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On a Class of Theorems Equivalent to Farkas's Lemma

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Abstract

We give a list of 28 theorems which are all equivalent to Farkas's Lemma. This list includes Gordan's Theorem, Stiemke's Theorem (Fundamental Theorem of Asset Pricing), Slater's Theorem, Gale's Theorem, Tucker's Theorem, Ville's Theorem (von Neumann's Theorem of the Alternative), von Neumann's Minimax Theorem for matrix games, Motzkin's Theorem, the Strong Duality Theorem in linear programming, and Broyden's Theorem. For convenience of exposition, we also mention three versions of Separating Hyperplane Theorems which are equivalent to Farkas's Lemma.

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Keywords: Farkas's lemma, Gordan's theorem, Slater's theorem, Stiemke's theorem, Gale's theorem, Tucker's theorem, von Neumann's minimax theorem, the strong duality theorem of linear programming, separating hyperplane theorem

1 Introduction

Farkas's lemma is one of the theorems of the alternative that arise naturally in solving linear inequalities. A typical approach in studying these theorems is to develop a main theorem and then derive other results as its consequence.

For example, Farkas's lemma and other theorems of the alternative can be derived by applying the strong duality theorem of linear programming ([4],[11]). It has been a folklore knowledge that many such related theorems are in fact equivalent. In this note, we make this precise by closely studying their mutual relationship: we collect a list of such theorems and prove their equivalence. This approach offers a way to understand these theorems better, and it offers a way to bypass the difficulty of attacking a problem directly. For example, if one knows that Farkas's lemma is equivalent to Gordan's theorem, then in order to prove Farkas's lemma, it suffices to prove Gordan's theorem ([12]). Similarly, by going through Sep I $\Rightarrow$ Mangasarian $\Rightarrow$ Theorem 17 $\Rightarrow$ Broyden (see Section 3 (d)), this gives a new proof of Broyden's theorem. A more striking example is that the approach of this paper confirms the assertion about the equivalence of LP strong duality theorem and the Minimax theorem (by combining (1) the equivalence of Minimax and Ville, (2) the equivalence of LP strong duality and Farkas, and (3) the equivalence of Ville and Farkas. For a direct (but technical) reduction of LP strong duality theorem to Minimax, we refer to [1].

We briefly explain the notations we will be using. For simplicity we will be working on vectors or matrices over the real numbers even though the results may still hold true for more general ordered fields. For a matrix A, we use $A' = A^T$ to denote the transpose of A. For two column vectors v, w, the writing of vw means that $vw = v^T w = v \cdot w$. For two vectors v, w, $v \geqq w$ (resp. $v > w$) means that $v_i \geqq w_i$ (resp. $v_i > w_i$) for each i, where v_i means the i-th component of v, while $v \geq w$ means that $v_i \geqq w_i$ for each i and for at leat one i, $v_i > w_i$. Similar notations hold for the other inequalities. For a vector subspace V in $\mathbb{R}^d$, we use $V^\perp$ to denote the orthogonal complement of V in $\mathbb{R}^d$. A linear polyhedral cone is a space V generated by taking linear combinations of a finite number of vectors using nonnegative coefficients. It is convenient to identify a linear polyhedral cone V with a matrix A, thinking of the row vectors or the column vectors to generate the polyhedral cone, thus it makes sense to write $y^T V > 0$ to mean $y^T A > 0$, where V is the polyhedral cone generated by the column vectors of A. A polyhedral cone A is *pointed* if $Ax = 0$ for no $x \geq 0$. In Section 2, we list the theorems which are all equivalent. The proofs are given in Section 3.

2 A List of Theorems

1. Theorem. (Supporting Hyperplane Theorem) Let V be a pointed linear polyhedral cone generated by $v_1, \cdots, v_k$. Then there exists a vector y such that $y \cdot v_i < 0$ for $1 \leq i \leq k$ (for simplicity we will write this as $y^T V < 0$).

2. Theorem. (Separation I) Let V be a pointed linear polyhedral cone and S be a vector subspace such that $V \cap S = \{0\}$. Then there exists a hyperplane H containing S such that H is a supporting hyperplane of V at $\{0\}$.

3. Theorem. (Separation II) Let V_1, V_2 be two nontrivial pointed linear polyhedral cones such that $V_1 \cap V_2 = \{0\}$. Then there exists a hyperplane defined by y such that $y^T V_1 > 0$ and $y^T V_2 < 0$.

4. Theorem (Gordan, 1873, [8]) For each given matrix A, exactly one of the following is true.
I. $Ax > 0$ has a solution x.
II. $A'y = 0, y \geq 0$ has a solution y.

5. Theorem (Farkas, 1902, [8]) For each given matrix A and each given vector b, exactly one of the following is true.
I. $Ax = b$ has a solution $x \geqq 0$.
II. $A'y \geqq 0, y^T b < 0$ has a solution y.

6. Theorem. (Gale, Variant of Farkas, [3]) Exactly one of the following is true.
I. $\exists x, Ax \leqq b, x \geqq 0$
II. $\exists y, y^T A \geqq 0, y \geqq 0$ and $y^T b < 0$.

7. Theorem. (Gale, 1960, [7]) Exactly one of the following is true.
I. $\exists x, Ax \leqq b$
II. $\exists y, y \geqq 0, y^T A = 0$ and $y^T b < 0$.

8. Theorem (Slater, 1951, [8]). Let A, B, C and D be given matrices, with A and B being nonvacuous. Then exactly one of the following is true.
I. $Ax > 0 \qquad Bx \geq 0 \qquad Cx \geqq 0 \qquad Dx = 0$ has a solution x.

$$\text{II.} \left\langle \begin{array}{ll} A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0 & \\ \text{with} & \\ y_1 \geq 0, y_2 \geqq 0, y_3 \geqq 0 & \text{or} \\ y_1 \geqq 0, y_2 > 0, y_3 \geqq 0 & \end{array} \right\rangle \text{has a solution } y_1, y_2, y_3, y_4.$$

9. Theorem (Tucker's First Existence Theorem, 1956, [8]) For any given matrix A, the systems
I. $Ax \geqq 0$ and
II. $A'y = 0, y \geqq 0$
possess solutions x and y satisfying $Ax + y > 0$.

10. Theorem (Tucker's Second Existence Theorem, 1956, [8]) The systems (with A nonvacuous)

I. $Ax \geqq 0, Bx = 0$ and
II. $A'y_1 + B'y_2 = 0, y_1 \geqq 0$ possess solutions x, y_1, y_2 such that $Ax + y_1 > 0$.

11. Theorem (Motzkin, 1936, [8]) Let A, C and D be given matrices, with A being nonvacuous. Then exactly one of the following is true.
I. $Ax > 0 \qquad Cx \geqq 0 \qquad Dx = 0$ has a solution x.
II. $\left\langle \begin{array}{c} A'y_1 + C'y_3 + D'y_4 = 0 \\ y_1 \geq 0, y_3 \geqq 0 \end{array} \right\rangle$ has a solution y_1, y_3, y_4.

12. Theorem. (Tucker's theorem of the alternative, [8]) Exactly one of the following is true (where B is nonvacuous).
I. $Bx \geq 0, Cx \geqq 0, Dx = 0$ has a solution x.
II. $\left\langle \begin{array}{c} B'y_2 + C'y_3 + D'y_4 = 0 \\ y_2 > 0, y_3 \geqq 0 \end{array} \right\rangle$ has a solution y_2, y_3, y_4.

13. Theorem. (Stiemke, 1915, [8]) For each given matrix B, exactly one of the following is true.
I. $Bx \geq 0$ has a solution x.
II. $B'y = 0, y > 0$ has a solution.

14. Theorem. (Nonhomogenous Farkas, Duffin 56, [8]) Exactly one of the following is true (where β is a scalar).
I. $bx > \beta, Ax \leqq c$ has a solution x.
II. $\left\langle \begin{array}{c} A'y = b, cy \leqq \beta, y \geqq 0, \text{ or} \\ A'y = 0, cy < 0, y \geqq 0 \end{array} \right\rangle$ has a solution y.

15. Theorem. (Minimax Theorem, von Neumann, 1928, [1]) Given $B \in \mathbb{R}^{m \times d}$, and denoting $S(n) = \{s \in \mathbb{R}^n | e^T s = 1, s \geq 0\}$, where e is a vector of 1's,

$$\max_{x \in S(d)} \min_{y \in S(m)} y^T Bx = \min_{y \in S(m)} \max_{x \in S(d)} y^T Bx.$$

Definition. A sign matrix is a diagonal matrix whose diagonal elements are equal to either plus one or minus one.

16. Theorem. (Broyden, 1998, [5]) Let Q be an orthogonal matrix. Then there exists a vector $x > 0$ and a unique sign matrix S such that $Qx = Sx$.

17. Theorem. Let Q be an orthogonal matrix. Then

$$\exists x > 0 \text{ such that } (I + Q)x \geqq 0 \text{ and } (I - Q)x \geqq 0.$$

18. Theorem. (Tucker, 1956, [4]) Let A be a skew-symmetric matrix, i.e. $A^T = -A$. Then there exists $u \geqq 0$ such that $Au \geqq 0$ and $u + Au > 0$.

Definition. For a given $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m, c \in \mathbb{R}^d$, the linear programming problem (in the canonical form) consists of the primal program

$$\begin{array}{lll} \text{(P)}: & \max & c^T x \\ & \text{subject to} & Ax \leqq b \\ & & x \geqq 0. \end{array}$$

and the dual program

$$\begin{array}{lll} \text{(D)}: & \min & b^T y \\ & \text{subject to} & A^T y \geqq c \\ & & y \geqq 0. \end{array}$$

Remark. It is often useful to use other presentation of the linear programs. For example, the primal linear program: max $c^T x$ subject to $Ax = b, x \geqq 0$ has dual program with min $b^T y$ subject to $A^T y \geqq c$.

19. Theorem. (LP Strong Duality Theorem) If both the primal problem (P) and the dual problem (D) are feasible, there exist a dual pair (x^*, y^*) of feasible solutions such that $c^T x^* = b^T y^*$.

20. Theorem. (Mangasarian, 1969, [8]) Exactly one of the following is true.
I. $Ax \leq 0, x \geqq 0$ has a solution x.
II. $A'y \geqq 0, y > 0$ has a solution y.

21. Theorem. (Ville, Problem 14 on page 35 of [8]) Exactly one of the following is true.
I. $Ax < 0, x \geqq 0$ has a solution x.
II. $A'y \geqq 0, y \geq 0$ has a solution y.

22. Theorem. (Problem 15 on page 36 of [8]) Exactly one of the following is true.
I. $Ax < 0, x > 0$ has a solution x.
II. $A'y \geqq 0, y \geq 0$ has a solution y.

23. Theorem. (Problem 17 on page 36 of [8]) Exactly one of the following is true.
I. $Ax \leqq 0, x \geq 0$ has a solution x.
II. $A'y > 0, y \geq 0$ has a solution y.

24. Theorem. (Antosiewicz, [2]) Either $A'u > 0, B'u \geqq 0$ for some u or $Ax + By = 0$ for some $x \geq 0, y \geqq 0$, but never both.

25. Theorem. (Antosiewicz, [2]) Either $A'u \geq 0, B'u \geqq 0$ for some u or $Ax + By = 0$ for some $x > 0, y \geqq 0$, but never both.

26. Theorem. (S. Morris, [9]) Let A be an $m \times n$ matrix. Let $\mathcal{S}$ be a family of nonempty subsets of $\{1, \cdots, m\}$. Exactly one of the following alternatives holds. Either there exists x and $S \in \mathcal{S}$ such that

$$Ax \geqq 0, A_i x > 0 \text{ for all } i \in S$$

or there exists p such that

$$pA = 0, p \geqq 0, \sum_{i \in S} p_i > 0 \text{ for all } S \in \mathcal{S}.$$

27. Theorem. (Selfdual Alternative Theorem, [6]) Let V be a vector subspace of $\mathbb{R}^d$, let $V^\perp$ be its orthogonal dual space, and let g be any fixed index in $[d]$. Then exactly one of the following is true.
I. $\exists x \in V : x \geq 0$ and $x_g > 0$.
II. $\exists y \in V^\perp : y \geq 0$ and $y_g > 0$.

28. Theorem. (Corollary to Tucker's Second Existence Theorem, [8]) Let A, B, C and D be matrices of the same column index, with A, B or C nonvacuous. Then the systems
I. $Ax \geqq 0, Bx \geqq 0, Cx \geqq 0, Dx = 0$ and
II. $A'y_1 + B'y_2 + C'y_3 + D'y_4 = 0, y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0$
possess solutions x, y_1, y_2, y_3, y_4 satisfying $Ax + y_1 > 0, Bx + y_2 > 0$ and $Cx + y_3 > 0$.

3 Proofs of Equivalence

Theorem. The list of 28 theorems in Section 2 are equivalent.

Proof. We remark first that for the proof of theorems of the alternative, it is very easy to show that both statements cannot be simultaneously true: for example, in the statement of Farkas (5), if x and y are the solutions, then $0 \leqq (y^T A)x = y^T(Ax) = y^T b < 0$ would yield a contradiction. *In what follows, we suppress the argument for this part.* The notation $\bar{\text{I}}$ means the negation of the statement I, etc.

(**a**) That 1, 2, 3, 4, and 5 are equivalent was proven in [12], where 1 and 4 are essentially tautology.
(**b**) $5 \Rightarrow 7 \Rightarrow 6 \Rightarrow 5$. ($5 \Rightarrow 7$) Assume 5. If 7 (II) has no solutions, then there does not exist $z \geqq 0$ such that $z^T[A\ b] = [0\ -1]$. By 5, there exists y such that

$$[A\ b]y \geqq 0, \text{ and } [0\ \ -1]y < 0.$$

By scaling with a positive factor, we may assume that $y = \begin{bmatrix} y_1 \\ 1 \end{bmatrix}$ and $[A\ b] \begin{bmatrix} y_1 \\ 1 \end{bmatrix} \geqq 0$, which implies $Ay_1 + b \geqq 0 \Rightarrow Ay_1 \geqq -b \Rightarrow A(-y_1) \leqq b$, hence we may let $x = -y_1$ to conclude. (7 $\Rightarrow$ 6) Assume 7. Note that

$$Ax \leqq b, x \geqq 0$$

$$\Leftrightarrow \begin{bmatrix} A \\ -I \end{bmatrix} x \leqq \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

If 6 (I) is not true, then by 7, there exists $z \geqq 0$, such that

$$z^T \begin{bmatrix} A \\ -I \end{bmatrix} = 0, z^T \begin{bmatrix} b \\ 0 \end{bmatrix} < 0.$$

Writing $z^T = [z_1^T, z_2^T] \geqq 0$, the above conditions mean that

$$z_1^T A = z_2^T \geqq 0, z_1^T b < 0,$$

which proves 6. (6 $\Rightarrow$ 5) Assume 6. Note that

$$Ax = b, x \geqq 0$$

$$\Leftrightarrow Ax \geqq b, Ax \leqq b, x \geqq 0$$

$$\Leftrightarrow \begin{bmatrix} A \\ -A \end{bmatrix} x \leqq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geqq 0.$$

If 5 (I) is not true, then by 6, there exists $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geqq 0$, such that

$$y^T \begin{bmatrix} A \\ -A \end{bmatrix} \geqq 0, y^T \begin{bmatrix} b \\ -b \end{bmatrix} < 0.$$

Letting $z = y_1 - y_2$, it is easy to see that the above conditions mean that $z^T A \geqq 0$, and $z^T b < 0$.

(**c**) 2 $\Rightarrow$ 13 $\Rightarrow$ 9 $\Rightarrow$ 10 $\Rightarrow$ 28 $\Rightarrow$ 8 $\Rightarrow$ 11 $\Rightarrow$ 4. (2 $\Rightarrow$ 13) Assume 2. We will show that 13 ($\bar{\text{I}}$) $\Rightarrow$ 13 (II). Let S be the vector space spanned by the column vectors of B, and V be the linear polyhedral cone represented by the positive orthant. Clearly V is pointed. But ($\bar{\text{I}}$) means precisely that $V \cap S = \{0\}$, so by Separation I, there exists a supporting hyperplane $H \supseteq S$ for V. Now it is straightforward to check that one of the normal vectors of H satisfies 13 (II). (13 $\Rightarrow$ 9) This was proven in [10]. (9 $\Rightarrow$ 10 $\Rightarrow$ 28 $\Rightarrow$ 8) These were proven in [8]. (8 $\Rightarrow$ 11) In 8, let $A = B$. (11 $\Rightarrow$ 4) In the statement of 11, let $C = D = 0$.

(**d**) 2 $\Rightarrow$ 20 $\Rightarrow$ 17 $\Rightarrow$ 16 $\Rightarrow$ 18 $\Rightarrow$ 9 $\Rightarrow$ 27 $\Rightarrow$ 5. (2 $\Rightarrow$ 20) Assume 2. It suffices to show that 20 ($\bar{\text{I}}$) $\Rightarrow$ 20 (II). Let V be the linear polyhedral cone determined by the column vectors of A. Then 20 ($\bar{\text{I}}$) says that V intersects the negative orthant N at 0. We may decompose V as $V = W + V_0$ (eg. see Lemma 4.1 of [12]), where W is a maximal linear subspace and V_0 is pointed (the case when $V = W$ is easy, so we assume here that $V_0 \neq \{0\}$). By the properties of $V \cap N = \{0\}$, it is straightforward to check that $N + (-1)V_0$ is pointed and $[N + (-1)V_0] \cap W = \{0\}$. So by Separation I, there exists a hyperplane H containing W and H forms a supporting hyperplane for $N + (-1)V_0$, i.e. one of the normal vectors y of H satisfies $v \cdot y < 0$ for any $v \in N + (-1)V_0$, in particular $y > 0$, and $v_0 \cdot y > 0$ for every $v_0 \in V_0$, hence $A'y \geq 0$. (20 $\Rightarrow$ 17) Let $A' = \begin{bmatrix} Q+I \\ -Q+I \end{bmatrix}$. If $A'y \geqq 0, y > 0$ has no solutions, then by 20, there exists a solution x such that $[Q'+I \mid -Q'+I]x \leq 0, x \geqq 0$. Letting $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the last condition is equivalent to

$$x_1 + x_2 \leq Q'(x_2 - x_1), x_1 \geqq 0, x_2 \geqq 0.$$

Necessarily this means that

$$\|x_1 + x_2\| < \|Q'(x_2 - x_1)\| = \|x_2 - x_1\|,$$

which is impossible, as it is clear that $\|x_2 - x_1\| \leqq \|x_1 + x_2\|$ under the conditions $x_1 \geqq 0$ and $x_2 \geqq 0$ (here $\|\cdot\|$ denotes the length of a vector and we have used the fact that orthogonal matrices preserve lengths). (17 $\Rightarrow$ 16) In fact, these two are equivalent. Note that $\exists x > 0, Qx = Sx \Leftrightarrow \exists x > 0, |Qx| = x$ (where $|\cdot|$ means taking absolute value in each component) $\Leftrightarrow \exists x > 0, |Qx| \leqq x$ (use the property of orthogonal matrices mentioned above) $\Leftrightarrow \exists x > 0$ such that $(I + Q)x \geqq 0$ and $(I - Q)x \geqq 0$. See also [13], where the authors gave two proofs of Broyden's theorem. (16 $\Rightarrow$ 18 $\Rightarrow$ 9) These were proven in [5]. (9 $\Rightarrow$ 27) In fact, these two are equivalent, which can be easily proven based on the observation: If $V = \{x \mid Ax = 0\}$, then $V^{\perp} = \{y \mid y = A^T z$ for some $z\}$. We omit the details. (27 $\Rightarrow$ 5) Assume 27. We will show that 5 ($\bar{\text{I}}$) implies 5 (II). Assume that $Ax = b, x \geqq 0$ has no solutions. Consider $B = [A \ \ -b]$. Then the system $By = 0$ has no solutions $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geqq 0$ with $y_2 > 0$, otherwise $x := y_2^{-1} y_1 \geqq 0$ would be a solution of $Ax = b$. Then by 27 there exists $z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \geqq 0$ in the orthogonal complement of the kernel of B such that $z_2 > 0$. Using the observation mentioned right above, such z is of the form $z = \begin{bmatrix} A^T \\ -b^T \end{bmatrix} u$ for some u. It follows that $A^T u \geqq 0$ and $b^T u < 0$, as required.

(**e**) 5 $\Rightarrow$ 19 $\Rightarrow$ 15 $\Rightarrow$ 21 $\Rightarrow$ 4. For (5 $\Rightarrow$ 19), see for example [8]. For (19 $\Rightarrow$ 15), see [4]. (15 $\Rightarrow$ 21) See [1]. (21 $\Rightarrow$ 4) Assume 21. It suffices to show that if 4 (II) does not have a solution, then 4 (I) has a solution. We note that

$A'y = 0, y \geq 0$ does not have solutions implies that

$$\begin{bmatrix} A' \\ -A' \end{bmatrix} y \leqq 0, y \geq 0$$

does not have solutions. But now by 21, this implies that

$$[A \ -A]z > 0, z \geqq 0$$

has a solution

$$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqq 0$$

such that

$$[A \ -A] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0, \text{ i.e. } A(x_1 - x_2) > 0.$$

It follows that $x := x_1 - x_2$ satisfies $Ax > 0$ so 4 (I) has a solution.
(**f**) Statements 21, 22, and 23 are equivalent. *Reasons:* In 21 (I), $Ax < 0, x \geqq 0$ has a solution $\Leftrightarrow$ $Ax < 0, x > 0$ has a solution (if $Au < 0, u \geqq 0$, then $x = u + \epsilon e > 0$ satisfies $Ax < 0$ if $\epsilon > 0$ is small enough, where e is the vector of all 1's). Similarly, in 23 (II), $A'y > 0, y \geq 0$ has a solution $\Leftrightarrow$ $A'y > 0, y > 0$ has a solution.
(**g**) 11 $\Rightarrow$ 14 $\Rightarrow$ 5. For 11 $\Rightarrow$ 14, See [8]. For 14 $\Rightarrow$ 5, take $\beta = 0$ and $c = 0$ in 14.
(**h**) $24 \Rightarrow 5; 25 \Rightarrow 5; 11 \Rightarrow 24; 12 \Rightarrow 25$. As noted in [2], each of 24 and 25 implies Farkas when $A = -a$ is a column vector. However it is also easy to show that 11 implies 24 and 12 implies 25.
(**i**) 9 $\Rightarrow$ 26 $\Rightarrow$ 13. (9 $\Rightarrow$ 26) Assume that 26 (II) is not true. This means that $\exists S \in \mathcal{S}$ such that $\nexists p$ such that $pA = 0, p \geqq 0$ with $\sum_{i \in S} p_i > 0$. In particular, $pA = 0, p \geqq 0 \Rightarrow p_i = 0, \forall i \in S$. But by 9, there exist $p \geqq 0$ and x such that $pA = 0, Ax \geqq 0$ and

$$p + x^T A^T > 0.$$

It follows that $(x^T A^T)_i = (Ax)_i > 0, \forall i \in S$, as required. (26 $\Rightarrow$ 13) Take $\mathcal{S} = \{\{1\}, \cdots, \{m\}\}$ containing subsets of the singletons. Then it is clear that the resulting statement of 26 becomes 13.
(**j**) 28 $\Rightarrow$ 12. Assume 28. In 12, we will show $\bar{\text{I}} \Rightarrow$ II. Following the method in [8], we have

$$\bar{\text{I}} \Rightarrow \langle < Bx \geqq 0, Cx \geqq 0, Dx = 0 > \Rightarrow < Bx = 0 > \rangle . \qquad (*)$$

But by 28, the systems $Bx \geqq 0, Cx \geqq 0, Dx = 0$ and $B'y_2 + C'y_3 + D'y_4 = 0, y_2 \geqq 0, y_3 \geqq 0$ have solutions $x, y_2 \geqq 0, y_3 \geqq 0, y_4$ such that $Bx + y_2 > 0$ and $Cx + y_3 > 0$. The condition $(*)$ then shows that $y_2 > 0, y_3 \geqq 0$, as required.

Combining items (a) through (j), we conclude that all 28 theorems in the list of Section 2 are equivalent.

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