

BSc Mathematics for Computer Scientists 2: VII

Continuity of real functions

Péter Hajnal

Bolyai Institute, TTIK, SZTE, Szeged, Hungary

2026 Spring semester

Reminder: Limit

Let $f : \text{dom } f \rightarrow \mathbb{R}$ and $h \in L_f \subset \widehat{\mathbb{R}}$. We write

$$\lim_{x \rightarrow h} f(x) = \ell \in \widehat{\mathbb{R}},$$

if for every ball B centered at ℓ there exists a ball B_0 centered at h such that for $x \in (B_0 - \{h\}) \cap \text{dom } f$ we have $f(x) \in B$.

Reminder: Limit of sequences

Let $f : \mathbb{N} \rightarrow \mathbb{R}$. We write

$$\lim_{x \rightarrow \infty} f(x) = \ell \in \widehat{\mathbb{R}},$$

if for every ball B centered at ℓ there exists a ball $B_0 =]N, \infty[$ such that for $x \in B_0 \cap \text{dom } f$ we have $f(x) \in B$.

Limit: Convergence of Sequences

Definition

The sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ is convergent if there exists $\ell \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} f(n) = \ell.$$

Definition

The sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ is divergent if it is not convergent.

Definition

The sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ is properly divergent if

$$\lim_{n \rightarrow \infty} f(n) = \infty, \text{ or } \lim_{n \rightarrow \infty} f(n) = -\infty.$$

Definition

The sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ is oscillatory if it is divergent but not properly divergent.

Definition: Cauchy sequence

A sequence $a : \mathbb{N} \rightarrow \mathbb{R}$ is a Cauchy sequence if for every positive ε there exists a ball B centered at ∞ such that for all $n, m \in B$ we have $|a_n - a_m| < \varepsilon$.

Theorem: Cauchy theorem

The following two properties of sequences are equivalent:

- (i) The sequence $s : \mathbb{N} \rightarrow \mathbb{R}$ is convergent,
- (ii) The sequence $s : \mathbb{N} \rightarrow \mathbb{R}$ is a Cauchy sequence.

Definition: Cauchy-0 sequence

A sequence $a : \mathbb{N} \rightarrow \mathbb{R}$ is a Cauchy-0 sequence if for every positive ε there exists a ball B centered at ∞ such that for all $n \in B$ we have $|a_n| < \varepsilon$.

Theorem: Cauchy theorem

The following two properties of sequences are equivalent:

- (i) The sequences $s : \mathbb{N} \rightarrow \mathbb{R}$ and $t : \mathbb{N} \rightarrow \mathbb{R}$ converge to the same number,
- (ii) The sequence $s - t : \mathbb{N} \rightarrow \mathbb{R}$ is a Cauchy-0 sequence.

The above equivalence also holds if we consider rational sequences $s : \mathbb{N} \rightarrow \mathbb{Q}$. Naturally, the limits of the sequences are real numbers. This theorem can serve as a foundation for the mathematically rigorous construction of the real numbers based on the rational numbers.

Break



Functions: Continuity

- Intuitively, continuity means that the graph of the function is "not broken", i.e., it can be drawn without lifting the pencil.

Definition: Local continuity without limits

The real function $f : \text{dom } f \rightarrow \mathbb{R}$ is continuous at the point $x_0 \in \text{dom } f$ if for every ball B centered at $f(x_0)$ there exists a ball B_0 centered at x_0 such that for all $x \in B_0 \cap \text{dom } f$ we have $f(x) \in B$.

Definition: Local continuity with limits

The function $f : \text{dom } f (\subset \mathbb{R}) \rightarrow \mathbb{R}$ is continuous at the point $x_0 \in \text{dom } f \cap L_f$ if:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

- If $x_0 \in \text{dom } f$ but $x_0 \notin L_f$, then we say that x_0 is an isolated point of the domain.

At isolated points, the function f is considered continuous!

Definition: Local continuity with sequences

The function $f : \text{dom } f (\subset \mathbb{R}) \rightarrow \mathbb{R}$ is continuous at the point $x_0 \in \text{dom } f \cap L_f$ if for every sequence $(s_n)_{n \in \mathbb{N}}$ with $s_n \rightarrow x_0$:

$$\lim_{n \rightarrow \infty} f(s_n) = f(x_0).$$

Definition: Global continuity

The function $f : \text{dom } f (\subset \mathbb{R}) \rightarrow \mathbb{R}$ is continuous (on its domain) if it is continuous at every point $x_0 \in \text{dom } f$.

Theorem

Assume that $a \in \mathbb{R}$ is not in the domain of f . Let $f^{\rightarrow a} : \text{dom } f (\subset \mathbb{R}) \cap]-\infty, a[\rightarrow \mathbb{R}, \quad x \mapsto f(x)$, and $f^{a \rightarrow} : \text{dom } f (\subset \mathbb{R}) \cap]a, \infty[\rightarrow \mathbb{R}, \quad x \mapsto f(x)$.

Then f is continuous if and only if both $f^{\rightarrow a}$ and $f^{a \rightarrow}$ are continuous.

Example: Constant function

$const_c : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto c$ is continuous.

Example: Power function with positive integer exponent

$p_n : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto x^n$ (n positive integer) is continuous.

Example: Power function with integer exponent

Let k be a nonzero integer. $p_k : \mathbb{R} - \{0\} \rightarrow \mathbb{R}, \quad x \mapsto x^k$ is continuous.

It is useful to think of (for example) $p_{-1}(x) = \frac{1}{x}$ as the combination of two continuous functions:

$p_{-1}^- :]-\infty, 0[\rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{x}$ and $p_{-1}^+ :]0, \infty[\rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{x}$. The domains of p_{-1}^- and p_{-1}^+ are intervals, in contrast to the domain of p_{-1} .

Example: Power function

For $\alpha > 0$: $p_\alpha : \mathbb{R}_+ = [0, \infty[\rightarrow \mathbb{R}$, $x \mapsto x^\alpha$ is continuous.

Example: Exponential function

For $a > 0$: $\exp_a : \mathbb{R}_+ \rightarrow \mathbb{R}$, $x \mapsto a^x$ is continuous.

Example: Sine function

$\sin : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \sin x$ is continuous.

Example: Cosine function

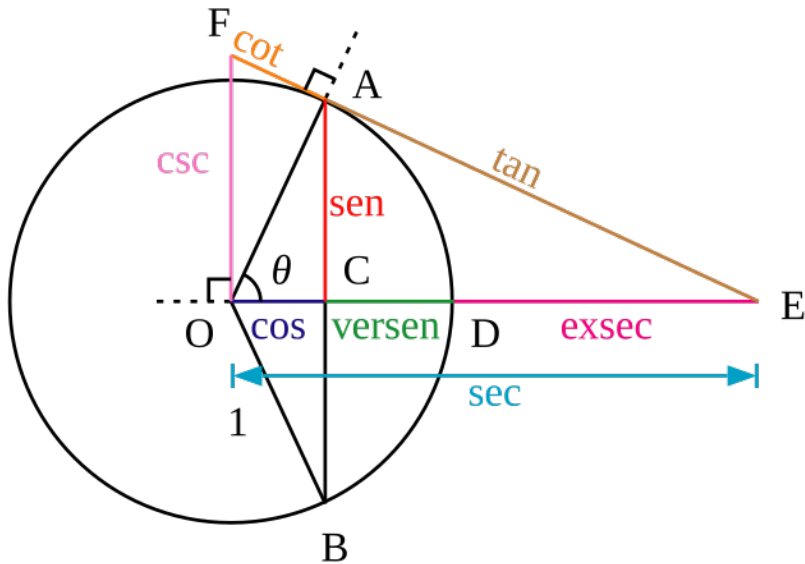
$\cos : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \cos x$ is continuous.

Example: Tangent function

$\tan : \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \rightarrow \mathbb{R}, \quad x \mapsto \tan x$ is continuous.

Again, we can think of the tangent function as being composed of functions $\tan_k :]\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi[\rightarrow \mathbb{R}, \quad x \mapsto \tan x$ ($k \in \mathbb{Z}$), each of which is continuous. The domains of the \tan_k functions are intervals.

Függvények: Folytonosság: Példák: Trigonometria: A kép



Theorem

If f and g are continuous at a point x_0 , then the following are also continuous at x_0 :

- (i) Sum, difference and product: $(f \pm g)$ and $(f \cdot g)$.
- (ii) Quotient: f/g is continuous if $g(x_0) \neq 0$.
- (iii) Composition: If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0 .
- (iv) Absolute value: $|f|$ is continuous.
- (v) Inverse: If moreover f is strictly increasing in a neighborhood of x_0 , then its inverse is also continuous at $f(x_0)$.

Naturally, under global continuity assumptions, global continuity results can be derived from this theorem.

Functions: Continuity: Further examples

Example

$\arcsin x : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a continuous function.

Example

$\arccos x : [-1, 1] \rightarrow [0, \pi]$ is a continuous function.

Example

$\arctan x : \mathbb{R} \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a continuous function.

Example

$\log_2 x : \mathbb{R}_{++} =]0, \infty[\rightarrow \mathbb{R}$ is a continuous function.

Example

$\ln x = \log_e x : \mathbb{R}_{++} =]0, \infty[\rightarrow \mathbb{R}$ is a continuous function.

Functions: Continuity: Further examples

Example

$e^{\frac{x^2}{2}}$ is a continuous function.

Example

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ is a continuous function.

Example

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ is a continuous function.

Example

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is a continuous function.

Example

$\operatorname{arsinh} x = \ln \left(x + \sqrt{x^2 + 1} \right) :] - \infty, \infty[\rightarrow \mathbb{R}$ is a continuous function.

Example

$\operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right) : [1, \infty[\rightarrow \mathbb{R}$ is a continuous function.

Example

$\operatorname{artanh} x = \frac{1}{2} \ln \frac{1-x}{1+x} :] - 1, 1[\rightarrow \mathbb{R}$ is a continuous function.

Break



Theorem [Weierstrass Theorem]

A continuous function defined on a compact interval is bounded, and moreover it attains its maximum and minimum on this interval.

Theorem [Weierstrass Theorem for compact sublevel sets]

Let f be a continuous function on its domain $\text{dom } f$. If for some $\tau \in \mathbb{R}$ the set

$$\{x \in \text{dom } f : f(x) \leq \tau\}$$

is a nonempty compact set, then f attains its minimum.

Theorem: Bolzano Theorem

If a continuous function on $[a, b] \subset \text{dom } f$ takes values of opposite sign at the endpoints ($f(a) \cdot f(b) < 0$), then it has at least one zero in the interior of the interval.

Corollary

If $p \in \mathbb{R}[x]$ is a real polynomial of odd degree, then p has a real root.

$$x^3 - 2x^2 + 5x - 3 = x^3 \left(1 - 2\frac{1}{x} + 5\frac{1}{x^2} - 3\frac{1}{x^3} \right)$$

$$1 - 2\frac{1}{x} + 5\frac{1}{x^2} - 3\frac{1}{x^3} \rightarrow 1, \text{ as } x \rightarrow \infty.$$

Thus $f(N) > 0$, $f(-M) < 0$.

Continuous functions on compact intervals: Properties

Theorem: Darboux Theorem (Intermediate Value Theorem)

A continuous function on $[a, b] \subset \text{dom } f$ takes every value between $f(a)$ and $f(b)$ at least once.

Corollary

The continuous image of an interval is also an interval.

Corollary

The continuous image of a compact interval is also a compact interval.

Example

Let $f(x) = \tan^2 x$ be a continuous function on the interval $I =] - \frac{\pi}{2}, \frac{\pi}{2}[$. Then the image of the bounded, open interval I , namely $f(I) = \{f(i) : i \in I\}$, is the unbounded, closed interval $[0, \infty[$.

Continuous functions on compact intervals: Properties

Definition

A function is uniformly continuous if for every positive ε there exists a positive $\delta \in \mathbb{R}$ such that for any two points in the domain ($a, b \in \text{dom } f$ with $|a - b| < \delta$), the distance of the function values is less than ε ($|f(a) - f(b)| \leq \varepsilon$).

- Thus there are no B, B_0 balls here. The definition works with radii.
- Uniform continuity is a global concept.

Theorem: Heine Theorem

Every continuous function on a compact interval is uniformly continuous.

Example

$f :]0, 1] \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{x}$ is continuous, but not uniformly continuous.

Break



Discontinuity: Example

Example

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \lfloor x \rfloor$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \mathbb{Z}$.

- 1st statement: If $x_0 \in \mathbb{Z}$, then f is not continuous. Use the definition.
- 2nd statement: If $x_0 \notin \mathbb{Z}$, then f is continuous. Continuity at a point is a local property.

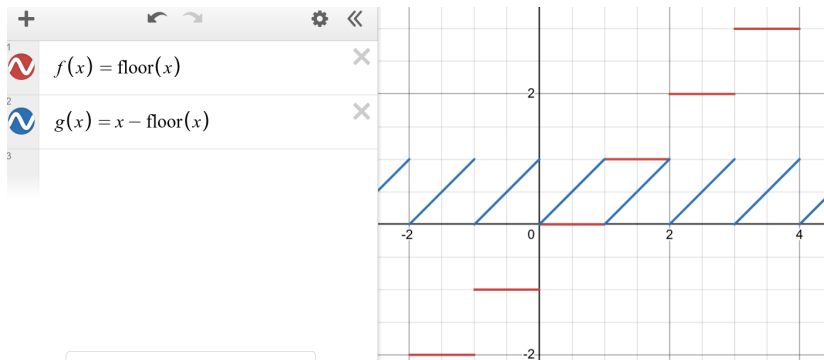
Theorem

Let f and g be two real functions. Assume that $f|_{B_\varepsilon(x_0)} = g|_{B_\varepsilon(x_0)}$, i.e., f and g coincide in a neighborhood of x_0 .
If f is continuous at x_0 , then so is g .

Example

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \{x\}$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \mathbb{Z}$.

Discontinuity: Illustration



Discontinuity: Example

Example

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \sin \frac{1}{x}$. Then f is continuous at $x_0 \in \mathbb{R} - \{0\}$.

- 1st statement: If $x_0 = 0$, then f is not continuous. Use the definition.
- 2nd statement: If $x_0 \neq 0$, then f is continuous. Continuity at a point is a local property.

Example

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Then f is continuous at $x_0 \in \mathbb{R}$.

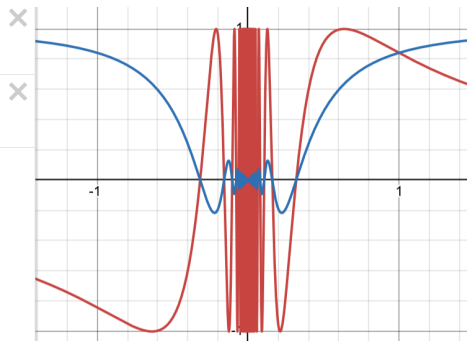
Discontinuity: Illustration



$$\sin \frac{1}{x}$$



$$x \sin \frac{1}{x}$$



Discontinuity: Example

Example

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$.

Then f is nowhere continuous.

Statement

For every center $c \in \mathbb{R}$ and every radius $\varepsilon > 0$, the ball $B_\varepsilon(c)$ contains both rational and irrational numbers.

No picture. Why?

Example [Thomae function / popcorn function / Riemann function]

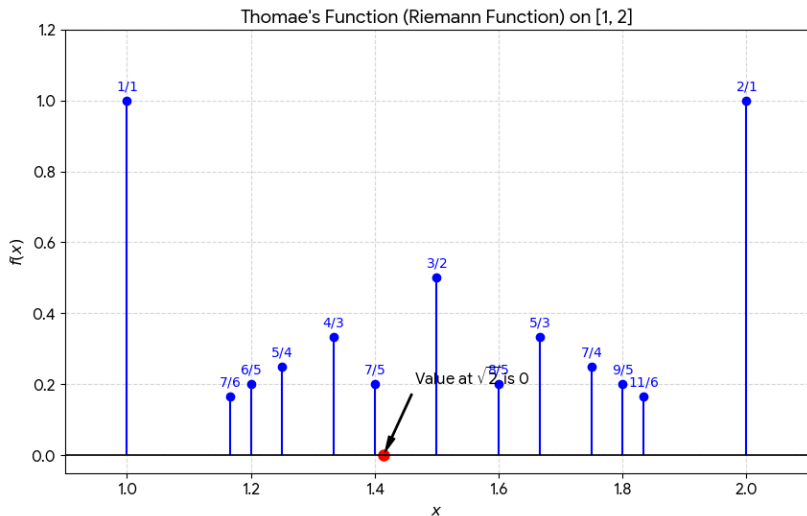
Let

$$f(x) : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ is rational, } \gcd(p, q) = 1 \end{cases}$$

f is continuous at x_0 if and only if x_0 is irrational.

- 1st statement: If x_0 is rational, then f is not continuous. Use the definition.
- 2nd statement: If x_0 is irrational, then f is continuous. Draw a picture.

Discontinuity: Illustration



Discontinuity: Points of discontinuity

Definition

A point x_0 is a point of discontinuity of f if $x_0 \in \text{dom } f$ and f is not continuous at x_0 .

Notation

$$D_f = \{x_0 \in \text{dom } f : x_0 \text{ is a point of discontinuity}\}.$$

- We have seen examples where D_f is one of the following sets: $\emptyset, \{0\}, \mathbb{Z}, \mathbb{Q}$.

Theorem

There is no function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $D_f = \mathbb{I} = \mathbb{R} - \mathbb{Q}$.

Discontinuity: Classification of discontinuities

Definition

Let $x_0 \in D_f$ be a point of discontinuity.

x_0 is a discontinuity of the first kind if $\lim_{x \rightarrow x_0^-} f(x)$ and $\lim_{x \rightarrow x_0^+} f(x)$ both exist.

Otherwise, x_0 is of the second kind.

Definition

A discontinuity of the first kind is called a jump discontinuity if

$\lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0^+} f(x)$.

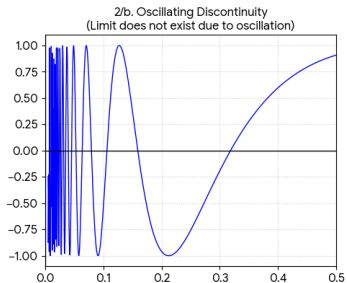
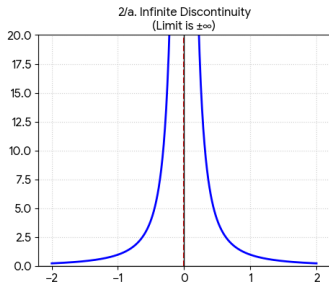
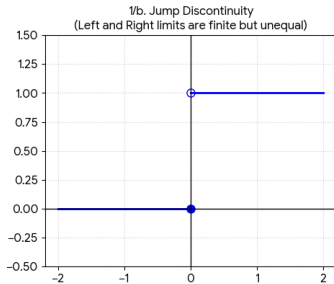
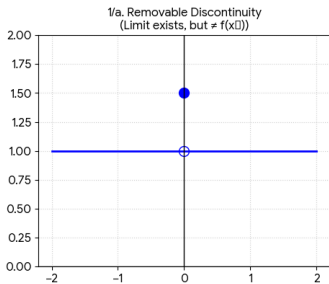
Otherwise, it is called removable.

Definition

A discontinuity of the second kind is called a pole if $\lim_{x \rightarrow x_0^-} f(x)$ and $\lim_{x \rightarrow x_0^+} f(x)$ are equal and their common value is ∞ or $-\infty$.

Otherwise, it is called an oscillatory discontinuity.

Discontinuity: Classification: Illustration



Break



Example

$$\lim_{x \rightarrow 2} e^{x^2+x} = ?$$

$f(x) = e^{x^2+x}$ is continuous.

$2 \in \text{dom } f$.

$$\lim_{x \rightarrow 2} e^{x^2+x} = f(2) = e^{2^2+2} = e^6.$$

Example

$$\lim_{x \rightarrow 2} (\sin^2(x^2 + x) + \cos x^3) = ?$$

$f(x) = \sin^2(x^2 + x) + \cos x^3$ is continuous.

$2 \in \text{dom } f$.

$$\lim_{x \rightarrow 2} (\sin^2(x^2 + x) + \cos x^3) = f(2) = \sin^2 6 + \cos 8.$$

This is the end!

Thank you for your attention!