Greedy algorithms

Peter Hajnal

Bolyai Institute, University of Szeged, Hungary

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- Let us see an example.

Break

The minimal cost spanning tree problem

The problem

Given a connected graph, for each edge we have a positive cost ($c : E(G) \to \mathbb{R}_{++}$). This cost function can be naturally extended to subsets of $E(G)$ (the cost of an edge set is the sum of the costs of its elements).

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Find a cheapest spanning tree of the input graph (the tree is considered as a set of edges).

The following algorithm, first described by Kruskal is a „prototype" of the greedy algorithm design.

Kruskal's algorithm

Kruskal's algorithm (1956)

(SORTING STEP) Sort the edges of the input graph in ascending order of cost. Let $E(G) : e_1, e_2, \ldots, e_m$, i.e. e_1 is the cheapest edge, e_m is the most expensive edge $(c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m))$.

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(TESTS) In the i^{th} step, we examine e_i. If $F \cup \{e_i\}$ is cycle-free, then we extend F: $F \leftarrow F \cup \{e_i\}$. If $F \cup \{e_i\}$ contains a cycle, then we do not change F.

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(OUTPUT) After examining the last edge, we announce the current F as the output.

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One can say: all our choices are the best decision at the time. Later some of the edges have been discarded. After that it is possible that we need to throw away an edge. This is a questionable choice. It was based on the fact that the previously selected edges form a part of the output. If our previous decisions are overruled, then we could have chosen to use the currently discarded edge (the cheapest edge of the remaining edge set). The cost of the calculated spanning tree cannot simply be compared to the tree calculated above.

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Theorem (Kruskal's Theorem)

The output of the above algorithm is a minimum cost spanning tree of the input graph.

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- Let F be an arbitrary spanning tree. We list the edges in increasing order of cost: $f_1, \ldots, f_{n-1}$, i.e.

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- Based on the connectivity of the input graph it is easy to see that Kruskal's algorithm computes a spanning tree, i.e. $\ell = n - 1$.

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For $i = 1, 2, \ldots, n-1$ we have

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- We have performed an analysis of a naive implementation. There are more clever solutions.

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Determine the distance of s and t.

Refreshing memory

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$\overrightarrow{uv}$-walk in $\overrightarrow{G}$:

$$\overrightarrow{\mathcal{S}} : u = w_0, \overrightarrow{e}_1, w_1, \overrightarrow{e}_2, \ldots, w_{L-1}, \overrightarrow{e}_L, w_L = v,$$

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Observation

The shortest $\overrightarrow{uv}$-walk will be a path.

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- Each edge between S and $\overline{S} = V(G) - S$ are oriented from $\overline{S}$ to S.

The case of graph theoretical distance: unweighted case

- We solve a harder problem. The input will be $(\overrightarrow{G}, s)$. We determine S, the set of vertices that are reachable from s.
- The set of reachable vertices will be $S = S_0 \dot\cup S_1 \dot\cup \ldots \dot\cup S_L$, where S_i contains exactly those vertices that are at distance i from s.
- L denotes the length of longest path starting at s.
- Each edge between S and $\overline{S} = V(G) - S$ are oriented from $\overline{S}$ to S. This property will be a proof of the fact that the elements of $\overline{S}$ are not reachable from s.

Breadth first search algorithm

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(E) // Extension cycle // While $S_i \neq \emptyset$

- $$S_{i+1} = \{x \in V(G) - (S_0 \cup \ldots \cup S_i) : \text{ there is } \sigma \in S_i, \text{ such that } \overrightarrow{\sigma x} \in E(G)\}$$
- $i \leftarrow i + 1$.

end-while

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For each vertex $x \in S$ there is exactly one $\overrightarrow{sx}$-path in $\mathcal{F}$. This one of the (graph theoretically) shortest $\overrightarrow{sx}$-path.

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- One should think about the promised information as a labeling, $c : V(\overrightarrow{G}) \to \mathbb{R} \cup \{\infty\}$, of the vertices.

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If $t \notin S$ (or $\overline{S} \neq \emptyset$) back to (E).

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• The above argument followed a naive implementation. There are cleverer ways to implement Dijkstra's high level description.

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• Changing the value of the label/updating: changing the "responsible" node too.

• If we keep track of these edges responsible for the actual value we will obtain a rooted, directed spanning tree of the original graph (see the computation of breadth first search tree). This tree maintains/contains for each vertex a shortest path leading to that vertex.

Break

Coding texts

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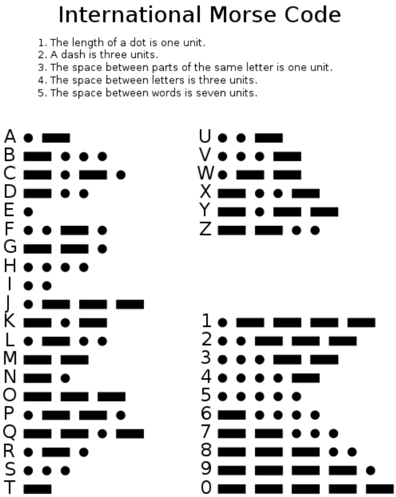
Character based coding: $c_0 : \Sigma \to \{0,1\}^\star$ coding of characters. The code of a text is obtained by "putting together" the codes of its characters.

Fixed-length codes, Example: ASCII (1972) (source: wiki)

Bits $b_7 b_6 b_5$ →					000	001	010	011	100	101	110	111
b_4 ↓	b_3 ↓	b_2 ↓	b_1 ↓	Column → Row ↓	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	´	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	¦
1	1	0	1	13	CR	GS	—	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

Variable-length codes, Example: Morse code (1837–44) (source:wiki)



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Definition: Prefix tree for Σ

Let (T, r) be a rooted binary plane tree. Let L be the set of leaves of (T, r). pause (T, r, ℓ) is a prefix tree for Σ, iff $\ell : \Sigma \to L$ is a bijection.

The coding of the characters based on a prefix tree

$$k(\in \Sigma) \mapsto \text{labels of the } r\text{-}\ell(k) \text{ path in } T$$

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- The natural idea: choose the two trees with lowest frequencies.

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The correctness of Huffman's algorithm: the main idea

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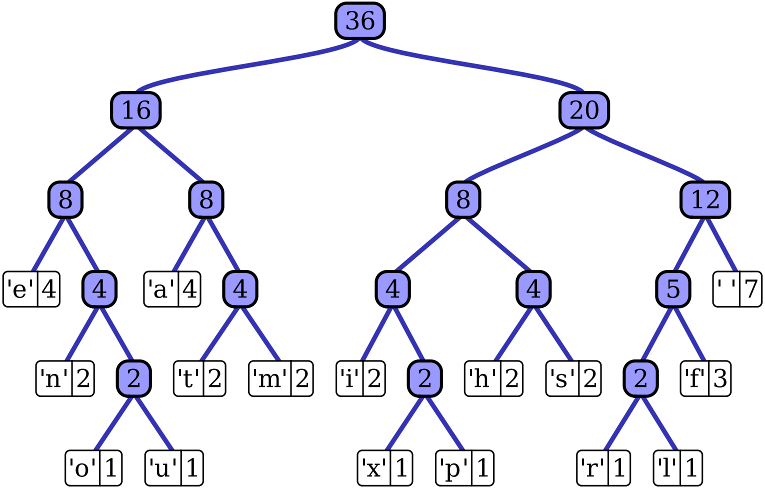
Observation

The length of β' is at most the length of β.

"this is an example of a huffman tree"



Source: wikipedia

Break

The greedy algorithm for matchings

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- We know that int the case of halting the output can't be augmented by extensions (by adding further edges to the output).

Example

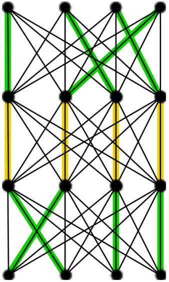

Figure: Our graph has four disjoint levels of equal sized vertex sets (let n be the size of the levels, in our example $n = 4$). Between two adjacent levels all possible edges are present and there are no further edges. It is possible that the greedy algorithm first chooses the yellow edges, matching the two middle levels. Then it halts. The green edges form a perfect matching.

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The second inequality is obvious since our algorithm computes a matching.

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Analysis: the proof

- Let M_{greedy} denote the output matching of the greedy algorithm.
- $L = V(M_{\text{greedy}})$ is the set of matched vertices.
- It is obvious that L is a covering vertex set, and $|L| = 2\nu_{\text{greedy}}(G)$.
- The size of L gives an upper bound on the size of an arbitrary matching, hence $\nu(G) \leq |L| = 2\nu_{\text{greedy}}(G)$.

This is the end!

Thank you for your attention!