \in \mathcal{H}\} \cup \{p\}$ as an edge in $\widetilde{\mathcal{H}}$. Also, add the edge $\{p, z\}$.

The Reduction in Pictures

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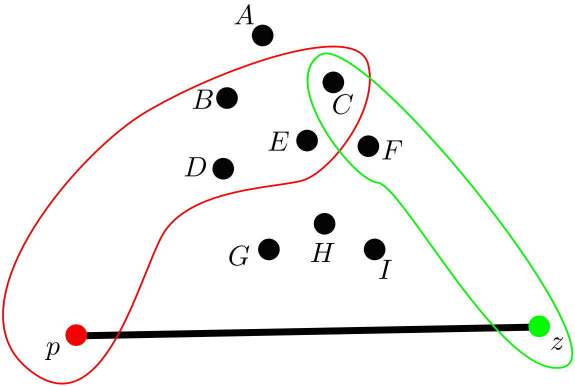

$A, B, C, \ldots, H, I$ precisely represent the edges of our set system. B, C, D, E precisely represent edges containing the element a. C and F are intersecting edges. The edges inferred from the above information are drawn in the figure, which includes a fraction of the reduction.

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In a 2-coloring of $\widetilde{V}, \widetilde{\mathcal{H}}$, let p be colored red and z be colored green (the edge $\{p, z\}$ enforces using the entire palette). The green color on the corresponding vertices of the original edges picks an edge set.

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The reasoning is reversible, completing the proof.

Break

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It is also easy to see that the resulting inequality system has an integer solution if and only if the conjunctive normal form is satisfiable.

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For $E \in \mathcal{H}$, let $a_E = \sum_{v:v\in E} w(v)$. Let $A = \{a_E : E \in \mathcal{H}\}$ and $b = 11\dots1_a = \sum_{v:v\in V} w(v)$. This describes an input for the subset sum problem.

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The problem's interpretation: The set A describes the volumes of items. Given b boxes, each of which can hold at most c total volume. Can we pack the items into the boxes?

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The volume of our knapsack is b.

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Clearly, a subset of items can fit in the knapsack and achieve the value limit if and only if the sum is exactly b.

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The knapsack can be filled with items such that both subsets have the same total value if and only if A can be partitioned into two equal-sum subsets.

This is the end!

Thank you for your attention!