BSc Mathematics for Computer Scientists 2: XII Applications of Riemann Integration

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Riemann Integral: Antiderivatives: Reminder

Example

$\int 0\ dx = c$

$\int e^x\ dx = e^x$

$\int \frac{1}{x}\ dx = \ln x + c \quad (x \in \mathbb{R}_{++})$

$\int \cos x\ dx = \sin x$

$\int \sec^2 x\ dx = \tan x + c$

$\int \sec x \tan x\ dx = \sec x + c$

$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + c$

$\int \frac{1}{1+x^2}\ dx = \arctan x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}}\ dx = \operatorname{arcsec} x + c$

$\int \cosh x\ dx = \sinh x + c$

$\int \operatorname{sech}^2 x\ dx = \tanh x + c$

$\int \operatorname{sech} x \tanh x\ dx = -\operatorname{sech} x + c$

$\int \frac{1}{\sqrt{1+x^2}}\ dx = \operatorname{arsinh} x + c$

$\int \frac{1}{1-x^2}\ dx = \operatorname{artanh} x + c \quad (|x| < 1)$

$\int \frac{1}{x\sqrt{1-x^2}}\ dx = -\operatorname{arsech} x + c$

$\int x^\alpha\ dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$

$\int a^x\ dx = \frac{1}{\ln a} a^x \quad (a > 0,\ a \neq 1)$

$\int \frac{1}{x}\ dx = \ln(-x) + c\ (x \in \mathbb{R}_{--})$

$\int \sin x\ dx = -\cos x + c$

$\int \csc^2 x\ dx = \cot x + c.$

$\int \csc x \cot x = -\csc x + c$

$\int \frac{1}{\sqrt{1-x^2}}\ dx = -\arccos x + c$

$\int \frac{1}{1+x^2}\ dx = -\operatorname{arccot} x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}} = -\operatorname{arccsc} x + c$

$\int \sinh x\ dx = \cosh x + c$

$\int \operatorname{csch}^2 x\ dx = -\coth x + c$

$\int \operatorname{csch} x \coth x\ dx = -\operatorname{csch} x + c$

$\int \frac{1}{\sqrt{x^2-1}}\ dx = \operatorname{arcosh} x + c \quad (x > 1)$

$\int \frac{1}{1-x^2}\ dx = \operatorname{arcoth} x + c \quad (x > 1)$

$\int \frac{1}{|x|\sqrt{1+x^2}}\ dx = \operatorname{arcsch} x + c.$

Further Integration Techniques: Trigonometric Substitutions

Example

$\int \sqrt{1-x^2}\ dx.$

• On the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we have $\sqrt{1-\sin^2 x} = \cos x$. The derivative of sin is cos, its trigonometric "pair". Therefore it is worth trying the substitution $x = \sin\theta$.

$$\int \sqrt{1-x^2}\ dx \quad \left[\substack{x=\sin\theta \\ dx=\cos\theta d\theta \\ \sqrt{1-x^2}=\cos\theta}\right] = \int \cos\theta \cdot \cos\theta d\theta = \int \cos^2\theta d\theta$$

$$\left(\cos^2\theta = \tfrac{1+\cos 2\theta}{2}\right) = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$$

$$\left(\sin 2\theta = 2\sin\theta\cos\theta\right) \quad = \quad \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + c.$$

$$\int_0^1 \sqrt{1-x^2}\ dx = \left[\frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}\right]_0^1 = \frac{\pi}{4} \approx 0.785.$$

Further Integration Techniques: Trigonometric Substitutions

Example

$\int \frac{1}{1+x^2}\ dx.$

If differentiation is already familiar, then this is a basic integral. OR we notice that $x = \tan\theta$ is a suitable substitution.

$$\int \frac{1}{1+x^2}\ dx \quad \begin{bmatrix} x=\tan\theta \\ dx=\frac{1}{\cos^2\theta}d\theta \\ 1+x^2=\frac{1}{\cos^2\theta} \end{bmatrix} = \int \cos^2\theta \frac{1}{\cos^2\theta} d\theta = \int d\theta = \theta + c$$

$$= \arctan x + c.$$

$$\int_0^{\sqrt{3}} \frac{1}{1+x^2}\ dx = [\arctan x]_0^{\sqrt{3}} = \frac{\pi}{3} \approx 1.047.$$

Further Integration Techniques: Trigonometric Substitutions

Example

$\int \frac{1}{(1+x^2)^2}\ dx.$

This is no longer a basic integral. HOWEVER, it is an important one. Partial fraction decomposition does not simplify the problem. On the other hand, the substitution $x = \tan\theta$ is again useful.

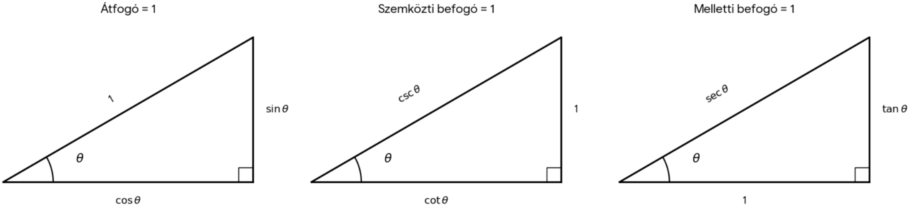
$$\int \frac{1}{(1+x^2)^2}\ dx \quad \begin{bmatrix} x=\tan\theta \\ dx=\frac{1}{\cos^2\theta}d\theta \\ 1+x^2=\frac{1}{\cos^2\theta} \end{bmatrix} = \int \cos^4\theta \frac{1}{\cos^2\theta} d\theta = \int \cos^2\theta\ d\theta$$

$$\left(\cos^2\theta = \frac{1+\cos 2\theta}{2}\right) = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$$

$$\left(\sin 2\theta = 2\sin\theta\cos\theta\right) \quad = \quad \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c = \frac{1}{2}\theta + \frac{1}{2}\frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta} + c$$

$$= \quad \frac{1}{2}\theta + \frac{1}{2}\frac{\tan\theta}{\tan^2\theta+1} + c = \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{1+x^2} + c.$$

Further Integration Techniques: Trigonometric Substitutions: The Geometric Picture



$$x = \sin\theta \qquad\qquad x = \cot\theta \qquad\qquad x = \tan\theta$$
$$\sqrt{1-x^2} = \cos\theta \qquad \sqrt{1+x^2} = \csc\theta \qquad \sqrt{1+x^2} = \sec\theta$$

Break

Improper Integrals: A First Step

- Assume that $f : [a, b[\to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. That is, $\int_a^{b-h}$ is defined for sufficiently small positive h.

Definition

Assume that $f : [a, b[\to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. If

$$\lim_{h \to 0+} \int_a^{b-h} f(x)\, dx$$

exists and is finite (i.e. belongs to $\mathbb{R}$), then the improper integral $\int_a^b f(x)\, dx$ is defined, and its value is the value of the limit:

$$\int_a^b f(x)\, dx = \lim_{h \to 0+} \int_a^{b-h} f(x)\, dx.$$

- Particularly important cases: (1) $b = \infty$, in which case $\int_a^{\infty - h}$ means that integration is performed outside a ball of radius h around ∞, (2) b is a point of discontinuity of a function.

Improper Integrals: A Second Step

- Assume that $f :]a, b] \to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. That is, $\int_{a+h}^{b}$ is defined for sufficiently small positive h.

Definition

Assume that $f :]a, b] \to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. If

$$\lim_{h \to 0+} \int_{a+h}^{b} f(x)\, dx$$

exists and is finite (i.e. belongs to $\mathbb{R}$), then the improper integral $\int_a^b f(x)\, dx$ is defined, and its value is the value of the limit:

$$\int_a^b f(x)\, dx = \lim_{h \to 0+} \int_{a+h}^{b} f(x)\, dx.$$

- Particularly important cases: (1) $a = -\infty$, (2) a is a point of discontinuity of a function $f : \mathbb{R} \to \mathbb{R}$.

Improper Integrals: The Definition

• Assume that $f :]a, b[\to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. That is, $\int_{a+h}^{b-h'}$ is defined for sufficiently small positive h, h'.

Definition

Assume that $f :]a, b[\to \mathbb{R}$ and that f is integrable on every closed subinterval of dom f. If for some $m \in]a, b[$ both

$$\int_a^m f(x)\,dx \qquad \text{AND} \qquad \int_m^b f(x)\,dx$$

exist, then the improper integral $\int_a^b f(x)\,dx$ is defined, and its value is

$$\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx.$$

• The point m appearing in the definition has no essential role. If one choice of m works, then any $m \in]a, b[$ may be chosen. Why?

Example

$$\int_1^\infty \frac{1}{x^2}\,dx = ?$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + c.$$

$$\int_1^N \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^N = -\frac{1}{N} - (-1) = 1 - \frac{1}{N}.$$

$$\lim_{N\to\infty}\left(1 - \frac{1}{N}\right) = 1.$$

Therefore the improper integral exists, and its value is 1.

Improper Integrals: Examples

Example

$$\int_1^\infty \frac{1}{x}\, dx = ?$$

$$\int \frac{1}{x}\, dx = \ln x + c.$$

$$\int_1^N \frac{1}{x}\, dx = [\ln x]_1^N = \ln N - \ln 1 = \ln N.$$

$$\lim_{N\to\infty} \ln N = \infty.$$

Therefore the improper integral diverges.

Improper Integrals: Examples

Example

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = ?$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c.$$

$$\int_\epsilon^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_\epsilon^1 = 2\sqrt{1} - 2\sqrt{\epsilon} = 2 - 2\sqrt{\epsilon}.$$

$$\lim_{\epsilon \to 0+} \left(2 - 2\sqrt{\epsilon}\right) = 2.$$

Therefore the improper integral exists, and its value is 2.

Improper Integrals: Examples

Example

$$\int_0^{\infty} e^{-3x}\,dx = ?$$

$$\int e^{-3x}\,dx = \frac{e^{-3x}}{-3} + c = -\frac{1}{3}e^{-3x} + c.$$

$$\int_0^N e^{-3x}\,dx = \left[-\frac{1}{3}e^{-3x}\right]_0^N = -\frac{1}{3}e^{-3N} + \frac{1}{3}e^{0} = \frac{1}{3} - \frac{1}{3}e^{-3N}.$$

$$\lim_{N\to\infty}\left(\frac{1}{3} - \frac{1}{3}e^{-3N}\right) = \frac{1}{3}.$$

Therefore the improper integral exists, and its value is $\frac{1}{3}$.

Example

$$\int_0^\infty xe^{-x}\,dx = ?$$

$$\int xe^{-x}\,dx \begin{bmatrix} u=x, & v'=e^{-x} \\ u'=1, & v=-e^{-x} \end{bmatrix} = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + c.$$

$$\begin{aligned} \int_0^N xe^{-x}\,dx &= \left[-xe^{-x} - e^{-x}\right]_0^N \\ &= (-Ne^{-N} - e^{-N}) - (-0e^0 - e^0) \\ &= 1 - Ne^{-N} - e^{-N}. \end{aligned}$$

$$\lim_{N\to\infty}\left(1 - Ne^{-N} - e^{-N}\right) = 1.$$

Therefore the improper integral exists, and its value is 1.

Improper Integrals: Examples

Example

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = ?$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + c.$$

$$\int_0^N \frac{1}{1+x^2}\, dx = [\arctan x]_0^N = \arctan N,$$

$$\int_{-N}^0 \frac{1}{1+x^2}\, dx = [\arctan x]_{-N}^0 = -\arctan(-N).$$

$$\lim_{N\to\infty} \arctan N = \frac{\pi}{2},$$

$$\lim_{N\to\infty} -\arctan(-N) = \frac{\pi}{2}.$$

Therefore the improper integral exists, and its value is

Improper Integrals: Examples

Example

$$\int_2^\infty \frac{2}{1-x^2}\,dx = ?$$

$$\begin{aligned}\int \frac{2}{1-x^2}\,dx &= \int \frac{(1-x)+(1+x)}{(1-x)(1+x)}\,dx = \int \frac{1}{1-x}+\frac{1}{1+x}\,dx \\ &= -\ln(x-1)+\ln(x+1)+c = \ln\frac{x+1}{x-1}+c.\end{aligned}$$

$$\int_2^N \frac{2}{1-x^2}\,dx = \left[\ln\frac{x+1}{x-1}\right]_2^N = \ln\frac{N+1}{N-1} - \ln\frac{2+1}{2-1} = \ln\frac{N+1}{N-1} - \ln 3.$$

$$\lim_{N\to\infty}\left(\ln\frac{N+1}{N-1} - \ln 3\right) = -\ln 3.$$

Therefore the improper integral exists, and its value is

$$-\ln 3.$$

Example

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = ?$$

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx \left[\substack{u=\sqrt{x} \\ du=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\, dx}\right] = \int e^{-u}\, 2du = -2e^{-u} + c = -2e^{-\sqrt{x}} + c.$$

$$\int_0^N \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \left[-2e^{-\sqrt{x}}\right]_0^N = -2e^{-\sqrt{N}} + 2e^{-\sqrt{0}} = -2e^{-\sqrt{N}} + 2.$$

$$\lim_{N\to\infty}\left(-2e^{-\sqrt{N}} + 2\right) = 2.$$

Therefore the improper integral exists, and its value is

$$2.$$


tea
time
coffee
break

Applications of the Integral: Geometry

- The original motivation for introducing the integral was the computation of areas.

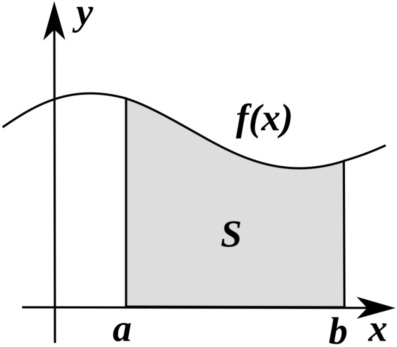
- The most fundamental geometric applications of integral calculus (in the one-variable setting) are:

(1) Area computation.
(2) Computation of arc lengths of curves.
(3) Solids of revolution:
    (a) computation of volume,
    (b) computation of surface area.

- In the following, we look at examples of these applications.

Let $f : [a, b] \to \mathbb{R}_+$ be an integrable function. Then the area enclosed by the graph of f and the x-axis above the interval $[a, b]$ is

$$\int_a^b f(x)\,dx.$$



Let $f : [a, b] \to \mathbb{R}_-$ be an integrable function. Then the area enclosed by the graph of f and the x-axis above the interval $[a, b]$ is

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx = \int_a^b |f(x)|\,dx.$$

• If the sign of f changes on $[a, b] = \mathrm{dom}(f)$, then decompose $[a, b]$ into a union of intervals $([a_i, b_i])_{i=1}^{\ell}$ with pairwise disjoint interiors, such that the sign of $f|_{[a_i, b_i]}$ does not change for $i = 1, 2, \ldots, \ell$.
The areas above the individual intervals can then be computed independently and added together.

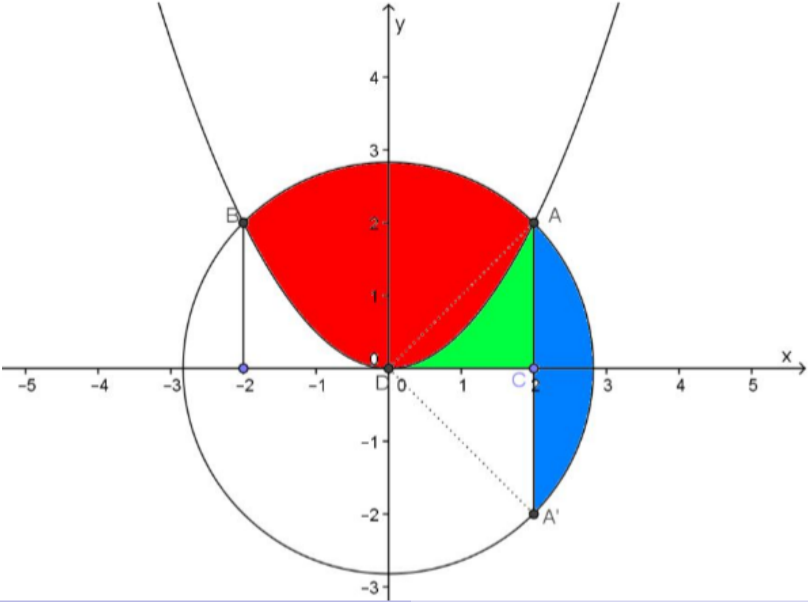
Applications of the Integral: Geometry: Area

• Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $f(x) \geq g(x)$ for every $x \in [a, b]$.
Then the area enclosed by the graphs of f and g above the interval $[a, b]$ is

$$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

• If the order of the functions changes on $[a, b] = \operatorname{dom}(f)$, then decompose $[a, b]$ into a union of intervals $([a_i, b_i])_{i=1}^{\ell}$ with pairwise disjoint interiors, such that on each interval one of the functions is everywhere greater than or equal to the other for $i = 1, 2, \dots, \ell$.
The areas above the individual intervals can then be computed independently and added together.

Applications of the Integral: Geometry: Area: The picture



- The three most common ways to describe curves are the following:
    - The graph of a differentiable function defined on an interval $[a, b]$ describes the curve.
    - The points $(x(t), y(t))$ describe the curve, where x and y are differentiable functions and $t \in [a, b]$ is a parameter interpreted as “time”.
    - The points $(r(\varphi)\cos\varphi, r(\varphi)\sin\varphi)$ describe the curve, where r is a differentiable function and $\varphi \in [\alpha, \beta]$ is an “angle” parameter. [Polar-coordinate representation]
- Each type of description has its own arc length formula.

Theorem

If $f(x) : [a, b] \to \mathbb{R}$ is continuously differentiable, then the arc length of the graph segment between $(a, f(a))$ and $(b, f(b))$ is

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Theorem

The arc length of a curve given in parametric form $(x(t), y(t))$, where $t \in [a, b]$, is

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Theorem

The arc length of a curve given in polar coordinates by $r(\varphi)$, where $\varphi \in [\alpha, \beta]$, is

$$s = \int_\alpha^\beta \sqrt{(r(\varphi))^2 + (r'(\varphi))^2}\, d\varphi$$

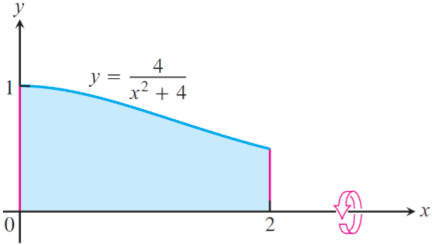
Applications of Integration: Geometry: Arc Length: Examples

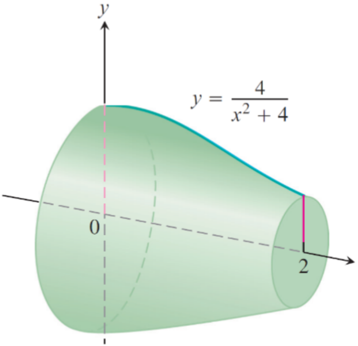
Example

Determine the length of the graph of $y = \cosh x$ on the interval $[0, a]$.

$$
\begin{aligned}
s = \int_0^a \sqrt{1 + (f')^2}\, dx &= \int_0^a \sqrt{1 + \sinh^2 x}\, dx = \int_0^a \sqrt{\cosh^2 x}\, dx \\
&= \int_0^a \cosh x\, dx = [\sinh x]_0^a = \sinh a.
\end{aligned}
$$

- Consider a curve represented as the graph of a function $f : [a, b] \to \mathbb{R}$. We regard the graph in the xy-plane as lying in three-dimensional space, and rotate it around the x-axis. In this way we obtain a surface enclosing a solid of revolution.





Applications of Integration: Geometry: Solids of Revolution: Surface Area: Theorems

Theorem

Suppose that the lateral curve of a solid of revolution symmetric about the x-axis is described by the graph of a continuous function $f(x) : [a, b] \to \mathbb{R}$. Then the surface area of the solid obtained by rotating the graph around the segment $[a, b]$ of the x-axis is

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Theorem

Suppose that a continuous curve given in parametric form by $(x(t), y(t))$, where $t \in [a, b]$, is rotated around the x-axis. Then the surface area of the resulting solid of revolution is

$$2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example

Determine the surface area of the solid obtained by rotating the graph of $y = \frac{1}{x}$ around the x-axis over the interval $[1, \infty[$.

$$A = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \;=\; 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left((\frac{1}{x})'\right)^2}\, dx$$

$$= \; 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(\frac{1}{x^2}\right)^2}\, dx.$$

We already know that $\int_1^\infty \frac{1}{x}\, dx$ diverges. The integrand above is bounded below by $\frac{1}{x}$. Therefore the surface area in this example is infinite.

Applications of Integration: Geometry: Solids of Revolution: Volume: Theorems

Theorem

Suppose that the graph of a continuous function $f : [a, b] \to \mathbb{R}$ describes the lateral curve of a solid of revolution around the x-axis. Then the volume of the part of the solid lying above the interval $[a, b]$ of the x-axis is

$$V = \pi \int_a^b f^2(x)\, dx.$$

Theorem

Suppose that a continuous curve given in parametric form by $(x(t), y(t))$, where $t \in [a, b]$, describes the lateral surface of a solid of revolution around the x-axis. Then the volume of the corresponding solid above the interval $[a, b]$ of the x-axis is

$$V = \pi \int_a^b y^2(t) x'(t)\, dt.$$

Applications of Integration: Geometry: Solids of Revolution: Volume: Examples

Example

Determine the volume of the solid obtained by rotating the graph of $y = \frac{1}{x}$ around the x-axis over the interval $[1, \infty[$.

$$
\begin{aligned}
V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx &= \pi \int_1^\infty \frac{1}{x^2}\, dx \\
&= \pi \left[-\frac{1}{x}\right]_1^\infty = \pi .
\end{aligned}
$$

The volume is finite. We also computed the surface area of this solid, and obtained that it is infinite. This apparently paradoxical phenomenon is known as the painter's paradox.

Break

Applications of Integration: Convergence of Series: The Idea

- Let $(a_n)_{n=1}^{\infty}$ be a sequence, and consider the infinite series

$$\sum_{i=1}^{\infty} a_i.$$

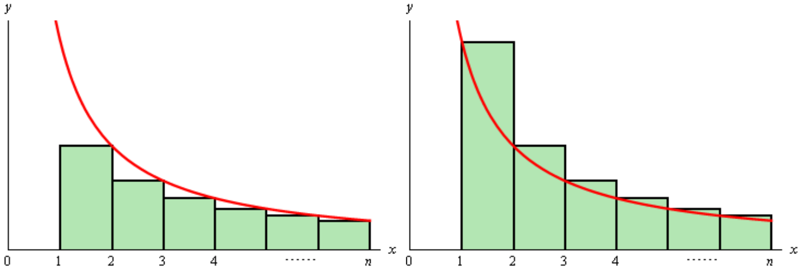
Assume that $a_n = f(n)$, where $f : [1, \infty[\to \mathbb{R}_+$ is a decreasing, non-negative function.

- Then the series $\sum_{i=1}^{\infty} a_i$ and the improper integral

$$\int_1^{\infty} f(x)\, dx$$

are closely related.

This relationship can be visualized in the following figures.



Applications of Integration: Convergence of Series: The Theorem

Theorem [Integral Test]

Assume that $a_n = f(n)$, where $f : [1, \infty[\to \mathbb{R}_+$ is a decreasing, non-negative function.

(i) If $\sum_{i=1}^{\infty} a_i \in \mathbb{R}$, then
$$\int_1^{\infty} f(x)\, dx < \infty.$$

(ii) If $\sum_{i=1}^{\infty} a_i = \infty$, that is, the series diverges, then
$$\int_1^{\infty} f(x)\, dx = \infty.$$

• In fact, both implications are equivalences. Thus, for series satisfying the assumptions of the theorem, convergence can be decided by evaluating an improper integral.

Applications of Integration: Convergence of Series: Examples

Example

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

- The sequence $a_n = \frac{1}{n}$ can be viewed as the values of the function

$$f(x) = \frac{1}{x} :]0, \infty[\to \mathbb{R}$$

at positive integers. The function f is positive and monotonically decreasing, so the integral test applies.

- $$\int \frac{1}{x}\, dx = \ln x + c.$$
- $$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{N\to\infty} \ln N = \infty.$$

Applications of Integration: Convergence of Series: Examples

Example

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

- The sequence $a_n = \frac{1}{n^2}$ can be viewed as the values of the function

$$f(x) = \frac{1}{x^2} :]0, \infty[\to \mathbb{R}$$

at positive integers. The function f is positive and monotonically decreasing, so the integral test applies.

- $$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c.$$

- $$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{N\to\infty}\left(1 - \frac{1}{N}\right) = 1.$$

Applications of Integration: Convergence of Series: Examples

- Therefore, by the integral test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Theorem [Euler]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Applications of Integration: Convergence of Series: Examples

Example

Let $\alpha > 0$ be a parameter. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

converges if and only if $\alpha > 1$.

• The sequence $a_n = \frac{1}{n^{\alpha}}$ can be viewed as the values of the function

$$f(x) = \frac{1}{x^{\alpha}} :]0, \infty[\to \mathbb{R}$$

at positive integers. The function f is positive and monotonically decreasing, so the integral test applies.

- The case $\alpha = 1$ has already been discussed. We assume $\alpha \in]0, 1[\cup]1, \infty[$.
- $$\int \frac{1}{x^\alpha}\, dx = \frac{x^{1-\alpha}}{1-\alpha} + c.$$
- The value of
$$\int_1^\infty \frac{1}{x^\alpha}\, dx$$
depends on the limit $\lim_{N\to\infty} N^{1-\alpha}$.
- If $\alpha \in]1, \infty[$, then
$$\lim_{N\to\infty} N^{1-\alpha} = \lim_{N\to\infty} \frac{1}{N^{\alpha-1}} = 0.$$
- If $\alpha \in]0, 1[$, then
$$\lim_{N\to\infty} N^{1-\alpha} = \infty.$$
- By the integral test, the statement is proved.

Applications of Integration: Convergence of Series: Examples

Example

The series

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$$

is divergent.

• The sequence $a_n = \frac{1}{n \ln n}$ can be viewed as the values of the function

$$f(x) = \frac{1}{x \ln x} :]3, \infty[\to \mathbb{R}$$

at positive integers. The function f is positive and monotonically decreasing, so the integral test applies.

Applications of Integration: Convergence of Series: Examples

- $$\int \frac{1}{x \ln x}\, dx \left[\begin{smallmatrix} u=\ln x \\ du=\frac{1}{x}\,dx \end{smallmatrix}\right] = \int \frac{1}{u}\, du = \ln u + c = \ln \ln x + c.$$
- $$\int_3^\infty \frac{1}{x \ln x}\, dx = \lim_{N\to\infty} (\ln \ln N - \ln \ln 3) = \infty.$$
- Therefore, by the integral test, the series

$$\sum_{n=1}^\infty \frac{1}{n \ln n}$$

is divergent.

Theorem [Euler]

$$\sum_{n=1}^\infty \frac{1}{p_n} = \infty,$$

where $(p_n)_{n=1}^\infty = (2, 3, 5, 7, 11, 13, 17, \dots)$ is the sequence of prime numbers.

Break

Definition

If during a straight-line motion we cover a distance L and apply a constant force F in the direction of motion, then the work performed is

$$W = L \cdot F.$$

• In a uniform gravitational field, the same constant force $\overrightarrow{F}$ acts at every point. If we move against $\overrightarrow{F}$, then work must be done. To move a distance L opposite to the direction of $\overrightarrow{F}$, we must perform work LF (to "overcome" the field).

• What happens in reality if we want to overcome the Earth's gravitational field?

• The main issue is that the Earth's gravitational field is not constant. The force F depends on the position x, i.e. $F = F(x)$.

Applications of Integration: Physics: Work

- We approximate the motion by small steps. During a small step of length dx starting at position x (with dx sufficiently small), the force does not change much, so it is well approximated by $F(x)$. Hence the work on this segment is $F(x)\,dx$.
- The total work is approximated by the sum $\sum F(x)\,dx$.
- In physics, the work done in a force field is defined as

$$\int_P^Q F(x)\,dx.$$

Question

With what speed must we launch an object from the Earth's surface to overcome Earth's gravitational attraction?

- To answer the question, we must understand the Earth's gravitational field.

Physical fact

In the gravitational field of a body with mass M_{Earth}, a body of mass m located at distance r experiences the force

$$\gamma \frac{mM_{\text{Earth}}}{r^2},$$

where γ is the gravitational constant, whose value is known.

- We are looking for the escape velocity v_{escape} such that an object launched with this speed can overcome the Earth's gravitational field.

- In other words, we seek an initial velocity such that the object can perform the work

$$\int_{R_{\text{Earth}}}^{\infty} \gamma \frac{mM_{\text{Earth}}}{x^2}\, dx.$$

- A body moving with speed v_{escape} has kinetic energy

$$\frac{1}{2}mv_{\text{escape}}^2.$$

- By the law of conservation of energy, this energy can be used by the object (e.g. a spacecraft) to perform the required work.

- We now have all the ingredients to formulate the mathematical model.

Question

Find the escape velocity v_{escape} such that

$$\frac{1}{2}mv_{\text{escape}}^2 = \int_{R_{\text{Earth}}}^{\infty} \gamma \frac{mM_{\text{Earth}}}{x^2}\, dx.$$

The mass m cancels out. The remaining physical constants are known:

$$\frac{1}{2}mv_{\text{escape}}^2 = \int_{R_{\text{Earth}}}^{\infty} \gamma \frac{mM_{\text{Earth}}}{x^2}\,dx.$$

Solution:

- Evaluation of the integral:

$$\begin{aligned}
\int_{R_{\text{Earth}}}^{\infty} \gamma \frac{mM_{\text{Earth}}}{x^2}\,dx &= \gamma mM_{\text{Earth}} \int_{R_{\text{Earth}}}^{\infty} \frac{1}{x^2}\,dx \\
&= \gamma mM_{\text{Earth}} \left[-\frac{1}{x}\right]_{R_{\text{Earth}}}^{\infty} \\
&= \gamma mM_{\text{Earth}} \cdot \frac{1}{R_{\text{Earth}}}.
\end{aligned}$$

- Solving the equation:

$$\begin{aligned} \frac{1}{2}mv_{\text{escape}}^2 &= \int_{R_{\text{Earth}}}^{\infty} \gamma \frac{mM_{\text{Earth}}}{x^2}\, dx, \\ \frac{1}{2}mv_{\text{escape}}^2 &= \gamma mM_{\text{Earth}} \frac{1}{R_{\text{Earth}}}, \\ v_{\text{escape}}^2 &= 2\gamma \frac{M_{\text{Earth}}}{R_{\text{Earth}}}, \\ v_{\text{escape}} &= \sqrt{2\gamma \frac{M_{\text{Earth}}}{R_{\text{Earth}}}} \approx 11.2\,\text{km/s}. \end{aligned}$$

Thank you for your attention!