BSc Mathematics for Computer Scientists 2: XI. Riemann Integration

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Integration: Motivation: Area computation

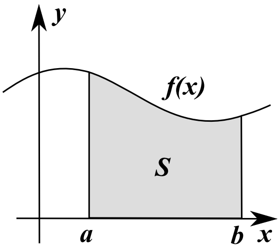
Basic problem

Given a region in the plane. Determine its area.

The following special case is important for us.

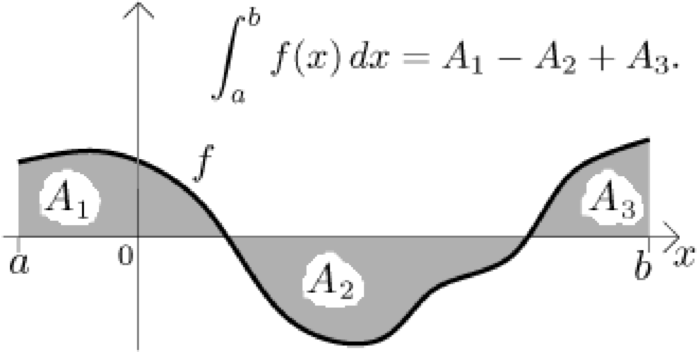
Basic problem

Given a non-negative function $f : [a, b] \to \mathbb{R}_+$. Determine the area of the region bounded by the graph and the x-axis over the interval $[a, b]$.



Integration: Motivation: Signed area

Generalization: Allow the function f to be arbitrary. The parts of the graph below the x-axis are counted with negative sign.



Integration: What is area?

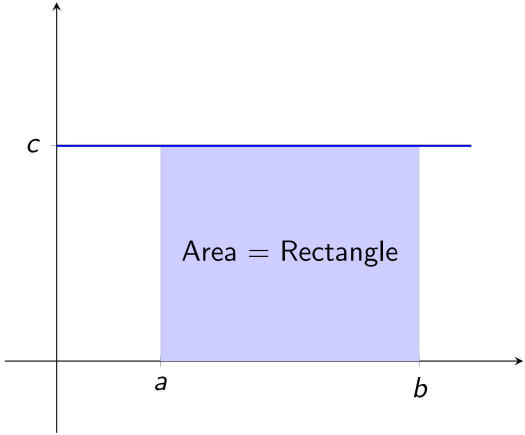
• We give a method to approximate and determine the area of a region. If the procedure leads to a well-defined non-negative value, we say that the region is "nice" and its area is the computed value.
• Thus the above method leads to a class of regions: $\mathcal{S}$, the set of nice regions, and to an area function $A : \mathcal{S} \to \mathbb{R}_+$.
• If the area corresponding to the graph of a function $f : [a, b] \to \mathbb{R}$ is well-defined, then we call the function integrable. The assigned area (which, according to the signed area convention, may also be negative) is denoted by

$$\int_a^b f(x)\, dx.$$

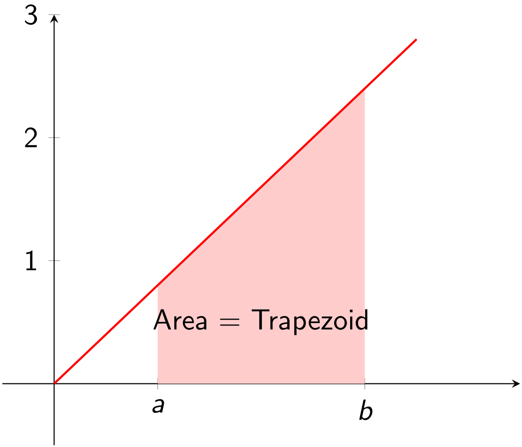
(Read as: "the integral from a to b of $f(x)$ dx".)
• To present the method mentioned above, we need to summarize some properties of area.

Integration: Area: Elementary examples

$$f(x) = c \quad \to \quad \int_a^b c\,dx = c(b-a)$$

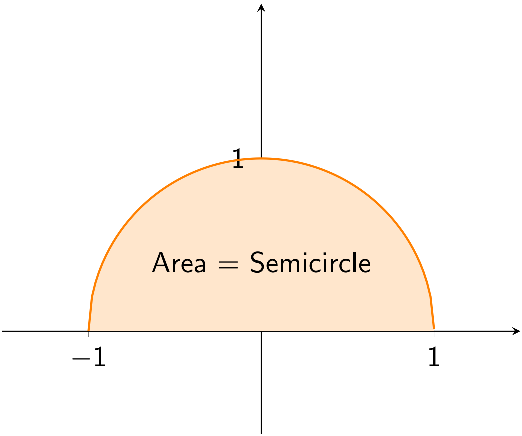


$$f(x) = mx \quad \rightarrow \quad \int_a^b mx\,dx = \frac{mb^2}{2} - \frac{ma^2}{2}$$



Integration: Area: Elementary examples

$$f(x) = \sqrt{1-x^2} \quad \rightarrow \quad \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{r^2\pi}{2} = \frac{\pi}{2}$$



Integration: Area: Properties

Property (additivity)

Let T be a region with area, which is partitioned by lines into regions $T_1, T_2, \ldots, T_\ell$. Then

$$A(T) = A(T_1) + A(T_2) + \ldots + A(T_\ell).$$

Property (monotonicity)

Let T be a region with area and R be a subregion of T with area. Then

$$A(R) \leq A(T).$$

Definition

A partition of the interval $[a, b]$ is a sequence $\beta = (t_i)_{i=0}^{\ell}$ such that

$$a = t_0 < t_1 < t_2 < \ldots < t_{\ell-1} < t_\ell = b.$$

The points t_i are called the partition points.

We think of the partition as cutting the interval $I = [a, b]$ into subintervals $I_i = [t_{i-1}, t_i]$.

Definition

The mesh (size) of a partition β is

$$\|\beta\| = \min\{t_i - t_{i-1} : i = 1, 2, \ldots, \ell\}.$$

$\|\beta\|$ small means that every interval in β is short. $\|\beta\|$ large means that at least one interval in β is long.

Integration: Partitions

Definition

If we add new points to the partition points of β, we obtain a refinement of the partition. If the refined partition is β', then we write $\beta' \preceq \beta$.

If $\beta' \preceq \beta$, then we say that β' is finer than β, and β is coarser than β'.

Definition

If we remove partition points of β (except a and b), then we obtain a coarser partition.

Lemma

If for two partitions $\beta' \preceq \beta$, then $\|\beta'\| \leq \|\beta\|$.

Lemma

For any two partitions β, β' there exists a common refinement $\beta' \wedge \beta$ which is the coarsest among common refinements.

Integration: Lower/upper sums

• Let β be a partition of the interval $[a, b]$, which is the domain of the function f. For each subinterval I_i we define

$$m_i = \inf\{f(x) : x \in I_i\} \qquad \text{and} \qquad M_i = \sup\{f(x) : x \in I_i\}.$$

• Thus $[m_i, M_i]$ is the smallest interval such that for all $x \in I_i$ we have $f(x) \in [m_i, M_i]$.

Definition

The lower sum corresponding to the partition β is

$$L_\beta(f) = m_1(t_1-t_0)+m_2(t_2-t_1)+\ldots+m_\ell(t_\ell-t_{\ell-1}) = \sum_{i=1}^{\ell} m_i(t_i-t_{i-1}).$$

The upper sum corresponding to the partition β is

$$U_\beta(f) = M_1(t_1-t_0)+M_2(t_2-t_1)+\ldots+M_\ell(t_\ell-t_{\ell-1}) = \sum_{i=1}^{\ell} M_i(t_i-t_{i-1}).$$

Observation

Let β be a partition of $[a, b]$ and β' be a refinement of it. Then

(i) $L_\beta(f) \leq L_{\beta'}(f)$,

(ii) $U_{\beta'}(f) \leq U_\beta(f)$,

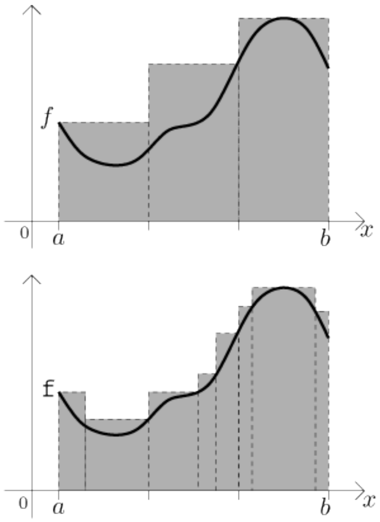
(iii) $L_\beta(f) \leq U_\beta(f)$.

Observation

Let β and γ be two arbitrary partitions of $[a, b]$. Then $L_\beta(f) \leq U_\gamma(f)$.

Proof. Consider their common refinement $\beta \wedge \gamma$.

Integration: Lower/Upper Sums: Geometry



Integration: Lower/Upper Integrals

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function. Then

$$\underline{\int_a^b} f(x)\,dx = \sup\{L_\beta(f) : \beta \text{ is a partition of } [a, b]\},$$

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function. Then

$$\overline{\int_a^b} f(x)\,dx = \inf\{U_\beta(f) : \beta \text{ is a partition of } [a, b]\},$$

Observation

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function. We say that f is Riemann integrable if

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

In this case we write

$$\int_a^b f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

Convention

If $a < b$, then

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$$

Integration: Riemann Integrability: Positive Examples

Example

The constant function $f(x) = c$ is integrable on every interval $[a, b]$ and
$$\int_a^b c\,dx = c(b-a).$$

Example

The linear function $f(x) = mx$ is integrable on every interval $[a, b]$ and
$$\int_a^b mx\,dx = \frac{m}{2}(b^2 - a^2).$$

Example

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then

$$\overline{\int_0^1} f(x)\,dx = 1 \qquad \text{and} \qquad \underline{\int_0^1} f(x)\,dx = 0.$$

Thus f is not Riemann integrable.

Break

Riemann Integral: Properties

Theorem

If f is Riemann integrable on $[a, b]$ and on $[b, c]$, then it is Riemann integrable on $[a, c]$, and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Theorem

If f and g are Riemann integrable functions on $[a, b]$, then $f + g$ and λf are also integrable, and

(i)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(ii)

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

Riemann Integral: Properties

Theorem

If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then

(i) $|f|$ is also Riemann integrable,

(ii) fg is also Riemann integrable.

Theorem

If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then

(i) If $f \geq 0$, then $\int_a^b f(x)\,dx \geq 0$,

(ii) If $f \leq g$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$,

(iii) If f is bounded, more precisely $\mu \leq f \leq M$, then $\mu(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Definition

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the average value of f is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Riemann Integrability: Sufficient Conditions

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Theorem

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is Riemann integrable.

Theorem

If $f : [a, b] \to \mathbb{R}$ is bounded and monotone on (a, b), then f is Riemann integrable.

Theorem

If $f : [a, b] \to \mathbb{R}$ is bounded and piecewise continuous (with respect to some partition β), then f is Riemann integrable.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let $(\beta_k)_{k=1}^{\infty}$ be a sequence of partitions of $[a, b]$ such that $\|\beta_k\| \to 0$ as $k \to \infty$. For each partition β_k, let $(\xi_i^{(k)})$ be points such that $\xi_i^{(k)} \in I_i^{(k)}$, where $I_i^{(k)}$ is the i-th subinterval.
Then

$$\sum_{i=1}^{\ell_k} f(\xi_i^{(k)})(t_i^{(k)} - t_{i-1}^{(k)}) \to \int_a^b f(x)\,dx \quad \text{as } k \to \infty.$$

Proof:

$$L_{\beta_k}(f) \le \sum_{i=1}^{\ell_k} f(\xi_i^{(k)})(t_i^{(k)} - t_{i-1}^{(k)}) \le U_{\beta_k}(f)$$

for all k.

Break

Riemann Integral: First Fundamental Theorem

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt.$$

(i) Then F is continuous.

(ii) If moreover f is continuous at x_0, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Definition

Let f be a function defined on an interval. A function F is called a primitive function (or antiderivative) of f if F is differentiable and $F' = f$.

Riemann Integral: Primitive Functions

Theorem

Let f be a function defined on an interval with primitive functions F_1 and F_2. Then $F_1 - F_2$ is a constant function.
Conversely, if F is a primitive function, then $F(x) + c$ is also a primitive function for any constant c.

Notation

If f is a function defined on an interval and F is its primitive function, then we write

$$\int f = \int f(x)\, dx = F(x) + c = F + c,$$

where c denotes an arbitrary constant.

Example

Let $f :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be the constant function 1.

Let $F_1 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by $F_1(x) = x$.

Let $F_2 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by

$$F_2(x) = \begin{cases} x-1, & \text{if } x < 0, \\ x+1, & \text{if } x > 0. \end{cases}$$

- Both F_1 and F_2 are primitive functions of f.

- $F_1 - F_2$ is not constant.

- The correct interpretation: the domain of f consists of two intervals. Thus f is composed of functions defined on separate intervals, which can (and should) be treated independently.

Reminder

$(c)' = 0 \ (c \in \mathbb{R})$	$(x^\alpha)' = \alpha x^{\alpha-1} \ (\alpha \in \mathbb{R})$
$(e^x)' = e^x$	$(a^x)' = a^x \ln a \ (a > 0,\ a \neq 1)$
$(\ln x)' = \frac{1}{x}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\arctan x)' = \frac{1}{1+x^2}$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$
$(\operatorname{arcsec} x)' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$(\operatorname{arccsc} x)' = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$(\sinh x)' = \cosh x$	$(\cosh x)' = \sinh x$
$(\tanh x)' = \operatorname{sech}^2 x$	$(\coth x)' = -\operatorname{csch}^2 x$
$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$	$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$
$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}}$	$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}} \ (x > 1)$
$(\operatorname{artanh} x)' = \frac{1}{1-x^2} \ (\lvert x\rvert < 1)$	$(\operatorname{arcoth} x)' = \frac{1}{1-x^2} \ (\lvert x\rvert > 1)$
$(\operatorname{arsech} x)' = -\frac{1}{x\sqrt{1-x^2}}$	$(\operatorname{arcsch} x)' = -\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

Example

$\int 0 \ dx = c$

$\int e^x \ dx = e^x$

$\int \frac{1}{x} \ dx = \ln x + c \quad (x \in \mathbb{R}_{++})$

$\int \cos x \ dx = \sin x$

$\int \sec^2 x \ dx = \tan x + c$

$\int \sec x \tan x \ dx = \sec x + c$

$\int \frac{1}{\sqrt{1-x^2}} \ dx = \arcsin x + c$

$\int \frac{1}{1+x^2} \ dx = \arctan x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}} \ dx = \operatorname{arcsec} x + c$

$\int \cosh x \ dx = \sinh x + c$

$\int \operatorname{sech}^2 x \ dx = \tanh x + c$

$\int \operatorname{sech} x \tanh x \ dx = -\operatorname{sech} x + c$

$\int \frac{1}{\sqrt{1+x^2}} \ dx = \operatorname{arsinh} x + c$

$\int \frac{1}{1-x^2} \ dx = \operatorname{artanh} x + c \quad (|x| < 1)$

$\int \frac{1}{x\sqrt{1-x^2}} \ dx = -\operatorname{arsech} x + c$

$\int x^\alpha \ dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$

$\int a^x \ dx = \frac{1}{\ln a} a^x \quad (a > 0, \ a \neq 1)$

$\int \frac{1}{x} \ dx = \ln(-x) + c \ (x \in \mathbb{R}_{--})$

$\int \sin x \ dx = -\cos x + c$

$\int \csc^2 x \ dx = \cot x + c.$

$\int \csc x \cot x = -\csc x + c$

$\int \frac{1}{\sqrt{1-x^2}} \ dx = -\arccos x + c$

$\int \frac{1}{1+x^2} \ dx = -\operatorname{arccot} x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}} = -\operatorname{arccsc} x + c$

$\int \sinh x \ dx = \cosh x + c$

$\int \operatorname{csch}^2 x \ dx = -\coth x + c$

$\int \operatorname{csch} x \coth x \ dx = -\operatorname{csch} x + c$

$\int \frac{1}{\sqrt{x^2-1}} \ dx = \operatorname{arcosh} x + c \quad (x > 1)$

$\int \frac{1}{1-x^2} \ dx = \operatorname{arcoth} x + c \quad (x > 1)$

$\int \frac{1}{|x|\sqrt{1+x^2}} \ dx = \operatorname{arcsch} x + c.$

Riemann Integral: Primitive Function: A Remark

• We have seen that $\int \frac{1}{x} = \ln x + c$. There is a hidden interval in this formula, namely the common domain of the functions involved (due to the ln function): $\mathbb{R}_{++}$.

• More precisely, the previous formula gives a primitive function of $\frac{1}{x} : \mathbb{R}_{++} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.

• We may also look for a primitive function of $\frac{1}{x} : \mathbb{R}_{--} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.

• After a short attempt, we see that a primitive function is $\ln(-x)$. Indeed, $(\ln(-x))' = \frac{1}{-x} \cdot (-x)' = \frac{1}{x}$. The domains also match.

• In the literature one often finds the formula $\int \frac{1}{x} dx = \ln|x| + c$, which merges the two cases. This is somewhat misleading, in my personal opinion.

Riemann Integral: Fundamental Theorem (Second)

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let F be a primitive (antiderivative) of f. Then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Riemann Integral: Fundamental Theorem (Second): Algorithmic View

• The theorem suggests a two-phase algorithm for computing integrals:

(I) Find a primitive function of the integrand f.

(II) Use the fundamental theorem to compute the value (two evaluations and one subtraction).

• The good news: this scheme works very often. However, finding a primitive may require technical skill.

• The bad news: there exist nice functions that cannot be expressed with a simple primitive in closed form. For example: e^{-x^2}.

Riemann Integral: Fundamental Theorem (Second): Consequences

Theorem [Integration by Parts]

Let f and g be two Riemann integrable functions on $[a, b]$. Let F and G be their primitive functions. Then

(i)
$$\int f(x)G(x)\, dx = F(x)G(x) - \int F(x)g(x)\, dx,$$

(ii)
$$\int_a^b f(x)G(x)\, dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)\, dx,$$

Riemann Integral: Integration by Parts: Interpretation

• The theorem is somewhat strange. On the right-hand side we still have an integral, but evaluating the left-hand side also requires computing an integral / primitive function.

• Thus, applying the formula can be viewed as a "trade". $\int fG$ asks for integrating a product. According to the formula, the first factor is replaced by its primitive function, while the second factor is replaced by its derivative. Computing the new integral also solves the original one. So we replaced one integral with another. Do this if it is advantageous.

• Applying integration by parts (and sometimes even recognizing when to use it) requires intuition and mathematical insight.

Riemann Integral: Fundamental Theorem (Second): Consequences

Theorem [Substitution Rule]

Let $\alpha : [a, b] \to [\alpha(a), \alpha(b)]$ be a strictly increasing, differentiable function whose derivative is Riemann integrable. Let $f : [\alpha(a), \alpha(b)] \to \mathbb{R}$ be a Riemann integrable function. Then $f(\alpha(x))\alpha'(x)$ is Riemann integrable on $[a, b]$, and moreover

(i)

$$\int f(\alpha(x))\alpha'(x)\, dx = \int f(u)\, du\Big|_{u=\alpha(x)},$$

(ii)

$$\int_a^b f(\alpha(x))\alpha'(x)\, dx = \int_{\alpha(a)}^{\alpha(b)} f(u)\, du.$$

• Recognizing when to apply substitution is usually easier than recognizing integration by parts. We need to identify that the derivative of an essential (possibly repeating) subexpression appears in the formula. In such a case, it is worth introducing a new variable for that subexpression.

• If we use the substitution $u = \alpha(x)$, then $\alpha'(x)\,dx$ must be replaced by du. If we formally write the derivative as $\frac{du}{dx} = \alpha'(x)$, then by formal rearrangement we obtain $du = \alpha'(x)\,dx$.

◦ It is already apparent that integration is a more subtle task than differentiation. Intuition and clever algebraic manipulations and observations play a significant role. Effective integration requires a lot of practice. Further examples will help with this.

Example

$\int \frac{x^2}{x^3+1}\ dx$

• Notice that the numerator is almost the derivative of the denominator.

• Let us denote the denominator by u, i.e. $u = x^3 + 1$. Then $du = 3x^2\ dx$, hence $\frac{1}{3}\ du = x^2\ dx$.

$$\int \frac{x^2}{x^3+1}\, dx \left[\substack{u=x^3+1 \\ du=3x^2\ dx}\right] = \int \frac{1}{u}\cdot\frac{1}{3}\, du = \frac{1}{3}\ln u + c = \frac{1}{3}\ln(x^3+1) + c.$$

$$\int_0^1 \frac{x^2}{x^3+1}\, dx \left[\substack{u=x^3+1 \\ du=3x^2\ dx}\right] = \int_1^2 \frac{1}{u}\cdot\frac{1}{3}\ du = \left[\frac{1}{3}\ln u\right]_1^2 = \frac{1}{3}\ln 2 - \frac{1}{3}\ln 1 \approx 0.231.$$

Example

$\int \sin^4 x \cos x \; dx.$

- The function $\sin x$ appears with a power, and we also have a factor $\cos x$, which is the derivative of $\sin x$.
- Let $u = \sin x$. Then $du = \cos x \; dx$.

$$\int \sin^4 x \cos x \; dx \left[{}_{du=\cos x \; dx}^{u=\sin x}\right] = \int u^4 \, du = \frac{u^5}{5} + c = \frac{\sin^5 x}{5} + c.$$

$$\int_0^{\pi/2} \sin^4 x \cos x \; dx \left[{}_{du=\cos x \; dx}^{u=\sin x}\right] = \int_0^1 u^4 \, du = \left[\frac{u^5}{5}\right]_0^1 = \frac{1}{5}.$$

Riemann Integration: Examples: Substitution

Example

$\int x\sqrt{x^2-4}\ dx.$

• The expression x^2-4 is a subexpression whose derivative is $2x$, which appears almost in the integrand.

• We perform the substitution $u = x^2-4$. Then $du = 2x\ dx$, i.e. $\frac{1}{2}du = x\ dx$.

$$\int x\sqrt{x^2-4}\ dx \left[{u=x^2-4 \atop du=2x\ dx}\right] = \int \sqrt{u}\cdot\frac{1}{2}\ du = \frac{1}{2}\int u^{1/2}\ du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+c = \frac{1}{3}(x^2-4)^{3/2}+c.$$

$$\int_2^3 x\sqrt{x^2-4}\ dx \left[{u=x^2-4 \atop du=2x\ dx}\right] = \frac{1}{2}\int_0^5 u^{1/2}\ du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_0^5 = \frac{1}{3}\cdot 5^{3/2} \approx 3.726.$$

Example

$\int \ln x \, dx$.

• Here the integrand is not even a product. We use the idea of viewing $\ln x$ as the product $1 \cdot \ln x$. We know that $(\ln x)' = \frac{1}{x}$, while the antiderivative of 1 is x. Integration by part (in this form) is "a good deal".

$$\int \ln x \, dx \left[{u=\ln x,\ u'=\frac{1}{x} \atop v'=1,\ v=x} \right] = x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x + c.$$

$$\begin{aligned} \int_1^e \ln x \, dx \left[{u=\ln x,\ u'=\frac{1}{x} \atop v'=1,\ v=x} \right] &= [x \ln x]_1^e - \int_1^e 1 \, dx \\ &= (e \ln e - 1 \ln 1) - [x]_1^e = e - (e-1) = 1. \end{aligned}$$

Example

$\int (2x-3)e^x\ dx$.

• The derivative of e^x is itself, so differentiating does not complicate the expression. On the other hand, differentiating $2x-3$ simplifies it significantly.

$$\begin{aligned}\int (2x-3)e^x\ dx \left[\begin{smallmatrix} u=2x-3,\ u'=2 \\ v'=e^x,\ v=e^x \end{smallmatrix}\right] &= (2x-3)e^x - \int 2e^x\ dx \\ &= (2x-3)e^x - 2e^x + c = (2x-5)e^x + c.\end{aligned}$$

$$\begin{aligned}\int_0^1 (2x-3)e^x\ dx \left[\begin{smallmatrix} u=2x-3,\ u'=2 \\ v'=e^x,\ v=e^x \end{smallmatrix}\right] &= [(2x-3)e^x]_0^1 - \int_0^1 2e^x\, dx \\ &= (-e+3)-(2e-2) = 5-3e \approx -3.15\end{aligned}$$

Example

$\int x\sin(2x)\ dx$.

• The antiderivative of sin is $-\cos$, i.e. its trigonometric "pair". Differentiating x leads to a significantly simpler expression.

$$\int x\sin(2x)\ dx \left[{u=x,\ u'=1 \atop v'=\sin 2x,\ v=-\frac{\cos 2x}{2}}\right] = -\frac{x\cos 2x}{2} - \int -\frac{\cos 2x}{2}\ dx$$
$$= -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + c.$$

$$\int_0^{\pi/2} x\sin(2x)\ dx \left[{u=x,\ u'=1 \atop v'=\sin 2x,\ v=-\frac{\cos 2x}{2}}\right] = \left[-\frac{x\cos 2x}{2}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos 2x}{2}\ dx$$
$$= \frac{\pi}{4} + 0 = \frac{\pi}{4} \approx 0.785.$$

Riemann integration: Examples: Substitution and integration by parts

Example

$\int e^{\sqrt{x}}\ dx$.

- Let us perform the substitution $t = \sqrt{x}$. Then $x = t^2$, hence $dx = 2t\ dt$.
- The transformed integral is: $\int e^t \cdot 2t\ dt = 2\int te^t\ dt$.
- Integration by parts now leads to the solution of the problem.
- The remaining details are left to the students.

Riemann integration: Examples: Substitution and integration by parts

Example

$\int \frac{\ln(\ln x)}{x}\ dx.$

- Let $u = \ln x$. Then $du = \frac{1}{x}\ dx$.
- The transformed integral becomes $\int \ln u\ du$.
- We have already seen how to compute this using integration by parts.
- The remaining details are left to the students.

Example

$\int \sin(\sqrt{x})\ dx$.

- We perform the substitution $t = \sqrt{x}$. Then $dx = 2t\ dt$.
- The new form is $2\int t \sin t\ dt$.
- We have already seen a similar expression. Using integration by parts, we replace t by its derivative and $\sin t$ by its antiderivative.
- The remaining details are left to the students.

Example

$\int \frac{5x-3}{x^2-2x-3}\, dx.$

- We factorize the denominator: $x^2-2x-3=(x-3)(x+1)$.
- We rewrite the fraction as $\frac{A}{x-3}+\frac{B}{x+1}$.

Eliminating denominators yields: $5x-3=A(x+1)+B(x-3)$. Substituting $x=3$ yields $12=4A$, hence $A=3$. Substituting $x=-1$ gives $-8=-4B$, hence $B=2$.

$$\begin{aligned}\int \frac{5x-3}{x^2-2x-3}\, dx &= \int \frac{3}{x-3}\, dx+\int \frac{2}{x+1}\, dx\\ &= 3\ln(x-3)+2\ln(x+1)+c.\end{aligned}$$

Example

$\int \frac{x^2+1}{x(x-1)^2}\, dx.$

- The denominator is factorized: one factor x and a squared factor $(x-1)^2$.
- We aim for the decomposition: $\frac{x^2+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$.
- Eliminating denominators yields:

$x^2+1 = A(x-1)^2 + Bx(x-1) + Cx$. Substituting $x=0$ gives $1=A$. Substituting $x=1$ gives $2=C$. Comparing coefficients of x^2 gives $1=A+B$, hence $B=0$.

$$\int \frac{x^2+1}{x(x-1)^2}\, dx = \int \left(\frac{1}{x} + \frac{2}{(x-1)^2}\right) dx$$

$$= \ln x - \frac{2}{x-1} + c.$$

Example

$\int \frac{2x^2+x+4}{x^3+4x}\ dx.$

- The denominator can be factored as: $x(x^2+4)$. The factor x^2+4 cannot be further decomposed over the real polynomials.
- We aim to rewrite the expression in the form:

$$\frac{2x^2+x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

- Eliminating denominators yields::

$$2x^2+x+4 = A(x^2+4) + (Bx+C)x.$$

Substituting $x=0$ gives $A=1$. Comparing coefficients yields $B=1$ and $1=C$.

$$\begin{aligned}\int \frac{2x^2+x+4}{x^3+4x}\ dx &= \int \frac{1}{x}\ dx + \int \frac{x+1}{x^2+4}\ dx \\ &= \ln x + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c.\end{aligned}$$

This is the end!

Thank you for your attention!