BSc Mathematics for Computer Scientists 2: IX. Applications of Differential Calculus: Function Analysis, L'Hospital's Rule

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Reminder: Basic Derivatives

Reminder

$(c)' = 0 \quad (c \in \mathbb{R})$	$(x^\alpha)' = \alpha x^{\alpha-1} \quad (\alpha \in \mathbb{R})$
$(e^x)' = e^x$	$(a^x)' = a^x \ln a \quad (a > 0,\ a \neq 1)$
$(\ln x)' = \frac{1}{x}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\arctan x)' = \frac{1}{1+x^2}$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$
$(\operatorname{arcsec} x)' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$(\operatorname{arccsc} x)' = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$(\sinh x)' = \cosh x$	$(\cosh x)' = \sinh x$
$(\tanh x)' = \operatorname{sech}^2 x$	$(\coth x)' = -\operatorname{csch}^2 x$
$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$	$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$
$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}}$	$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
$(\operatorname{artanh} x)' = \frac{1}{1-x^2} \quad (\lvert x\rvert < 1)$	$(\operatorname{arcoth} x)' = \frac{1}{1-x^2} \quad (\lvert x\rvert > 1)$
$(\operatorname{arsech} x)' = -\frac{1}{x\sqrt{1-x^2}}$	$(\operatorname{arcsch} x)' = -\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

Function Analysis: Theorems

Theorem

Let f be a differentiable function defined on an interval.

(i) If $f'(x) \geq 0$ ($x \in \text{dom } f$), then f is monotone increasing.

(ii) If $f'(x) > 0$ ($x \in \text{dom } f$), then f is strictly increasing.

Theorem

Let f be a differentiable function defined on an interval.

(i) If $f'(x) \leq 0$ ($x \in \text{dom } f$), then f is monotone decreasing.

(ii) If $f'(x) < 0$ ($x \in \text{dom } f$), then f is strictly decreasing.

Theorem

Let f be a differentiable function defined on an interval.

(i) If $f''(x) \geq 0$ ($x \in \text{dom } f$), then f is convex.

(ii) If $f''(x) > 0$ ($x \in \text{dom } f$), then f is strictly convex.

Theorem

Let f be a differentiable function defined on an interval.

(i) If $f''(x) \leq 0$ ($x \in \text{dom } f$), then f is concave.

(ii) If $f''(x) < 0$ ($x \in \text{dom } f$), then f is strictly concave.

Function Analysis: Theorems (continued)

Theorem

Let f be a differentiable function defined on an open interval. If f has a local extremum at x_0, then $f'(x_0) = 0$.

Example

$f(x) = x^3$

Theorem

Let f be a differentiable function defined on an open interval, containing x_0. If $f'(x_0) = 0$ and $f''(x_0) > 0$ then has a local minimum point at x_0.

Theorem

Let f be a differentiable function defined on an open interval, containing x_0. If $f'(x_0) = 0$ and $f''(x_0) < 0$ then has a local maximum point at x_0.

Function Analysis: Theorems (continued)

Theorem

Let f be a differentiable function defined on an open interval, containing x_0. If $f'(x_0) = 0$ and for a suitable $\delta > 0$ $f'(x) < 0$ if $x \in]x_0 - \delta, x_0[$, furthermore $f'(x) > 0$ if $x \in]x_0, x_0 + \delta[$ then f has a local minimum point at x_0.

Theorem

Let f be a differentiable function defined on an open interval, containing x_0. If $f'(x_0) = 0$ and for a suitable $\delta > 0$ $f'(x) > 0$ if $x \in]x_0 - \delta, x_0[$, furthermore $f'(x) < 0$ if $x \in]x_0, x_0 + \delta[$ then f has a local minimum point at x_0.

Example

$$f(x) = \frac{x^2 - 1}{x^2 - 4}.$$

- This is a rational function.
- x appears only as x^2. Therefore f is an even function.
- $f(x) = \frac{x^2-1}{x^2-4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$.
- In its definition, only the division is problematic. Thus it is undefined exactly at $x = -2$ and $x = 2$ (since we would divide by 0).
- Domain: $]-\infty, -2\, [\ \cup\ ]-2, 2\, [\ \cup\ ]2, \infty\, [$.

Function Analysis: Example 1: Signs

• Its value is 0 exactly when it is defined and the numerator is 0: its roots are $\{-1, 1\}$. That is, the graph intersects the x-axis at these points.

• The numerator is positive if $x \in]-\infty, -1\ [\ \cup\ ]\ 1, \infty\ [$. The numerator is negative if $x \in]-1, 1\ [$. The denominator is positive if $x \in]-\infty, -2\ [\ \cup\ ]\ 2, \infty\ [$. The denominator is negative if $x \in]-2, 2\ [$.

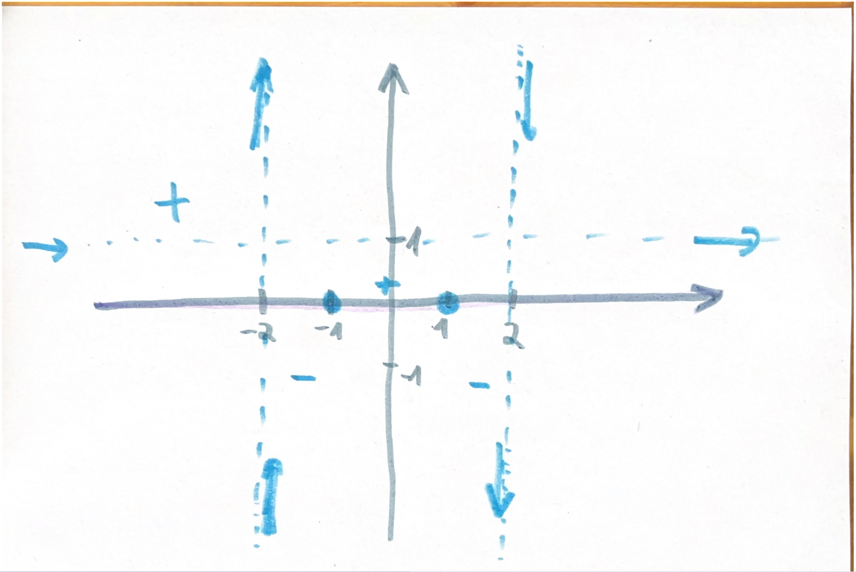
• The function is positive on $]-\infty, -2\ [\ \cup\ ]-1, 1\ [\ \cup\ ]2, \infty\ [$. The function is negative on $]-2, -1\ [\ \cup\ ]\ 1, 2\ [$.

Function Analysis: Example 1: Limits

The boundary points of the domain $]-\infty,-2\,[\ \cup\ ]-2,2\,[\ \cup\ ]2,\infty\,[$ in $\widehat{\mathbb{R}}$ are: $\{-\infty,-2,2,\infty\}$. At these points we determine the one-sided limits: $\lim_{x\to-\infty} f(x)$, $\lim_{x\to-2-} f(x)$, $\lim_{x\to-2+} f(x)$, $\lim_{x\to2-} f(x)$, $\lim_{x\to2+} f(x)$, $\lim_{x\to\infty} f(x)$.

- $\lim_{x\to-\infty}\frac{x^2-1}{x^2-4}=\lim_{x\to-\infty}\frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}}=1.$
- $\lim_{x\to-2-}\frac{x^2-1}{x^2-4}=\infty$ (if $x\to-2-$, then $x^2\to4+$).
- $\lim_{x\to-2+}\frac{x^2-1}{x^2-4}=-\infty$ (if $x\to-2+$, then $x^2\to4-$).
- $\lim_{x\to2-}\frac{x^2-1}{x^2-4}=-\infty$ (if $x\to2-$, then $x^2\to4-$).
- $\lim_{x\to2+}\frac{x^2-1}{x^2-4}=\infty$ (if $x\to2+$, then $x^2\to4+$).
- $\lim_{x\to\infty}\frac{x^2-1}{x^2-4}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}}=1.$

Function Analysis: Example 1: First Summary



First derivative (quotient rule):

$$\begin{aligned} f'(x) &= \frac{2x(x^2-4)-(x^2-1)2x}{(x^2-4)^2} \\ &= \frac{2x\big[(x^2-4)-(x^2-1)\big]}{(x^2-4)^2} \\ &= \frac{2x(-3)}{(x^2-4)^2} = \frac{-6x}{(x^2-4)^2} \end{aligned}$$

First derivative ($(g^{-1})' = -g^{-1} \cdot g'$):

$$f'(x) = \left(\frac{x^2-1}{x^2-4}\right)' = \left(1+\frac{3}{x^2-4}\right)' = 3\cdot(-1)\frac{1}{(x^2-4)^2}\cdot 2x.$$

The denominator is positive for all x in the domain, thus the sign of $f'(x)$ is the same as the sign of $-6x$.

Function Analysis: Example 1: Monotonicity

- On the interval $]-\infty,-2\,[$, $f'(x)>0$: f is strictly increasing.
- On the interval $]-2,0\,[$, $f'(x)>0$: f is strictly increasing.
- On the interval $]\,0,2\,[$, $f'(x)<0$: f is strictly decreasing.
- On the interval $]\,2,\infty\,[$, $f'(x)<0$: f is strictly decreasing.

Function Analysis: Example 1: Derivative, Extremum

The derivative is zero exactly when $x = 0$.

• In a neighborhood of $x = 0$, f is increasing from the left and decreasing from the right.

• There is a local maximum at $x = 0$.

• The maximal value on the interval $]-2, 2[$ is

$$f(0) = \frac{-1}{-4} = \frac{1}{4}.$$

Function Analysis: Example 1: Range

• On the interval $(-\infty, -2)$, f is increasing and continuous, $\lim_{x\to-\infty} f = 1^+$, $\lim_{x\to-2^-} f = +\infty$. The set of values taken here is $]1, \infty[$.

• On the interval $(-2, 0)$, f is continuous and increasing, $\lim_{x\to-2^+} f = -\infty$, $f(0) = 1/4$. The set of values taken here is $(-\infty, 1/4]$.

• On the interval $(0, 2)$, f is continuous and decreasing, $f(0) = 1/4$, $\lim_{x\to 2^-} f = -\infty$. The set of values taken here is $(-\infty, 1/4]$.

• On the interval $(2, \infty)$, f is continuous and decreasing, $\lim_{x\to 2^+} f = +\infty$, $\lim_{x\to\infty} f = 1^+$. The set of values taken here is $(1, \infty)$.

• Summarizing:

$$\mathrm{im}_f =]-\infty, \tfrac{1}{4}] \cup]1, \infty[$$

Function Analysis: Example 1: Second Derivative

Second derivative:

$$f''(x) = \frac{d}{dx}\left(\frac{-6x}{(x^2-4)^2}\right)$$

We apply the quotient rule again:

$$\begin{aligned} f''(x) &= \frac{(-6)(x^2-4)^2-(-6x)\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4} \\ &= \frac{-6(x^2-4)^2+24x^2(x^2-4)}{(x^2-4)^4} \\ &= \frac{(x^2-4)\left[-6(x^2-4)+24x^2\right]}{(x^2-4)^4} \\ &= \frac{-6x^2+24+24x^2}{(x^2-4)^3} = \frac{18x^2+24}{(x^2-4)^3} = \frac{6(3x^2+4)}{(x^2-4)^3}. \end{aligned}$$

• Since $3x^2+4>0$ for all x, the sign of $f''(x)$ is determined by $(x^2-4)^3$, which has the same sign as x^2-4 (raising to the third power preserves the sign).

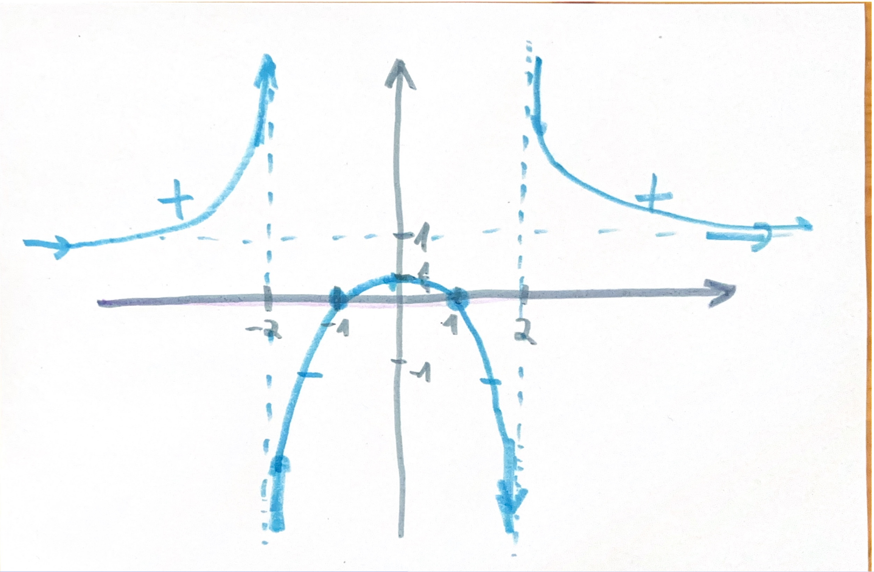
Function Analysis: Example 1: Convexity

• On the interval $]-\infty,-2[$, $x^2-4>0$, hence $f''(x)>0$. Thus on $]-\infty,-2[$, f is strictly convex.

• On the interval $]-2,2[$, $x^2-4<0$, hence $f''(x)<0$. Thus on $]-2,2[$, f is strictly concave.

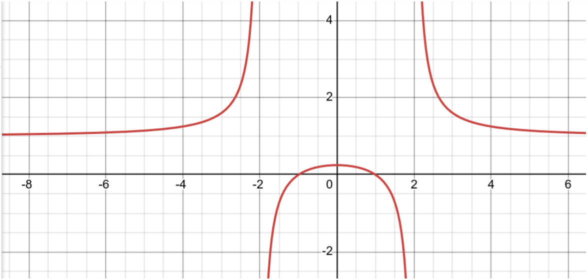
• On the interval $]\,2,\infty\,[$, $x^2-4>0$, hence $f''(x)>0$. Thus on $]2,\infty[$, f is strictly convex.

Function Analysis: Example 1: Final Summary



Function Analysis: Example 1: The Exact Graph

$$\frac{x^2-1}{x^2-4}$$



Function Analysis: Example 2: Domain, Signs, Roots

Example

$$g(x) = \frac{1}{1 + e^{\frac{1}{1-x^2}}}$$

• x appears only as x^2. Thus g is an even function.

• In its definition, only the division is problematic. The denominator of the "big" fraction is a shifted power of e. Thus we divide by a positive number, never by zero.

• In the exponent we divide by $1 - x^2$. Hence the function is undefined exactly at $x = -1$ and $x = 1$ (since we would divide by 0).

• Domain: $]-\infty, -1\,[\ \cup\ ]-1, 1\,[\ \cup\ ]1, \infty\,[$.

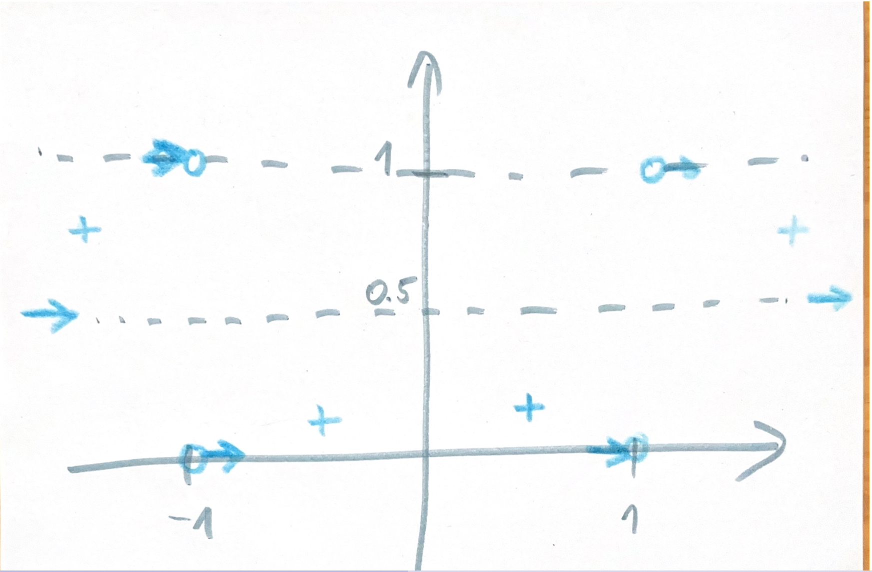
• The function value is the reciprocal of a positive number. Thus it is always positive. Hence it is never 0, it has no roots.

Function Analysis: Example 2: Limits

The boundary points of the domain $]-\infty,-1\,[\ \cup\ ]-1,1\,[\ \cup\ ]1,\infty\,[$ in $\widehat{\mathbb{R}}$ are: $\{-\infty,-1,1,\infty\}$. At these points we determine the one-sided limits: $\lim_{x\to-\infty} g(x)$, $\lim_{x\to-1-} g(x)$, $\lim_{x\to-1+} g(x)$, $\lim_{x\to1-} g(x)$, $\lim_{x\to1+} g(x)$, $\lim_{x\to\infty} g(x)$.

- $\lim_{x\to-\infty} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{1+e^0} = \frac{1}{2}$ (if $x\to-\infty$, then $\frac{1}{1-x^2}\to 0$).
- $\lim_{x\to-1-} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{1+0} = 1$ (if $x\to-1-$, then $\frac{1}{1-x^2}\to -\infty$).
- $\lim_{x\to-1+} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{+\infty} = 0$ (if $x\to-1+$, then $\frac{1}{1-x^2}\to +\infty$).
- $\lim_{x\to1-} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{+\infty} = 0$ (if $x\to1-$, then $\frac{1}{1-x^2}\to +\infty$).
- $\lim_{x\to1+} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{1+0} = 1$ (if $x\to1+$, then $\frac{1}{1-x^2}\to -\infty$).
- $\lim_{x\to\infty} \frac{1}{1+e^{\frac{1}{1-x^2}}} = \frac{1}{1+e^0} = \frac{1}{2}$ (if $x\to\infty$, then $\frac{1}{1-x^2}\to 0$).

Function Analysis: Example 2: First Summary



Function Analysis: Example 2: Differentiation

First derivative (quotient / chain rule): Let $k(x) = \frac{1}{1-x^2}$. Then $g(x) = \left(1 + e^{k(x)}\right)^{-1}$. First derivative:

$$\begin{aligned} g'(x) &= -\frac{1}{\left(1+e^{k(x)}\right)^2} \cdot e^{k(x)} \cdot k'(x) = -\frac{1}{\left(1+e^{\frac{1}{1-x^2}}\right)^2} \cdot e^{\frac{1}{1-x^2}} \cdot k'(x) \\ &= -\frac{1}{\left(1+e^{\frac{1}{1-x^2}}\right)^2} \cdot e^{\frac{1}{1-x^2}} \cdot (-1)\frac{1}{(1-x^2)^2}(-2x) \\ &= -\frac{2x \cdot e^{\frac{1}{1-x^2}}}{\left(1+e^{\frac{1}{1-x^2}}\right)^2 (1-x^2)^2}. \end{aligned}$$

- Thus $\operatorname{sgn}(g'(x)) = -\operatorname{sgn}(x)$.

Function Analysis: Example 2: Differentiation, Monotonicity

- On the interval $]\infty, -1[$, $f'(x) > 0$: f is strictly increasing.
- On the interval $]-1, 0[$, $f'(x) > 0$: f is strictly increasing.
- On the interval $]0, 1[$, $f'(x) < 0$: f is strictly decreasing.
- On the interval $]1, \infty[$, $f'(x) < 0$: f is strictly decreasing.

Function Analysis: Example 2: Differentiation, Extremum

The derivative is zero exactly when $x = 0$.

• In a neighborhood of $x = 0$, f is increasing from the left and decreasing from the right.

• Thus, there is a local maximum at $x = 0$.

• The maximal value on the interval $]-1, 1[$ is

$$f(0) = \frac{1}{1} = \frac{1}{1 + e}.$$

Function Analysis: Example 2: Differentiation, Range

• On the interval $(-\infty, -1)$, f is increasing and continuous, $\lim_{x\to-\infty} f = \frac{1}{2}^{+}$, $\lim_{x\to-1^-} f = 1$. The set of values taken here is $]\frac{1}{2}, 1[$.

• On the interval $(-1, 0)$, f is continuous and increasing, $\lim_{x\to-1^+} f = 0$, $f(0) = \frac{1}{1+e}$. The set of values taken here is $]0, \frac{1}{1+e}]$.

• On the interval $(0, 1)$, f is continuous and decreasing, $f(0) = \frac{1}{1+e}$, $\lim_{x\to1^-} f = 0$. The set of values taken here is $]0, \frac{1}{1+e}]$.

• On the interval $(1, \infty)$, f is continuous and decreasing, $\lim_{x\to1^+} f = 1$, $\lim_{x\to\infty} f = \frac{1}{2}^{+}$. The set of values taken here is $(\frac{1}{2}, 1)$.

• Summarizing:

$$\mathrm{im}_f =]0, \tfrac{1}{1+e}] \cup]\tfrac{1}{2}, 1[.$$

The second derivative:

$$g''(x) = -\frac{e^k(1+e^k)}{(1-x^2)^4\left(1+e^k\right)^4}\,G(x),$$

where $k = \frac{1}{1-x^2}$ and

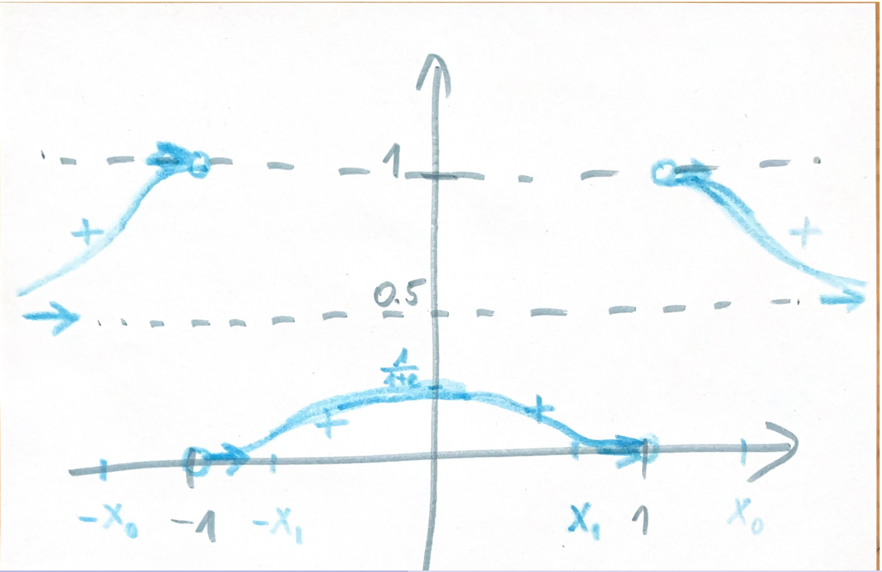
$$G(x) = 2 + 8x^2 - 6x^4 + (2-6x^4)e^k.$$

The denominator is positive, thus $\operatorname{sgn}(f'') = -\operatorname{sgn}(G)$.

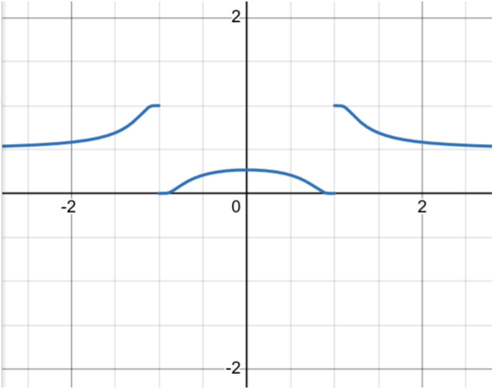
Function Analysis: Example 2: Convexity

Analyzing the sign of $G(x)$ yields the following result (inflection points determined numerically):

- On $]-\infty,-x_0[$: $G<0 \Rightarrow f''>0 \Rightarrow g$ is convex.
- On $]-x_0,-1[$: $G>0 \Rightarrow f''<0 \Rightarrow f$ is concave.
- On $(-1,-x_1)$: $G>0 \Rightarrow f''<0 \Rightarrow f$ is concave.
- On $(-x_1,x_1)$: $G>0 \Rightarrow f''<0 \Rightarrow f$ is concave.
- On $(x_1,1)$: $G>0 \Rightarrow f''>0 \Rightarrow f$ is convex.
- On $(1,x_1)$: $G<0 \Rightarrow f''<0 \Rightarrow f$ is concave.
- On (x_1,∞): $G>0 \Rightarrow f''>0 \Rightarrow f$ is convex.
- The inflection points are of the form $\pm x_1$ and $\pm x_2$, where $0<x_1<1$ and $x_0>1$.


1
0.5
$-x_0$
-1
$-x_1$
x_1
1
x_0

$$\frac{1}{1+e^{\frac{1}{1-x^2}}}$$



Break

Rolle's Theorem

Theorem [Rolle's Theorem]

Let f be continuous on the closed interval $[a, b]$, differentiable on the open interval $]a, b[$, and suppose that $f(a) = f(b)$. Then there exists at least one $\xi \in]a, b[$ such that

$$f'(\xi) = 0.$$

Mean Value Theorem (Lagrange)

Theorem [Lagrange's Theorem]

Let f be continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists $\xi \in]a, b[$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

(Equivalently: $f(b) - f(a) = f'(\xi)(b - a)$.)

Theorem [Cauchy's Theorem]

Let f and g be continuous on $[a,b]$, differentiable on $]a,b[$, and assume that $g'(x) \neq 0$ for all $x \in]a,b[$. Then there exists $\xi \in]a,b[$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Remark: From the assumptions it follows that $g(b) \neq g(a)$.

Darboux's Theorem: Intermediate Value Property of Derivatives

Theorem [Darboux's Theorem]

Let f be differentiable on an interval. Then f' (not necessarily continuous) has the intermediate value property: if $f'(a) < k < f'(b)$, then there exists a point c between a and b such that $f'(c) = k$.

Relationships

• Rolle's theorem is a special case of Lagrange's theorem ($f(a) = f(b)$).

• Lagrange's theorem is a special case of Cauchy's theorem ($g(x) = x$).

• The proof of Cauchy's theorem is based on Rolle's theorem. (Apply Rolle's theorem to the auxiliary function $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$.)

Break

L'Hospital's Rule: Theorem

Theorem [L'Hospital's Rule]

Let f and g be differentiable functions in a neighborhood of a (possibly excluding a itself), and assume:

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$.
2. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists $\widehat{\mathbb{R}}$.
3. $g'(x) \neq 0$ in a ball centered at a (possibly excluding a).

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof: Application of Cauchy's mean value theorem.

Remark: The rule also holds for one-sided limits and for $a = \pm\infty$ (with appropriate reparametrization).

L'Hospital's Rule: Example

Example ($\frac{0}{0}$ form)

$\lim_{x\to 0} \frac{\sin x}{x} =?$

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

L'Hospital's Rule: Example

Example ($\frac{0}{0}$ form)

$\lim_{x\to 0} \frac{e^x - 1}{x} = ?$

$$\lim_{x\to 0} \frac{e^x - 1}{x} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1.$$

L'Hospital's Rule: Example

Example ($\frac{0}{0}$ form)

$$\lim_{x\to 1} \frac{\ln x}{x-1} = ?$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\text{L'H.}}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

L'Hospital's Rule: Example

Example ($\frac{\infty}{\infty}$ form)

$\lim_{x\to\infty} \frac{x^2}{e^x} =?$

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

L'Hospital's Rule: Example

Example ($0 \cdot (-\infty)$ form $\to$ rewrite as $\frac{\infty}{\infty}$ or $\frac{0}{0}$)

$\lim_{x \to 0^+} x \ln x = ?$

Rewrite: $x \ln x = \frac{\ln x}{1/x}$.

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \left(\frac{-\infty}{\infty}\right) \overset{\text{L'H.}}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

L'Hospital's Rule: Example

Example ($0 \cdot (-\infty)$ form $\to$ rewrite $\to$ $\frac{\infty}{\infty}$ or $\frac{0}{0}$)

$$\frac{1}{\sin x} - \frac{1}{x} = ?$$

Rewrite:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}.$$

Now $\frac{0}{0}$ form:

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} \overset{\text{L'H.}}{=} \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}.$$

Still $\frac{0}{0}$, apply L'Hospital again:

$$= \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

L'Hospital's Rule: Example

Example (0^0 form $\to$ logarithm $\to$ $\frac{0}{0}$ form)

$\lim\limits_{x\to 0^+} x^x =?$

Let $y = x^x$, $\ln y = x \ln x$. From a previous example $\lim_{x\to 0^+} x \ln x = 0$. Since $\ln y \to 0$, we get $y = e^{\ln y} \to e^0 = 1$. Thus

$$\lim_{x\to 0+} x^x = 1.$$

L'Hospital's Rule: Example

Example (∞^0 form)

$\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x} = ?$

Let $y = (\tan x)^{\cos x}$, $\ln y = \cos x \cdot \ln(\tan x)$. Since $\cos x \to 0^+$, $\ln(\tan x) \to +\infty$, we have the form $0 \cdot \infty$.
Rewrite:

$$\ln y = \frac{\ln(\tan x)}{1/\cos x} = \frac{\ln(\tan x)}{\sec x}.$$

This is $\frac{\infty}{\infty}$. Apply L'Hospital:

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{\sec x \tan x} = \lim \frac{\sec x}{\tan^2 x} = \lim \frac{1/\cos x}{\sin^2 x/\cos^2 x} = \lim \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

Thus $\ln y \to 0 \Rightarrow y \to 1$.

L'Hospital's Rule: Counterexample

Counterexample

$$\lim_{x\to\infty} \frac{x+\sin x}{x} =?$$

- Numerator: $f(x) = x + \sin x$, denominator: $g(x) = x$.
- As $x \to \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.
- Both f and g are differentiable for $x > 0$.
- $g'(x) = 1 \neq 0$.
- Ratio of derivatives: $\frac{f'(x)}{g'(x)} = \frac{1+\cos x}{1} = 1 + \cos x$.
- $\lim_{x\to\infty}(1 + \cos x)$ DOES NOT EXIST, since $\cos x$ oscillates between -1 and 1.
- L'Hospital's rule requires that $\lim \frac{f'}{g'}$ exists (finite or infinite). This condition is not satisfied here.

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \left(1 + \frac{\sin x}{x}\right) = 1 + 0 = 1.$$

Thank you for your attention!