European Graduate Program
Berlin - Zürich
CGC Combinatorics

01M2 Lecture
Basics of Polyhedral Theory

Martin Grötschel

Block Course at TU Berlin
"Combinatorial Optimization at Work“

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Contents

Martin Grötschel

Contents

Martin Grötschel

Linear Programming

$$\max c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$subject\ \ to$$

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$

$$.$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

linear program
in standard form

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Linear Programming

$$\max c^T x \quad Ax = b \quad x \geq 0$$

linear program in standard form

$$\max c^T x \quad Ax \leq b \quad -Ax \leq -b \quad -x \leq 0$$

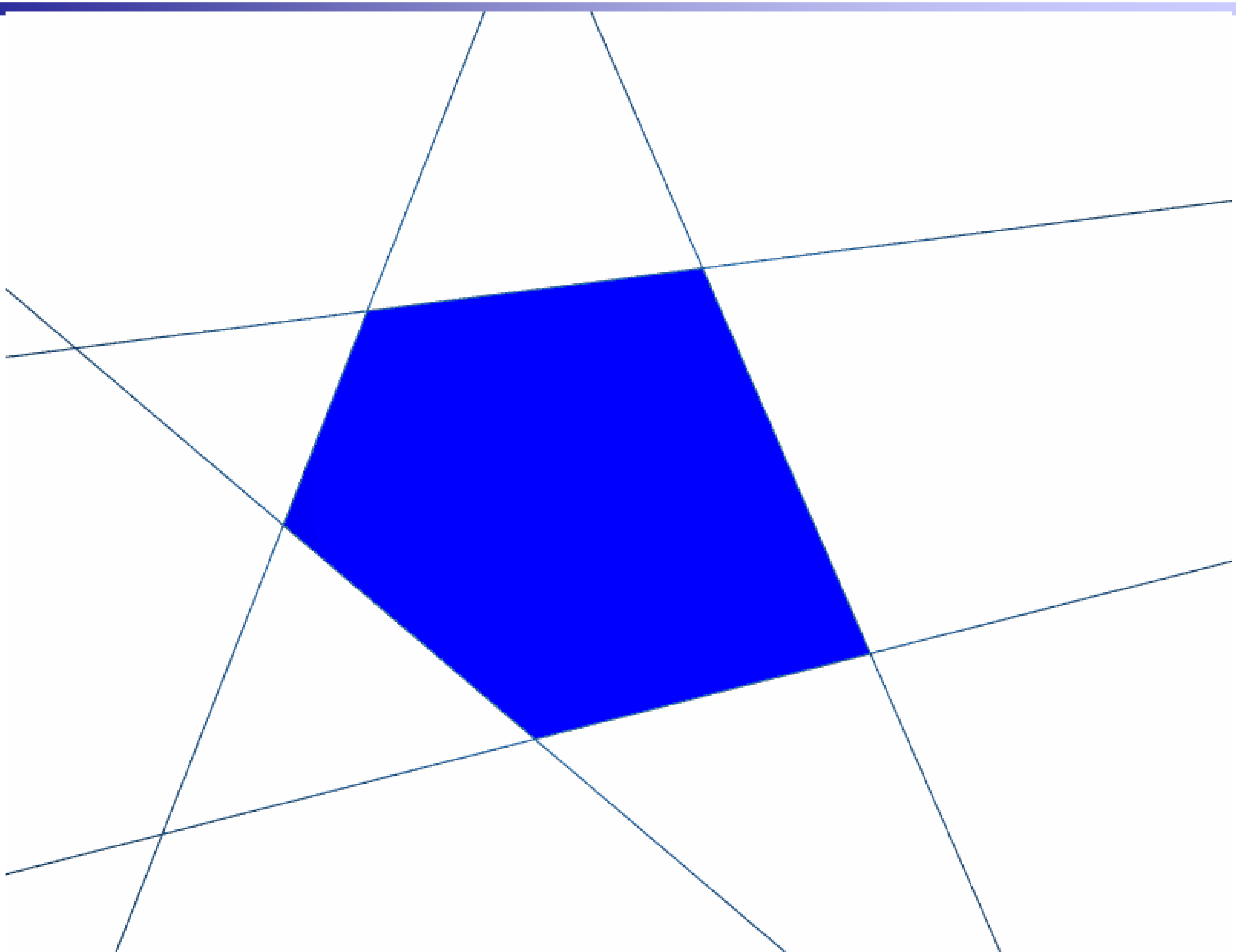
$$\max c^T x \quad Ax \leq b$$

linear program in "polyhedral form"

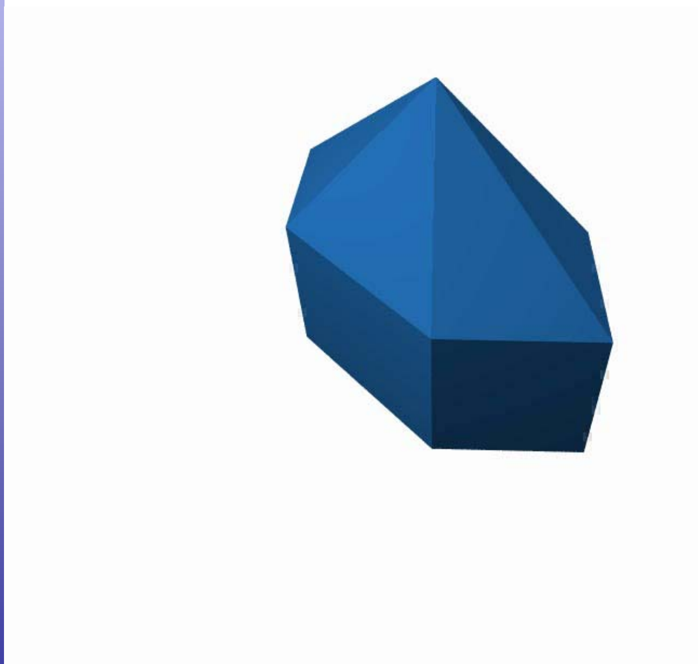
$$\max c^T x^+ - c^T x^- \quad Ax^+ + Ax^- + Is = b \quad x^+, x^-, s \geq 0 \quad (x = x^+ - x^-)$$

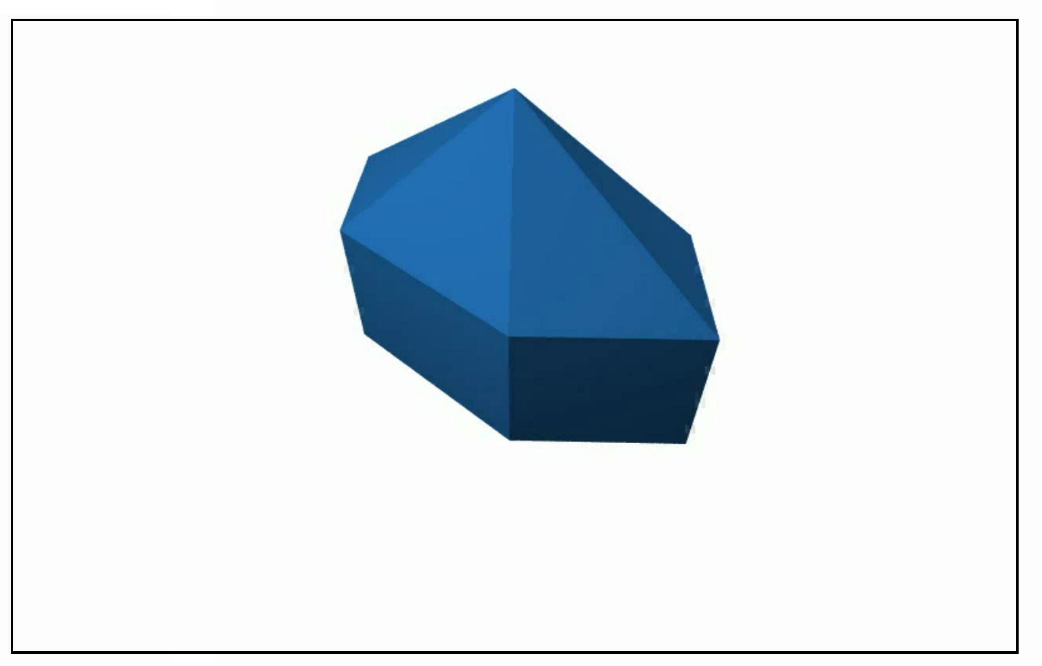
Contents

A Polytope in the Plane

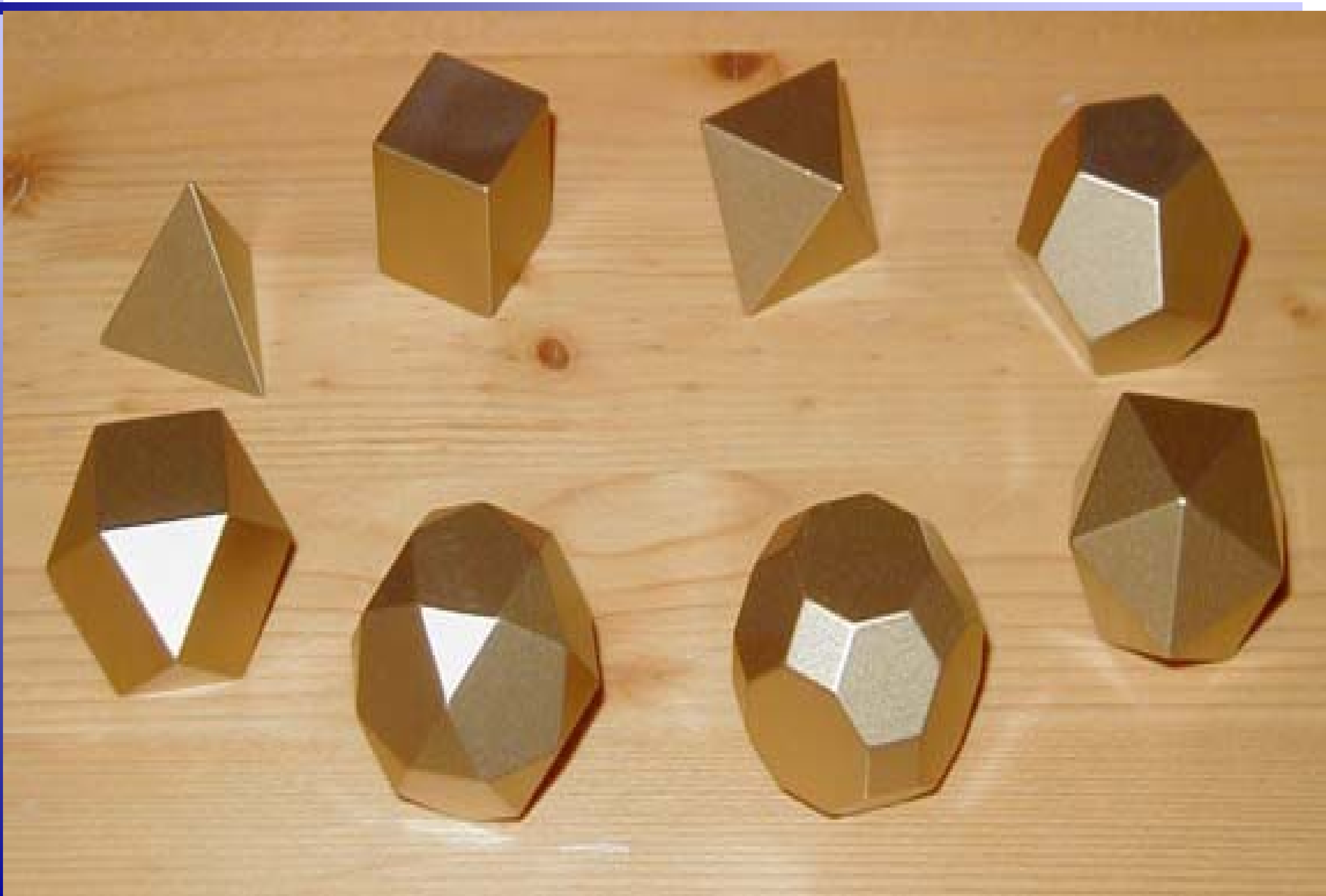


A Polytope in 3-dimensional space





„beautiful“ polyehedra

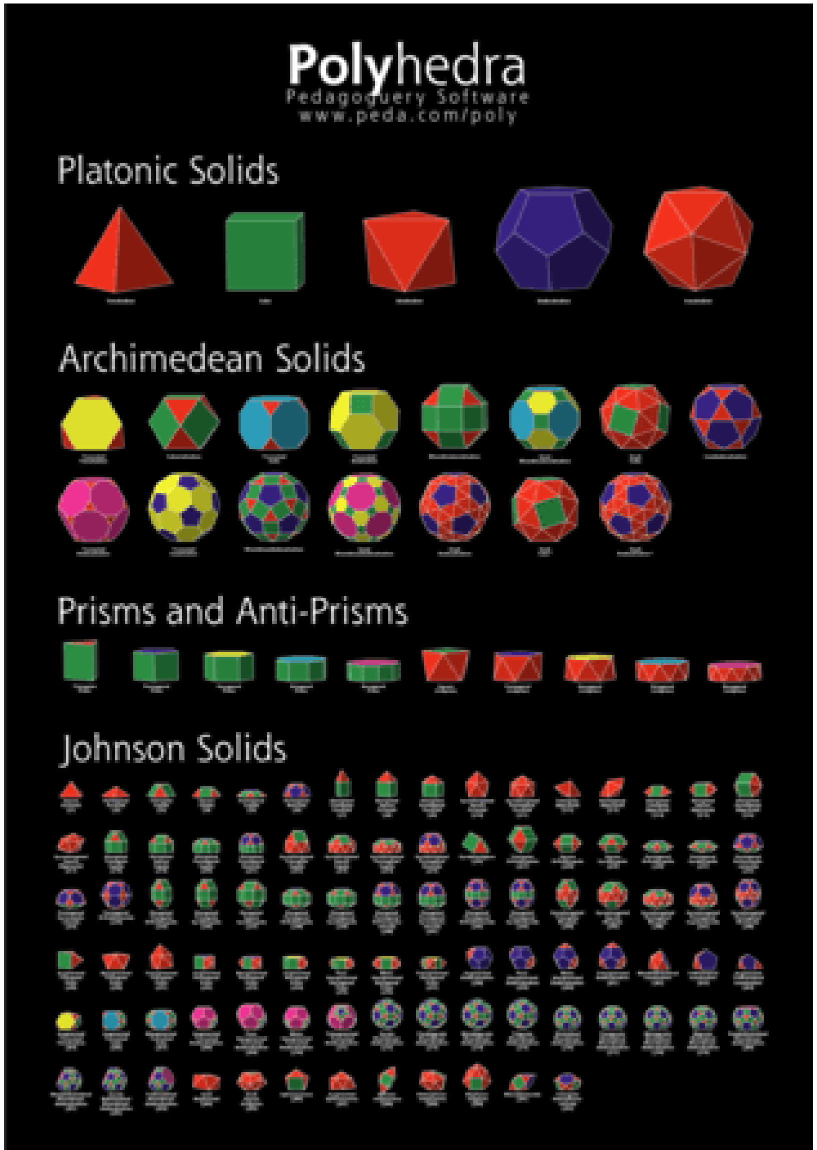


Polytopes in nature

- see examples

Polyhedra-Poster

http://www.peda.com/posters/Welcome.html



Poster
which displays all convex polyhedra with regular polygonal faces

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Electronic Geometry Models - Microsoft Internet Explorer

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Zurück Suchen Favoriten Medien

Adresse http://www.eg-models.de/ Wechseln zu Links

EG-Models

EG-Models - a new archive of electronic geometry models
Internal Links: Upload Review

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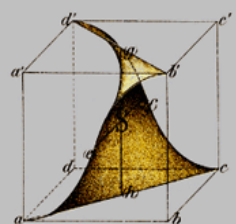


H.A. Schwarz Ges.Math.Abh
Springer Berlin 1890

Note: Some browser versions do not display Java applets. Please, press the 'No Applet' button in the navigation bar to avoid using Java.

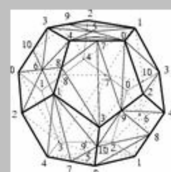
Anschauliche Geometrie - A tribute to Hilbert, Cohn-Vossen, Klein and all other geometers.

Electronic Geometry Models

This archive is open for any geometer to publish new geometric models, or to browse this site for material to be used in education and research. These geometry models cover a broad range of mathematical topics from geometry, topology, and to some extent from numerics.

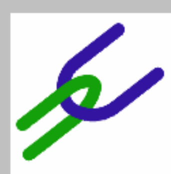
Click "Models" to see the full list of published models. See here for details on the submission and review process.

Selection of recently published models



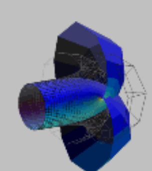
Model 2003.04.001 by Anders Björner and Frank H. Lutz: *A 16-Vertex Triangulation of the Poincaré Homology 3-Sphere and Non-PL Spheres with Few Vertices.*
Section: *Discrete Mathematics / Simplicial Manifolds*

We present a 16-vertex triangulation of the Poincaré homology 3-sphere that can be taken as the starting point for a series of non-PL d-spheres with d+13 vertices in dimensions $d \geqslant 5$.



Model 2001.11.001 by John M. Sullivan: *Tight Clasp.*
Section: *Curves / Space Curves*

This model simulates the shape of a tight clasp, that is, a ropelength-minimizing configuration of two linked arcs with endpoints fixed in parallel planes.



Model 2002.03.001 by Shimpei Kobayashi: *Bubbletons and their parallel surfaces in Euclidean 3-space.*
Section: *Surfaces / Mean Curvature Surfaces*

We show one of the cylinder bubbletons in Euclidean 3-space which are constant mean curvature surfaces derived by applying the Backlund-Bianchi transformation to the cylinder. We also show the parallel constant mean curvature surface of this cylinder bubbleton.

Internet

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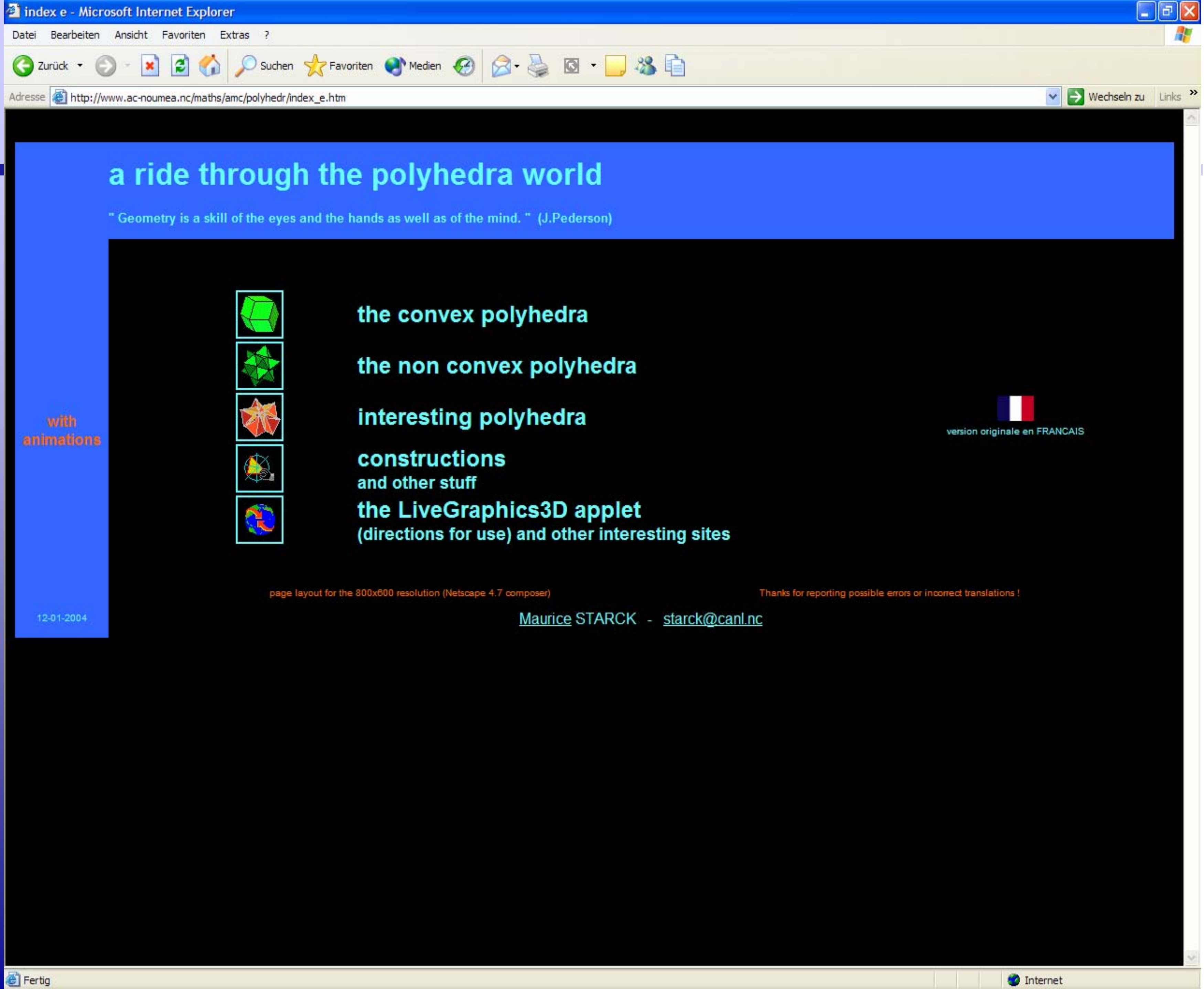
index e - Microsoft Internet Explorer
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Zurück Suchen Favoriten Medien
Adresse http://www.ac-noumea.nc/maths/amc/polyhedr/index_e.htm
Wechseln zu Links
a ride through the polyhedra world
" Geometry is a skill of the eyes and the hands as well as of the mind. " (J.Pederson)
with animations
the convex polyhedra
the non convex polyhedra
interesting polyhedra
constructions
and other stuff
the LiveGraphics3D applet
(directions for use) and other interesting sites
version originale en FRANCAIS
page layout for the 800x600 resolution (Netscape 4.7 composer)
Thanks for reporting possible errors or incorrect translations !
12-01-2004
Maurice STARCK - starck@canl.nc
Fertig
Internet

CO at Work

convex polyhedra 1 - Microsoft Internet Explorer

Datei Bearbeiten Ansicht Favoriten Extras ?

Zurück Suchen Favoriten Medien

Adresse http://www.ac-noumea.nc/maths/amc/polyhedr/convex1_.htm Wechseln zu Links

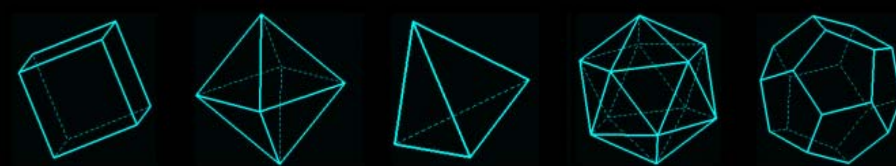
Plato's five regular polyhedra

The regular polyhedra are, in the space, the analogues of the regular polygons in the plane ; their faces are regular and identical polygons, and their vertices, regular and identical, are regularly distributed on a sphere. Their analogues in dimension four are the regular polytopes.

As we do for the polygons, we recognize a convex polyhedron by the very fact that all its diagonals (segments which join two vertices not joined by an edge) are inside the polyhedron.

Whereas there exist an infinity of regular convex polygons, the regular convex polyhedra are only five.

proof : The angle of a regular polygon with n sides is 180°(n-2)/n : 60° (triangle), 90° (square), 108° (pentagon), 120° (hexagon)...
On a vertex of a regular polyhedron the sum of the face's angles (there are at least three) must be smaller than 360°.
Since 6x60° = 4x90° = 3x120° = 360° < 4x108°, there are only five possibilities: 3, 4, or 5 triangles, 3 squares or 3 pentagons.

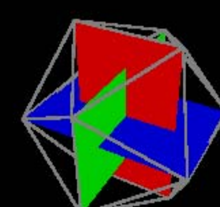


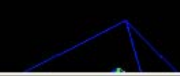
name	cube	octahedron	tetrahedron	icosahedron	dodecahedron
faces	6 squares	8 equil.triangles	4 equil.triangles	20 equil.triangles	12 regul.pentagons
vertices	8	6	4	12	20
edges	12	12	6	30	30
faces angle	90°	109°28'	70°32'	138°11'	116°34'

The LiveGraphics3D applet by Martin Kraus (University of Stuttgart) allows you to move these polyhedra with your mouse.

The regular octahedron's edges are the sides of three squares with the same centre and orthogonal by pairs.

The regular icosahedron's vertices are the vertices of three golden rectangles (sides in golden ratio 1.618...) with the same centre and orthogonal by pairs.

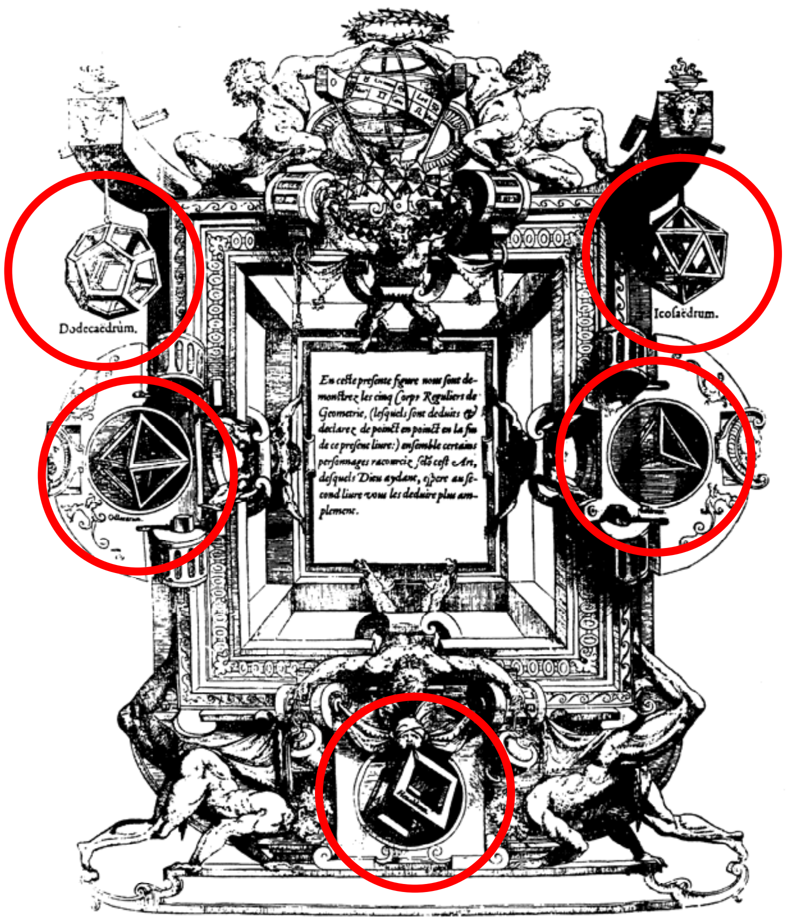




Four vertices of a cube are the vertices of a regular tetrahedron ; so we can make a regular tetrahedron by cutting four "corners" of a cube.

LiveGraphics3D 1.54: Please drag to rotate. Internet

Polyhedra have fascinated people during all periods of our history



From *Livre de Perspective* by Jean Cousin, 1568.

- book illustrations
- magic objects
- pieces of art
- objects of symmetry
- models of the universe

Definitions

Linear programming lives (for our purposes) in the n-dimensional real (in practice: rational) vector space.

- convex polyhedral cone: conic combination (i. e., nonnegative linear combination or conical hull) of finitely many points
  K = cone(E)
- polytope: convex hull of finitely many points:
  P = conv(V)
- polyhedron: intersection of finitely many halfspaces

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

Theorem of the alternative

Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

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Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

Minkowski (1896), Weyl (1935), Steinitz (1916) Motzkin (1936)

Theorem: For a subset P of $\mathbf{R}^n$ the following are equivalent:

(1) P is a polyhedron.

(2) P is the intersection of finitely many halfspaces, i.e., there exist a matrix A und ein vector b with

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}.$$ (exterior representation)

(3) P is the sum of a convex polytope and a finitely generated (polyhedral) cone, i.e., there exist finite sets V and E with

$$P = \text{conv(V)+cone(E)}.$$ (interior representation)

Representations of polyhedra

Carathéodory's Theorem (1911), 1873 Berlin – 1950 München

Let $x \in P = \text{conv}(V) + \text{cone}(E)$, there exist

$$v_0, \ldots, v_s \in V,\ \lambda_0, \ldots, \lambda_s \in \mathbf{R}_+, \sum_{i=0}^{s} \lambda_i = 1$$

$$\text{and } e_{s+1}, \ldots, e_t \in E,\ \mu_{s+1}, \ldots, \mu_t \in \mathbf{R}_+ \text{ with } t \leq n \text{ such that}$$

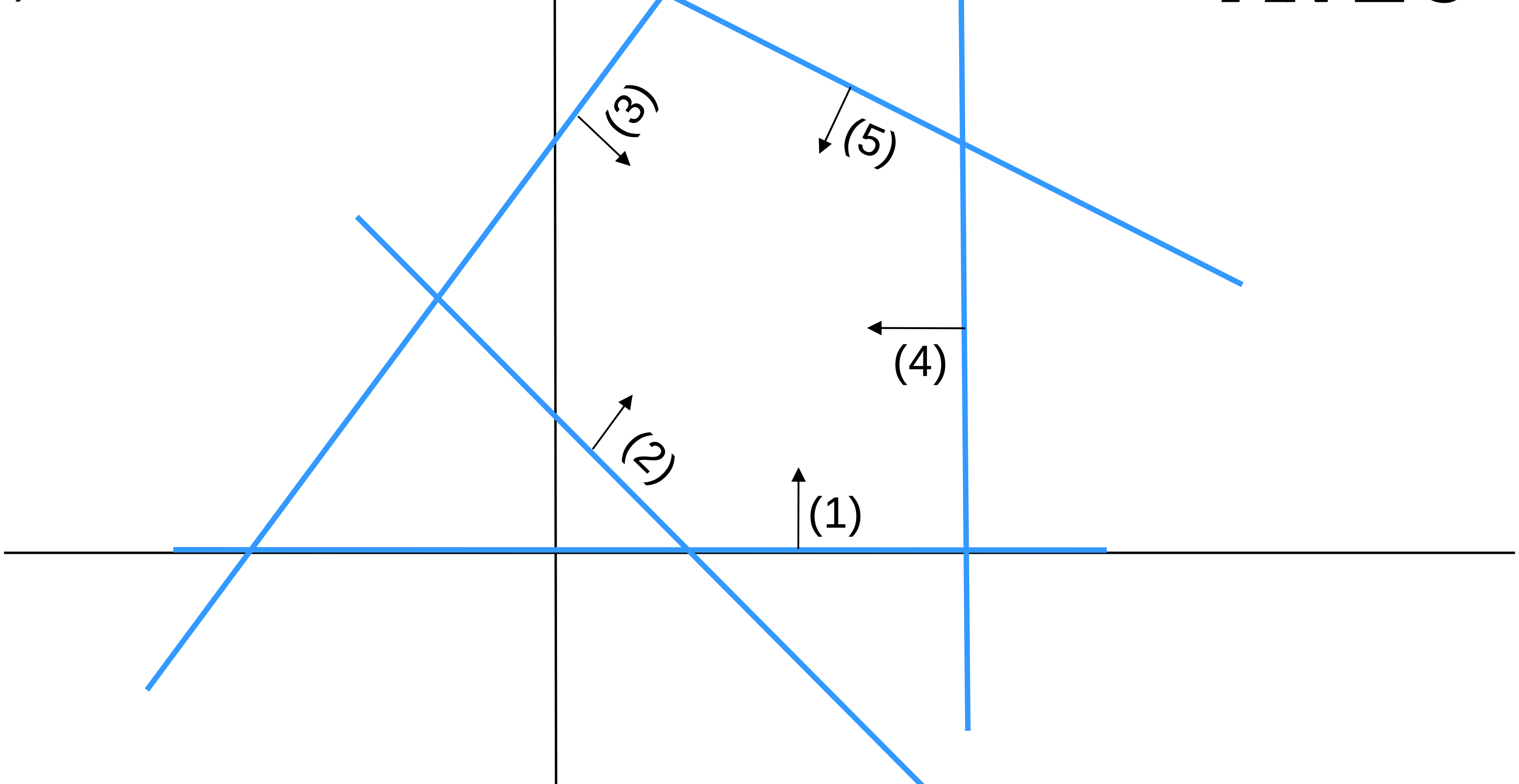
$$x = \sum_{i=1}^{s} \lambda_i v_i + \sum_{i=s+1}^{t} \mu_i e_i$$

Representations of polyhedra

```
(1)          -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

The H-representation
(exterior representation)

$$Ax \leq b$$



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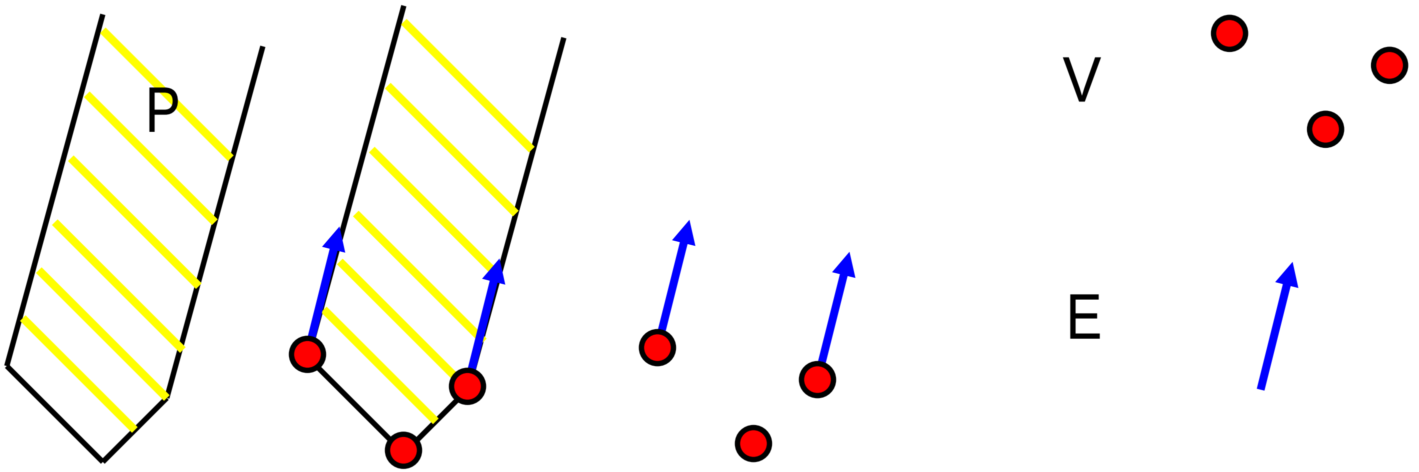
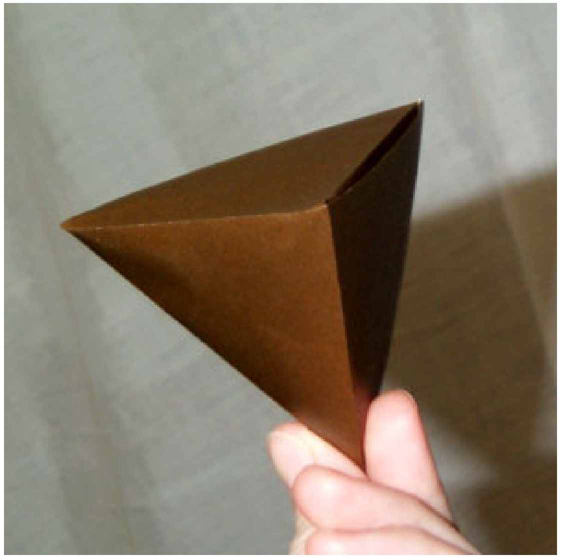
Representations of polyhedra

The ς-representation (interior representation)

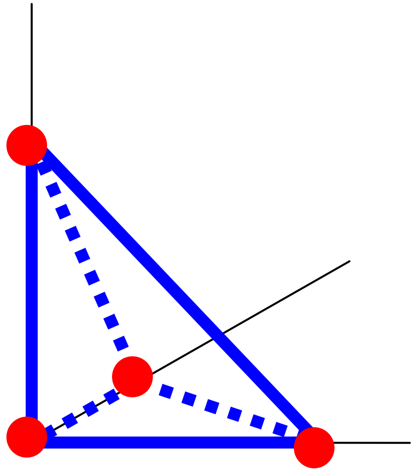
$$P = \mathrm{conv}(V) + \mathrm{cone}(E).$$



Example: the Tetrahedron



$$y \in conv\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$



$$y_1 + y_2 + y_3 \leq 1$$

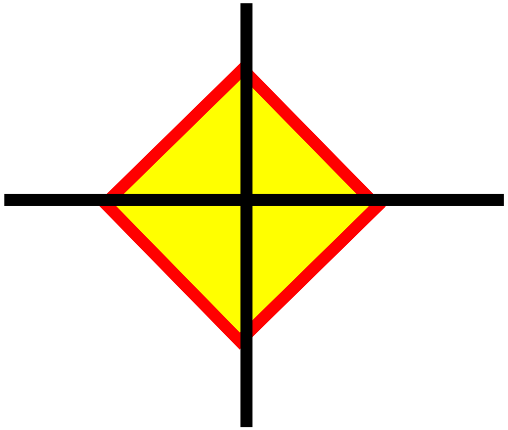
$$y_1 \geq 0$$

$$y_2 \geq 0$$

$$y_3 \geq 0$$

Example: the cross polytope

$$P = conv\{e_i, -e_i \mid i = 1, \ldots, n\} \subseteq \mathbf{R}^n$$



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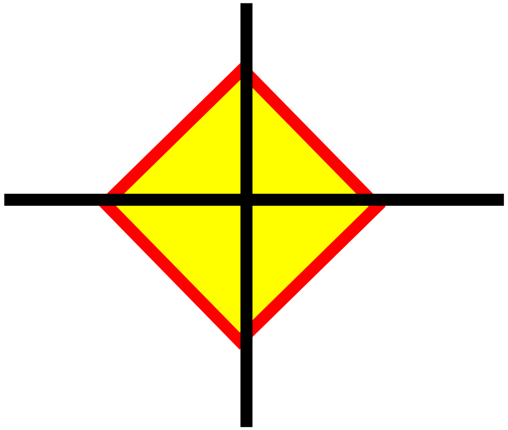
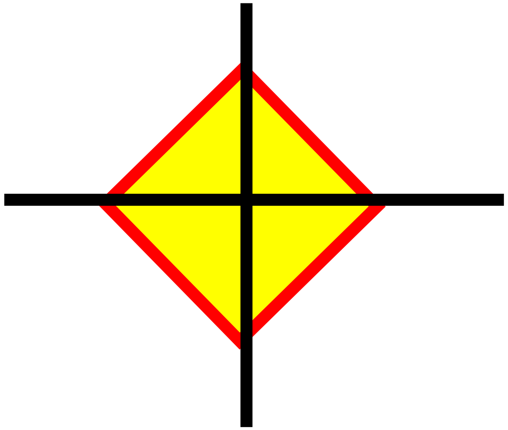


$$P = \{x \in \mathbf{R}^n \mid a^T x \leq 1 \ \forall \ a \in \{-1,1\}^n\}$$

Example: the cross polytope

$$P = conv\{e_i, -e_i \mid i = 1, \ldots, n\} \subseteq \mathbf{R}^n$$

$$P = \left\{ x \in \mathbf{R}^n \mid \sum_{i=1}^{n} |x_i| \leq 1 \right\}$$



$$P = \left\{ x \in \mathbf{R}^n \mid a^T x \leq 1 \ \forall \ a \in \{-1, 1\}^n \right\}$$

Contents

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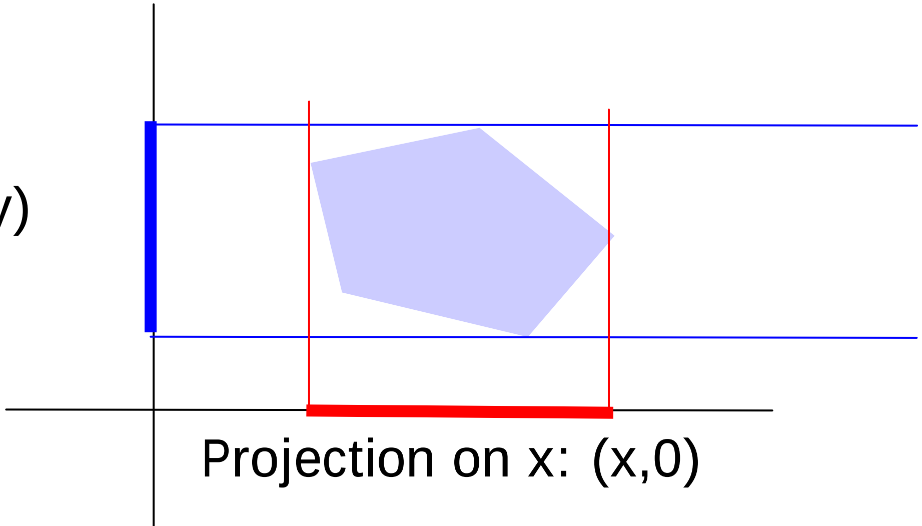
Polyedra in linear programming

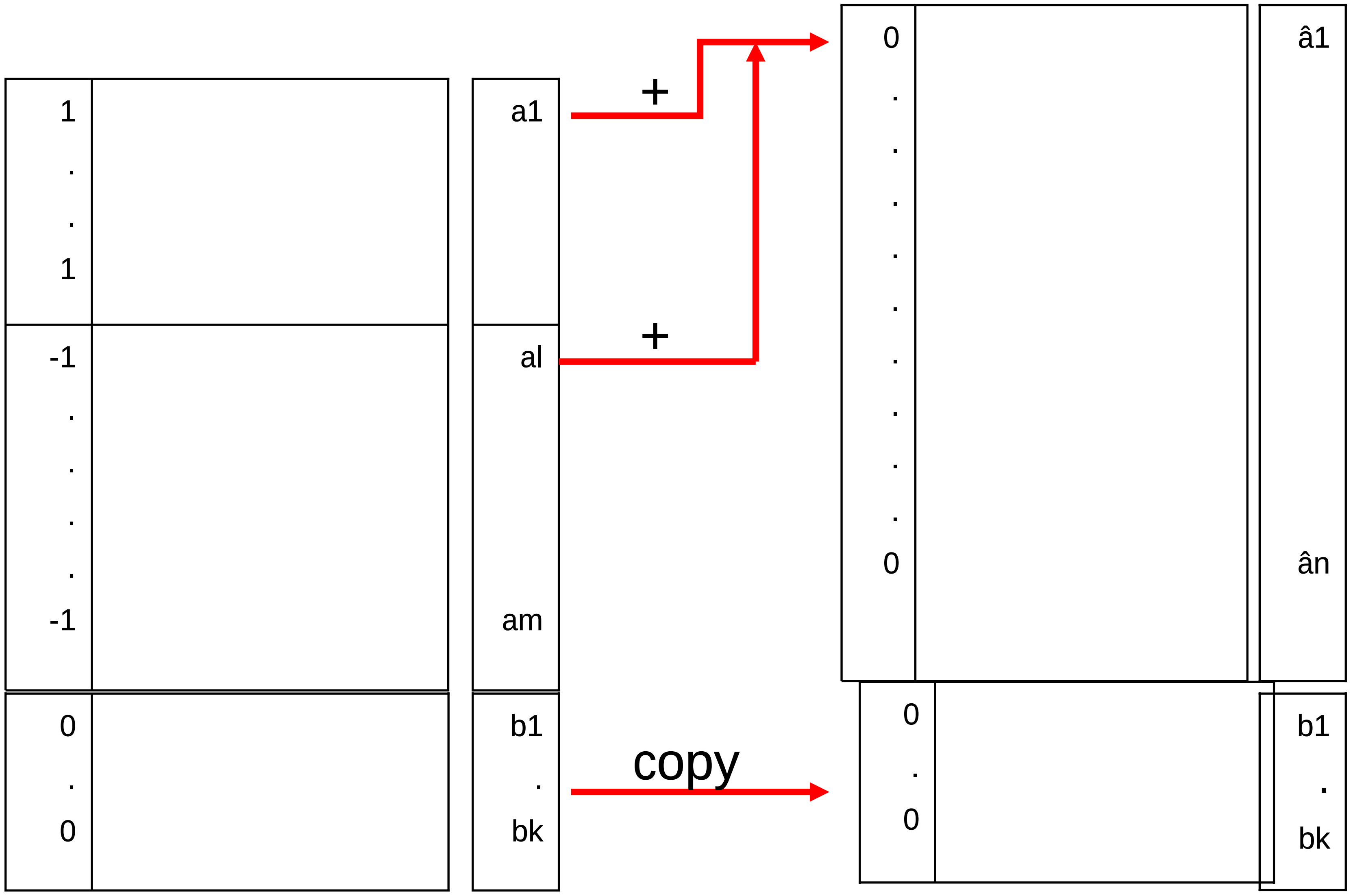
- The solution sets of linear programs are polyhedra.
- If a polyhedron $P = \text{conv(V)+cone(E)}$ is given explicitly via finite sets V und E, linear programming is trivial.
- In linear programming, polyhedra are always given in H-representation. Each solution method has its „standard form“.

Fourier-Motzkin Elimination

- Fourier, 1847
- Motzkin, 1938
- **Method**: successive projection of a polyhedron in n-dimensional space into a vector space of dimension n-1 by elimination of one variable.



A Fourier-Motzkin step



Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

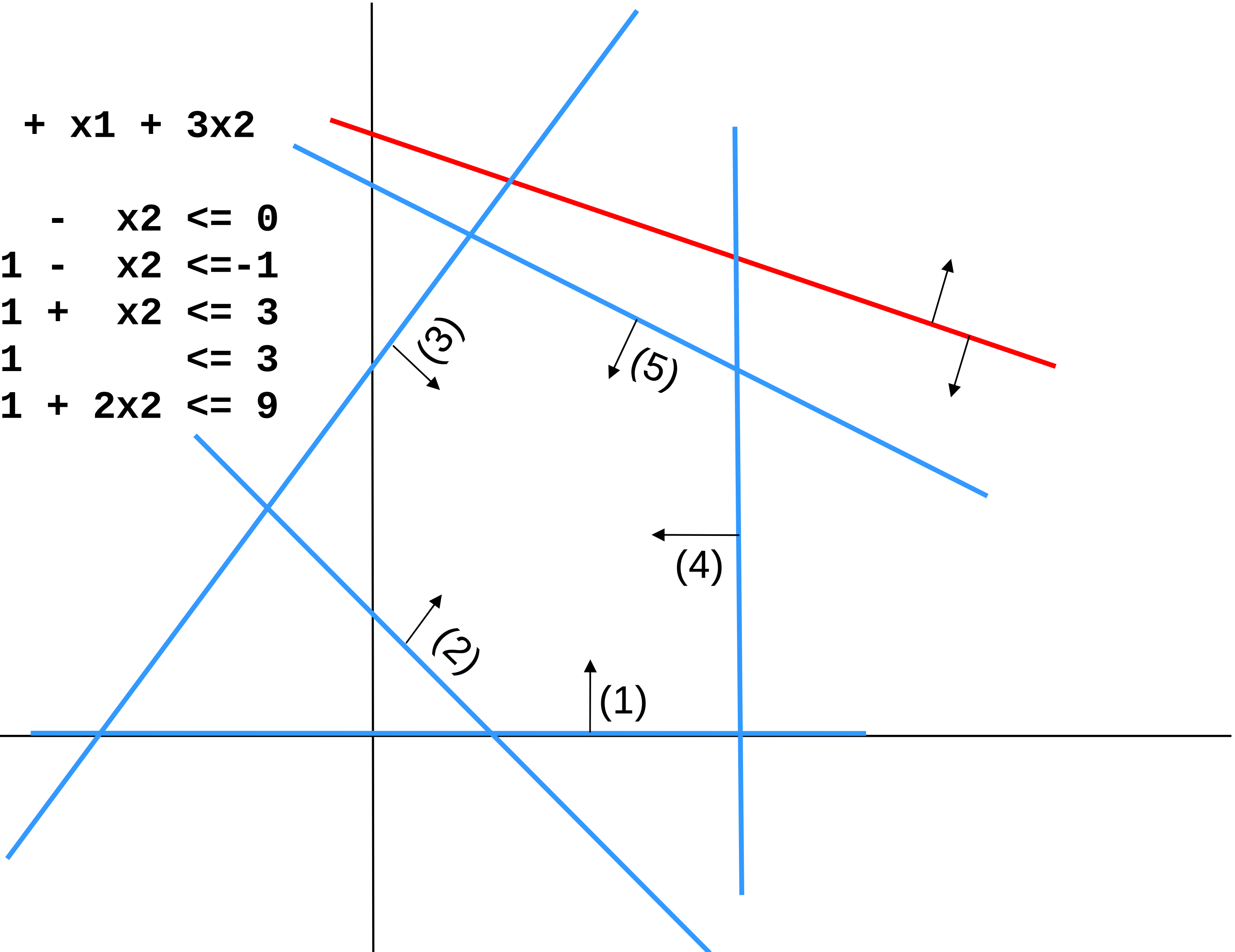
is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

Fourier-Motzkin Elimination: an example

```
min/max + x1 + 3x2

(1)        -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```



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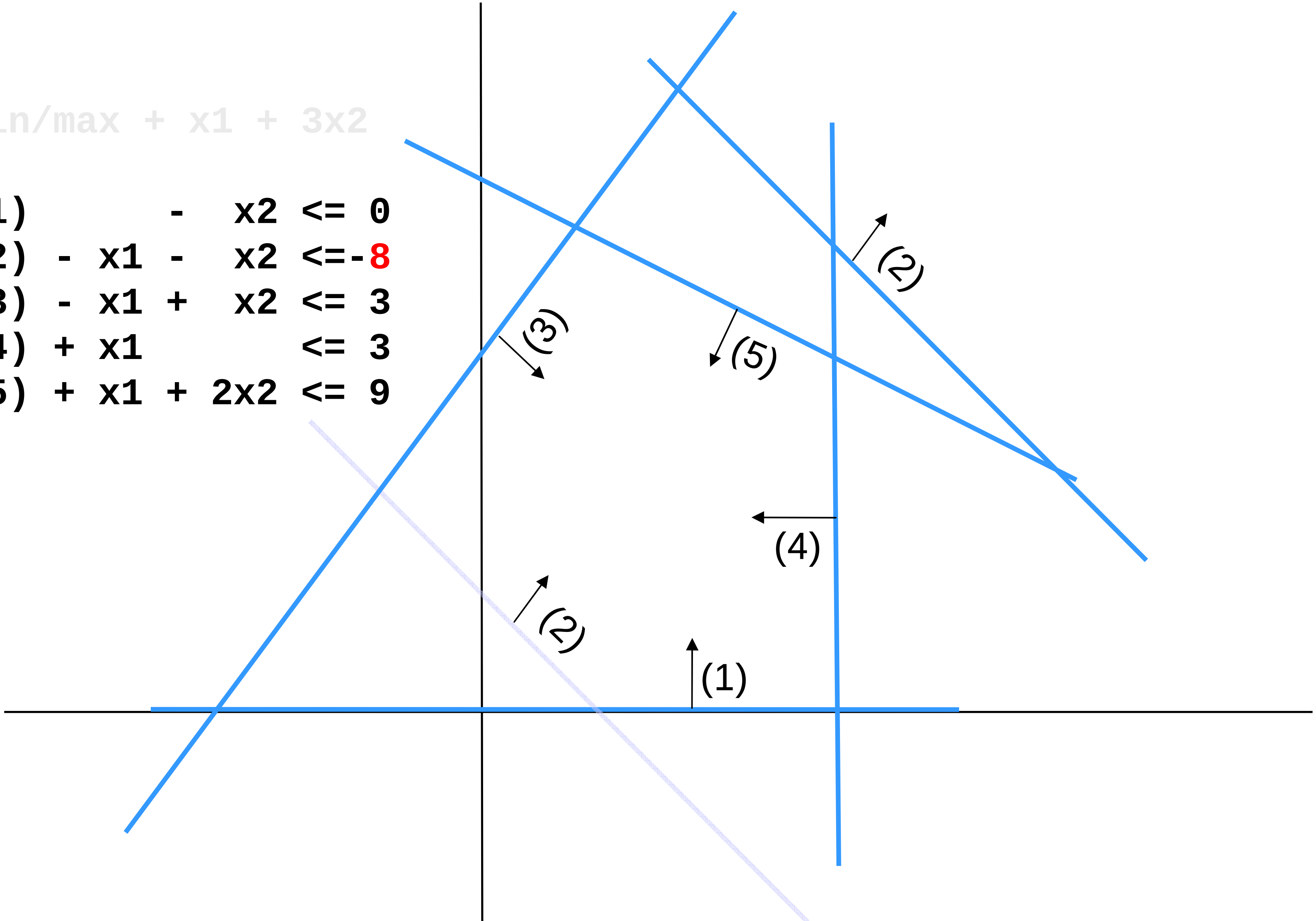
Fourier-Motzkin Elimination: an example

min/max + x1 + 3x2

```
(1)        -   x2 <= 0
(2) - x1 -   x2 <=-8
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```



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Fourier-Motzkin Elimination: an example, call of PORTA

```
(1)       -   x2 <= 0
(2) - x1 -   x2 <=-8
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

```
DIM = 3

INEQUALITIES_SECTION

(1)       -   x2       <= 0
(2) - x1 -   x2       <=-8
(3) - x1 +   x2       <= 3
(4) + x1              <= 3
(5) + x1 + 2x2       <= 9


ELIMINATION_ORDER
1 0
```

Fourier-Motzkin Elimination: an example, call of PORTA

```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2        <=  0
(2,4) (2) - x2        <=  -5
(2,5) (3) + x2        <=  1
(3,4) (4) + x2        <=  6
(3,5) (5) + x2        <=  4
```

```
DIM = 3

INEQUALITIES_SECTION

(1)        -   x2      <= 0
(2) - x1 -   x2      <=-8
(3) - x1 +   x2      <= 3
(4) + x1             <= 3
(5) + x1 + 2x2      <= 9


ELIMINATION_ORDER
1 0
```

Fourier-Motzkin Elimination: an example, call of PORTA

```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2      <=  0
(2,4) (2) - x2      <=  -5
(2,5) (3) + x2      <=  1
(3,4) (4) + x2      <=  6
(3,5) (5) + x2      <=  4

ELIMINATION_ORDER

0 1
```

```
DIM = 3

INEQUALITIES_SECTION

(2,3) 0 <= -4
```

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Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

Which LP solvers are used in practice?

- Fourier-Motzkin: hopeless
- Ellipsoid Method: total failure
- primal Simplex Method: good
- dual Simplex Method: better
- Barrier Method: for LPs frequently even better
- For LP relaxations of IPs: dual Simplex Method

Fourier-Motzkin works reasonably well for polyhedral transformations:

Example: Let a polyhedron be given (as usual in combinatorial optimization implicitly) via:

$$P = \text{conv(V)+cone(E)}$$

Find a non-redundant representation of P in the form:

$$P = \{x \in \mathbf{R}^d \mid Ax \leq b\}$$

Solution: Write P as follows

$$P = \{x \in \mathbf{R}^d \mid Vy + Ez - x = 0, \sum_{i=1}^{d} y_i = 1, y \geq 0, z \geq 0\}$$

and eliminate y und z.

Relations between polyhedra representations

- Given V and E, then one can compute A und b as indicated above.
- Similarly (polarity): Given A und b, one can compute V und E.
- The Transformation of a ς-representation of a polyhedron P into a H-representation and vice versa requires exponential space, and thus, also exponential running time.
- Examples: Hypercube and cross polytope.
- That is why it is OK to employ an exponential algorithm such as Fourier-Motzkin Elimination (or Double Description) for polyhedral transformations.
- Several codes for such transformations can be found in the Internet, e.g.. PORTA at ZIB and in Heidelberg.

The Polytope of stable sets of the Schläfli Graph

input file Schlaefli.poi
dimension : 27
number of cone-points : 0
number of conv-points : 208

sum of inequalities over all iterations : 527962
maximal number of inequalities : 14230

transformation to integer values
sorting system

number of equations : 0
number of inequalities : 4086

The Polytope of stable sets of the Schläfli Graph

FOURIER - MOTZKIN - ELIMINATION:

iter- ation	upper bound # ineq	# ineq	max bit- length	long arith metic	non- zeros in %	mem used in kB	time used in sec
180	29	29	1	n	0.04	522	1.00
179	30	29	1	n	0.04	522	1.00
10	8748283	13408	3	n	0.93	6376	349.00
9	13879262	12662	3	n	0.93	6376	368.00
8	12576986	11877	3	n	0.93	6376	385.00
7	11816187	11556	3	n	0.93	6376	404.00
6	11337192	10431	3	n	0.93	6376	417.00
5	9642291	9295	3	n	0.93	6376	429.00
4	10238785	5848	3	n	0.92	6376	441.00
3	3700762	4967	3	n	0.92	6376	445.00
2	2924601	4087	2	n	0.92	6376	448.00
1	8073	4086	2	n	0.92	6376	448.00

The Polytope of stable sets of the Schläfli Graph

INEQUALITIES_SECTION

(1) - x1 <= 0

(4086) +2x1+2x2+2x3+ x4+ x5+ x6 + x10+ x11+ x12+ x13+ x14+ x15
+x16+ x17+ x18+ x19+2x20 + x22+2x23 + x25+2x26 <= 3

8 different classes of inequalities found in total, among these, 5 classes have been unknown so far.

Web resources

Linear Programming: Frequently Asked Questions

http://www-unix.mcs.anl.gov/otc/Guide/faq/linear-programming-faq.html

- Q1. "What is Linear Programming?"
- Q2. "Where is there good software to solve LP problems?"
  - "Free" codes
  - Commercial codes and modeling systems
  - Free demos of commercial codes
- Q3. "Oh, and we also want to solve it as an integer program."
- Q4. "I wrote an optimization code. Where are some test models?"
- Q5. "What is MPS format?"

Web resources

- A Short Course in Linear Programming

  by Harvey J. Greenberg

  http://carbon.cudenver.edu/~hgreenbe/courseware/LPshort/intro.html

- *OR/MS Today* : 2003 LINEAR PROGRAMMING SOFTWARE SURVEY (~50 commercial codes)

  http://www.lionhrtpub.com/orms/surveys/LP/LP-survey.html

- INFORMS OR/MS Resource Collection

  http://www.informs.org/Resources/

- NEOS Server for Optimization

  http://www-neos.mcs.anl.gov/

Web resources (at ZIB)

- MIPLIB
- FAPLIB
- STEINLIB

ZIB offerings

- **PORTA -** POlyhedron Representation Transformation Algorithm
- **SoPlex -** The **S**equential **o**bject-oriented sim**plex** class library
- **Zimpl -** A mathematical modelling language
- **SCIP -** Solving constraint integer programs (IP & MIP)

Contents

Martin Grötschel

Semi-algebraic Geometry
Real-algebraic Geometry

$f_i(x), g_j(x), h_k(x)$ are polynomials in d real variables

$$S_{\geq} := \{x \in \mathbf{R}^{d} : f_1(x) \geq 0, \ldots, f_l(x) \geq 0\}$$ basic closed

$$S_{>} := \{x \in \mathbf{R}^{d} : g_1(x) > 0, \ldots, g_m(x) > 0\}$$ basic open

$$S_{=} := \{x \in \mathbf{R}^{d} : h_1(x) = 0, \ldots, h_n(x) = 0\}$$

$S := S_{\geq} \cup S_{>} \cup S_{=}$ is a semi-algebraic set

Martin Grötschel

Theorem of Bröcker(1991) & Scheiderer(1989) basic closed case

Every basic closed semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^{d}: f_1(x) \geq 0, \ldots, f_l(x) \geq 0\},$$

where $f_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \leq i \leq l,$ are polynomials,

can be represented by at most $d(d+1)/2$

polynomials, i.e., there exist polynomials

such that

$$p_1, \ldots, p_{d(d+1)/2} \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^d : p_1(x) \geq 0, \ldots, p_{d(d+1)/2}(x) \geq 0\}.$$

Theorem of Bröcker(1991) & Scheiderer(1989) basic open case

Every basic open semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^d : f_1(x) > 0, \ldots, f_l(x) > 0\},$$

where $f_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \le i \le l,$ are polynomials,

can be represented by at most d

polynomials, i.e., there exist polynomials

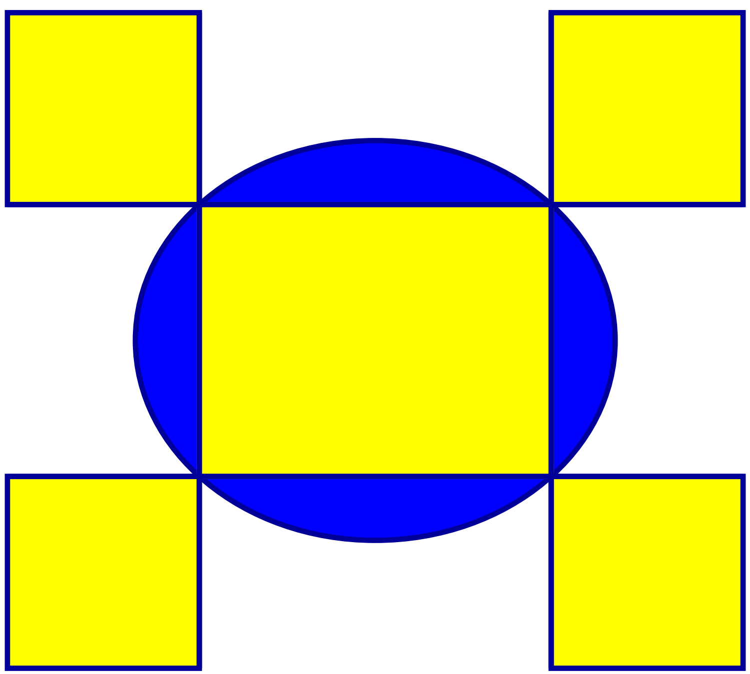
such that

$$p_1, \ldots, p_d \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^d : p_1(x) > 0, \ldots, p_d(x) > 0\}.$$

A first constructive result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.

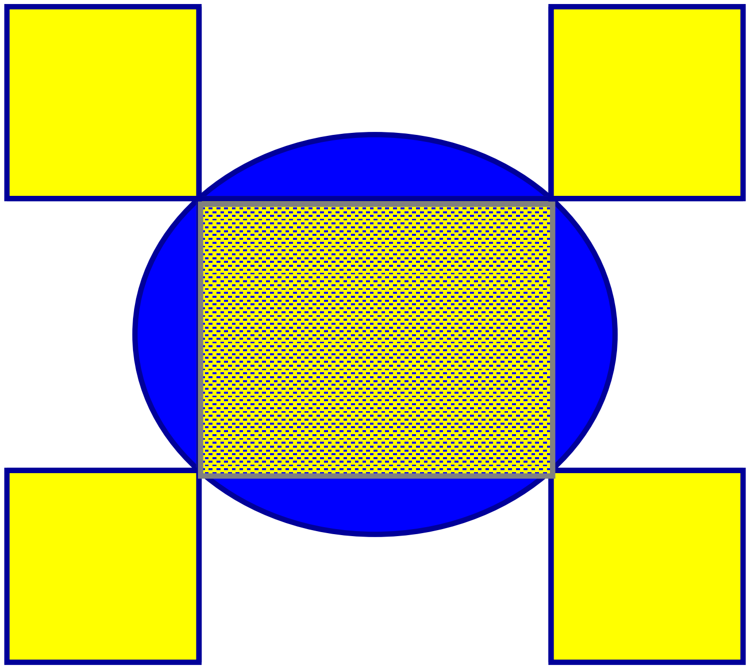


p(1)= product of all linear inequalities

p(2)= ellipse

A first Constructive Result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.



p(1)= product of all linear inequalities

p(2)= ellipse

Our first theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional polytope given by an inequality representation. Then

$k \star n^n$ polynomials $\quad p_i \in \mathbf{R}[x_1,\ldots,x_n]$

can be constructed such that

$$P = \mathrm{P}(p_1,\ldots,p_k).$$

Martin Grötschel, Martin Henk:

The Representation of Polyhedra by Polynomial Inequalities

Discrete & Computational Geometry, 29:4 (2003) 485-504

Our main theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional polytope given by an inequality representation. Then

2n polynomials $p_i \in \mathbf{R}[x_1, \ldots, x_n]$

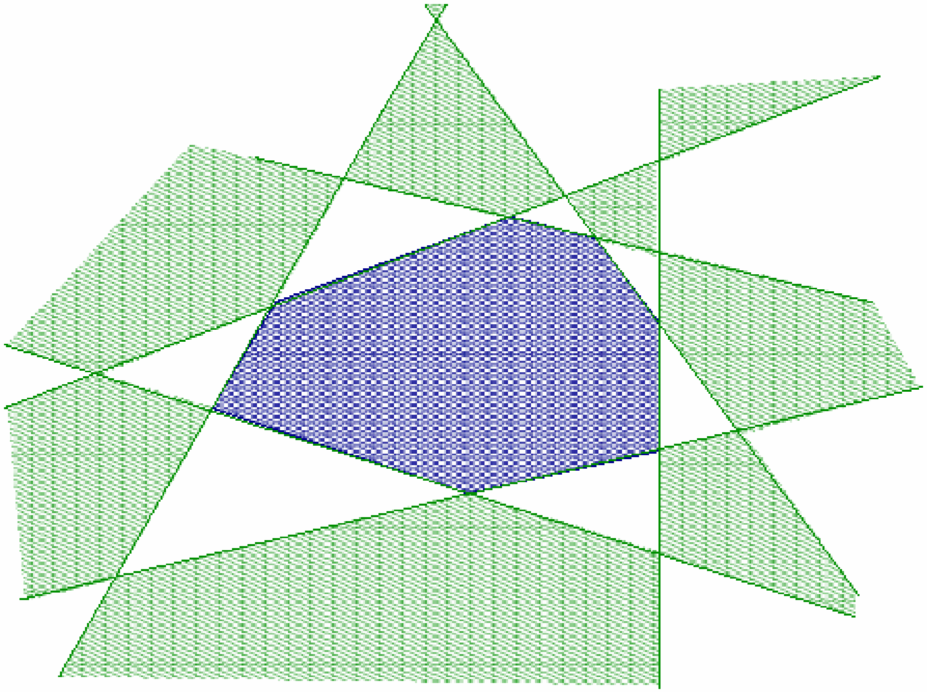
can be constructed such that

$$P = \mathrm{P}(p_1, \ldots, p_{2n}).$$

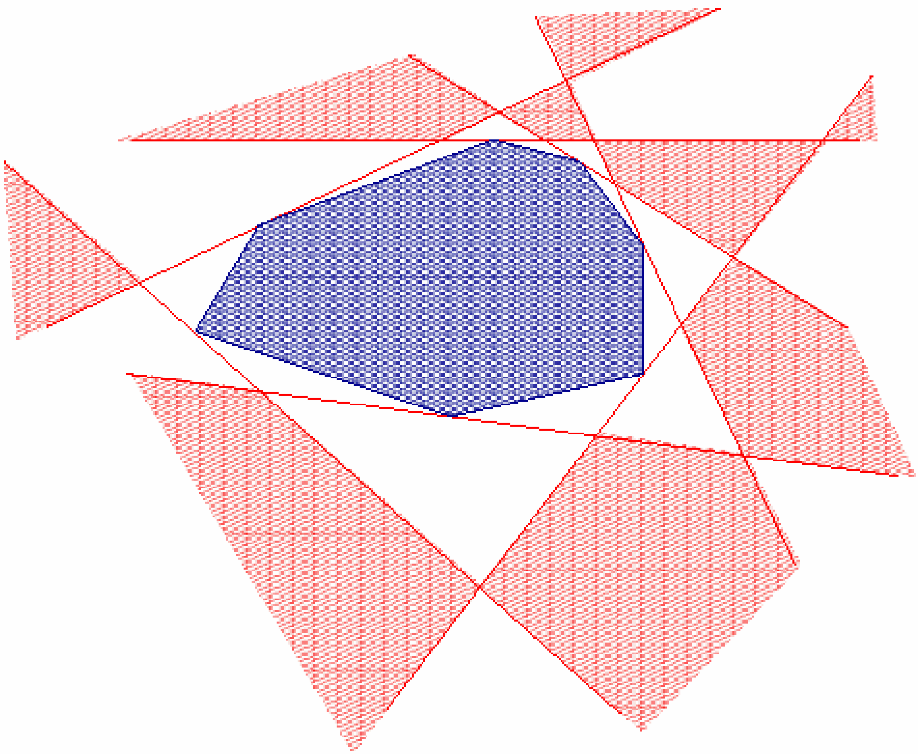
Hartwig Bosse, Martin Grötschel, Martin Henk:
Polynomial inequalities representing polyhedra
Mathematical Programming 103 (2005)35-44

http://www.springerlink.com/index/10.1007/s10107-004-0563-2

The construction in the 2-dimensional case

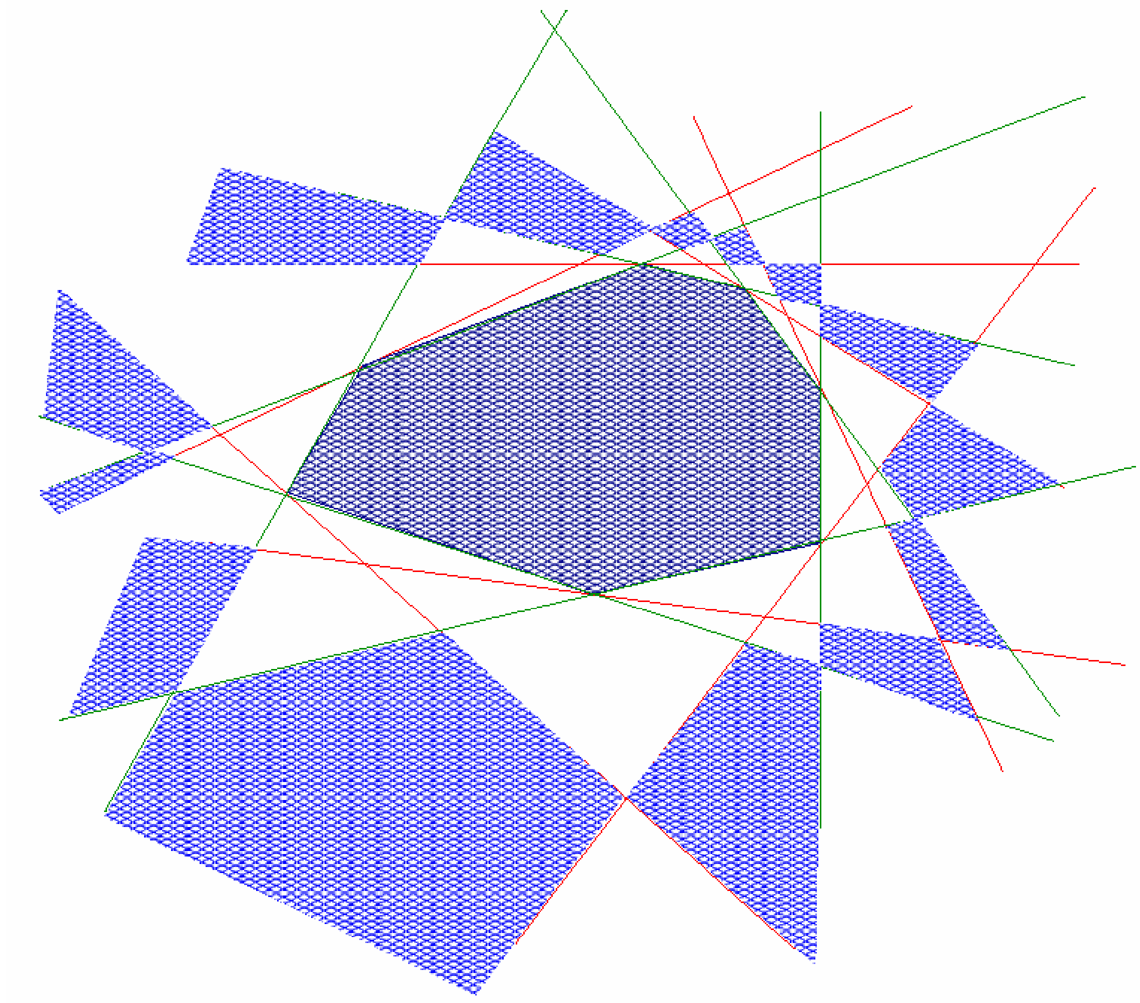


$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0\}$



$\{x \in \mathbb{R}^d : \mathfrak{p}_0(x) \geq 0\}$

The construction in the 2-dimensional case



$$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0 \text{ and } \mathfrak{p}_0(x) \geq 0\}$$

Martin Grötschel

Contents

Faces etc.

- Important concept: dimension

- face
- vertex
- edge
- (neighbourly polytopes)
- ridge = subfacet
- facet

European Graduate Program Berlin - Zürich
CGC Combinatorics

01M2 Lecture

Basics of Polyhedral Theory

The End

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