Complexity classes

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Definition

The *space complexity* of a Turing-machine T on an input ω is $s := SPACE(\omega, T)$, if during its execution on ω the largest index of the cells under the work hand is s, or ∞, if the working eye/hand moves arbitrarily far from the left border of the work tape.

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$$SPACE(\omega, T) \leq TIME(\omega, T)$$

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Let $t : \mathbb{N} \to \mathbb{R}$ be an arbitrary function. We say that a Turing machine T is an element of the set $\texttt{TIME}(t(n))$ if for every $\omega \in \Sigma^*$

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Definition: Decidable languages in polynomial time

$$\mathcal{P} := \bigcup_{p \in \mathbb{R}[x]} \mathcal{TIME}(p(n)) = \bigcup_{a \in \mathbb{N}} \mathcal{TIME}(an^a + a).$$

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The complete list of complexity classes is much longer
http://qwiki.stanford.edu/index.php/Complexity_Zoo.

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$$\mathcal{SPACE}(s(n)) \subseteq \cup_{c\in\mathbb{N}} \mathcal{TIME}(c^{s(n)+\log(n+1)}).$$

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The previous sentence is a characteristic first line in proofs on complexity. We introduce a spacial notation for that: $L \in_T \mathcal{SPACE}(s(n))$.

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Let $\kappa_0(\omega) \to \kappa_1(\omega) \to \kappa_2(\omega) \to \ldots \to \kappa_\ell(\omega)$ be the run on input ω. This is a sequence of configurations of length $\ell \geq 1$, where the first one ($\kappa_0(\omega)$) is the initial configuration (specially the state is START) and the last configuration is ($\kappa_\ell(\omega)$) the first one, where the state is ACCEPT or REJECT.

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$$(n+2) \cdot (s(n)+1) \cdot |\Gamma|^{s(n)} \cdot |S| \leq \alpha_T (n+1) \alpha_T^{s(n)} \leq \beta_T^{s(n)+\log(n+1)},$$

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Indeed: The number of the possible position of the input eye is $n+2$. The number of the possible position of the work eye/pen is $s(n)+1$. The number of the possible content of the work tape is $(|\Gamma|+1)^{s(n)}$. Finally the state of the machine comes from $|S|$ possibilities.

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We know that this is not the case. So we get the bound claimed.

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$\mathcal{P}$ is a robust class of languages.

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Similarly serious technical problems would be raised by the language class $\mathcal{E} := \{L(T) : T \in \cup_{a\in\mathbb{N}}\texttt{Time}(a^{n+a})\}$.

Break

The language PALINDROM

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Definition: PALINDROM language

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Two Turing machines/algorithms are sketched. For simplicity we assume that $\Sigma = \{0, 1\}$.

First model: Single tape model

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- The first decision is to choose an alphabet for the work tape: $\Gamma = \{0, 1, 0^{\checkmark}, 1^{\checkmark}\}$.

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Mark it and remember its value.

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If so, we continue the procedure, until we run out of unchecked bits.

First algorithm/TM: S, set of states

For the Turing machine's work so far, we used the following state set

$$S = \{\text{START, LEFT-MARK, RIGHT-FIND-0, RIGHT-FIND-1, RIGHT-TEST-0, RIGHT-TEST-1, LEFT-FIND, ACCEPT, REJECT}\}.$$

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$$(\text{START}, 0) \mapsto \text{„Who cares?''}$$

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Second model: Standard model with 1 work tape

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The input head goes to the beginning of the input.

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The the eyes move in different directions and test the palindrom property.

Second algorithm/TM: S, the set of states

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$$S = \{\text{START, COPY, TO-THE-LEFT, TEST, ACCEPT, REJECT}\}.$$

Second algorithm/TM: Transition function

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Second algorithm/TM: The time

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Observation

All ω input, the run length is at most $3|\omega| + 3 = \mathcal{O}(|\omega|)$.

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This result is sharp in terms of magnitude.

Break

The one-tape model is bad

Theorem

If T is a Turing machine that decides in the single-tape model the *PALINDROM* language, then $\forall\, n$, $\exists\, \omega \in \Sigma^n$:

$$TIME(\omega, T) \geq \alpha_T |\omega|^2,$$

for some positive constant α_T.

Proof: Notions

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The common boundary of two adjacent cells on the input tape is called *door*. The cells of the (input) tape can be imagined as an infinite series of rooms. We can say, that the head can only move through the doors.

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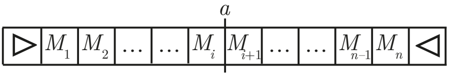

The cells M_i and M_{i+1} and the door a between them.

A run of T

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Consider the truncated run of the Turing machine T on ω input:

$$\kappa_0(\omega) \to \kappa_1 \to \kappa_2 \to \ldots \to \kappa_\ell,$$

where

$$\ell := \min\{n \mid \text{ state of } \kappa_n \text{ is ACCEPT or REJECT.}\}$$

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Definition: $\sigma(a, \omega)$

Now take those κ_j, κ_{j+1} configurations, in which the input eye is over M_i/M_{i+1}. Let s_j be the state of the Turing machine when passing through the a door separating the cells.
The sequence of these s_j states is denoted by $\sigma(a, \omega)$.

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Observation

Knowing $\omega \overset{a}{|}$ and the $\sigma(a, \omega)$ state sequence, then we are able to reconstruct how the Turing machine "works" when the eye is on the left hand side of the door a.

We cannot know how long the eye was a right but as soon as it crosses the door (and as long as it remains on the left) we are able to describe the run of T.

Corollary of the Observation

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Corollary

Let $\omega, \omega' \in \Sigma^n$ be arbitrary inputs and a be a a door. Suppose $\sigma(a, \omega) = \sigma(a, \omega')$ and Turing machine T has the same result on the two inputs.

Then T outputs the same on the input $\widetilde{\omega} = \left(\omega \overset{a}{|}\right)\left(\overset{a}{|} \omega'\right)$.

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Definition: I_0

Suppose that $3|n$. Let

$$I_0 := \{\alpha 0^{\frac{n}{3}} \overleftarrow{\alpha} : \alpha \in \Sigma^{\frac{n}{3}}\} \subseteq \text{PALINDROM} \cap \Sigma^n.$$

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Let ω, $\omega' \in I_0$ be distinct words and a be a middle door (i.e. one of the doors separating the middle $n/3$ zeros). Then $\sigma(a, \omega) \neq \sigma(a, \omega')$.

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The number of state sequences shorter than t

$$1 + |S| + \cdots + |S|^{t-1} = \frac{|S|^t - 1}{|S| - 1} < |S|^t - 1 < |S|^t.$$

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If $|I_0| = |\Sigma|^{\frac{n}{3}} \geq |S|^t$, then $\exists \omega \in I_0$ such that $\sigma(a, \omega)$ has length at least t, where a is a middle door.

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We have $|I_0| = |\Sigma|^{\frac{n}{3}} \geq |S|^t$ in the case when $t \sim \beta_T \cdot n$.

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Let a be an arbitrary middle door. Assume that $|I_0| \geq 2|S|^t$. There exists at least $|I_0|/2$ inputs in I_0 for which $|\sigma(a, \omega)| \geq t$.

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Let denote the set of inputs in the Corollary by $I_1(a)$, i.e.

$$I_1(a) = \{\omega \in I_0 : |\sigma(a, \omega)| \geq t\} \subseteq I_0.$$

We also know that $t \sim \gamma_T \cdot n$ is a suitable choice to guarantee $|I_1| \geq |I_0|/2$.

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After division by $|I_0|$:

$$\frac{1}{|I_0|} \sum_{\omega\in I_0} TIME(\omega, T) \geq \frac{\gamma_T}{6} \cdot n^2.$$

This is the end!

Thank you for your attention!