BSc Mathematics for Computer Scientists 2:
VI. Foundations of Analysis

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Real sets: Bounded sets: Definitions

Unless stated otherwise, our sets are subsets of $\mathbb{R}$.

Definition: Upper bounded set

A set $S \subset \mathbb{R}$ is upper bounded if there exists $U \in \mathbb{R}$ such that for all $s \in S$, $s \leq U$.

Definition: Lower bounded set

A set $S \subset \mathbb{R}$ is lower bounded if there exists $L \in \mathbb{R}$ such that for all $s \in S$, $L \leq s$.

Definition: Bounded set

A set $S \subset \mathbb{R}$ is bounded if there exists $B \in \mathbb{R}$ such that for all $s \in S$, $|s| \leq B$.

Real sets: Bounded sets: Examples

Notation

$B_r(c) = \{x \in \mathbb{R} : |x - c| \leq r\}$ is the closed ball with center c and radius r.
$B_r^o(c) = \{x \in \mathbb{R} : |x - c| < r\}$ is the open ball with center c and radius r.

Example

$B_r(c)$ and $B_r^o(c)$ are bounded sets.

Observation

If K is bounded and $R \subset K$, then R is also bounded.

Lemma

A set K is bounded if and only if it is contained in some ball.

Real sets: Upper bounded sets: Maximum

Reminder

Any two distinct real numbers a, b are comparable: one is smaller, the other is larger. Thus $\max\{a, b\}$ is well-defined.

Observation

If S is a finite set, then $\max S$ exists.

Observation

Every finite set is bounded and has both maximum and minimum.

What happens for infinite sets?

Real sets: Upper bounded sets: Maximum

Example

$\{x \in \mathbb{R} : x^2 \leq 2\}$: Does it have a maximum? Yes: $\sqrt{2}$.

Example

$\{x \in \mathbb{R} : x^2 < 2\}$: Does it have a maximum? No.If $m \in S$, then $m < \frac{m+\sqrt{2}}{2} < \sqrt{2}$.

Example

$\{x \in \mathbb{Q} : x^2 \leq 2\} = \{x \in \mathbb{Q} : x^2 < 2\}$: Does it have a maximum? No.

Real sets: Upper bounded sets: Supremum

Reminder

A number U is an upper bound of S if $s \leq U$ for all $s \in S$.

Theorem

For every non-empty set S, the set of upper bounds has a smallest element.

More precisely, there exists $u \in \mathbb{R}$ such that: (1) u is an upper bound of S, (2) for every upper bound U, $u \leq U$.

Definition: Supremum

For a non-empty set $S \subset \mathbb{R}$,

$$\sup S = \min\{U \in \mathbb{R} : U \text{ is an upper bound of } S\}.$$

The existence of the supremum is a fundamental property of $\mathbb{R}$. It does not hold in $\mathbb{Q}$.

Real sets: Lower bounded sets: Infimum

Reminder

A number L is a lower bound of S if $L \leq s$ for all $s \in S$.

Theorem

For every non-empty set S, the set of lower bounds has a largest element.
More precisely, there exists $\ell \in \mathbb{R}$ such that: (1) ℓ is a lower bound of S, (2) for every lower bound L, $L \leq \ell$.

Definition: Infimum

For a non-empty set $S \subset \mathbb{R}$,

$$\inf S = \max\{L \in \mathbb{R} : L \text{ is a lower bound of } S\}.$$

The existence of the infimum is an easy consequence of the existence of supremum: $\inf S = -\sup(-S)$.

Real sets: Nested intervals

Let $I_i = [a_i, b_i]$ be a nested sequence of closed intervals:

$$a_1 \leq a_2 \leq \cdots \leq a_i \leq \cdots \leq b_i \leq \cdots \leq b_2 \leq b_1.$$

Theorem

$$\bigcap_{i \in \mathbb{N}_+} I_i \neq \emptyset.$$

Proof idea:

- All a_i are lower bounds of $\{b_i\}$.
- $\sup\{a_i\}$ is also a lower bound of $\{b_i\}$.

$$\Rightarrow \sup\{a_i\} \in I_i \text{ for all } i.$$

Real sets: Infimum/Supremum: Examples

Example

$$\begin{aligned}\sup\{x\in\mathbb{R}: x^2\le 2\} &= \sup\{x\in\mathbb{R}: x^2<2\}\\ &= \sup\{x\in\mathbb{Q}: x^2<2\} = \sqrt{2}.\end{aligned}$$

Theorem

$$\inf\left\{\frac{1}{2^n} : n\in\mathbb{N}\right\} = \inf\left\{\frac{1}{n} : n\in\mathbb{N}_+\right\} = 0.$$

Theorem

$$\sup\left\{1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n} : n\in\mathbb{N}_+\right\} = 2.$$

Real sets: Infimum/Supremum: Examples

Theorem

$$\sup\left\{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} : n\in\mathbb{N}_+\right\}<3.$$

Theorem

$$\sup\left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}_+\right\}<3.$$

Theorem

The above two suprema coincide.

Definition

Their common value is denoted by e.

Real sets: Interior points, open sets

Definition

A point b is an interior point of S if there exists $\varepsilon > 0$ such that $B^o_\varepsilon(b) \subset S$.

- Since $b \in B^o_\varepsilon(b) \subset S$, interior points belong to S.

Definition: Open set

A set U is open if every point is an interior point.

Example

For $a < b$, the interval $]a, b[$ is open.

Example

$[a, b]$ is NOT open, since a is not an interior point.

Real sets: Accumulation points, closed sets

Definition

A point ℓ is an accumulation point of S if for every $\varepsilon > 0$, $B_\varepsilon(\ell) \cap S$ is infinite.

Definition

A set S is closed if it contains all its accumulation points.

Example

$[a, b]$ is closed.

Example

$]a, b[$ is NOT closed, since a is an accumulation point but not in the set.

Theorem

Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

Real sets: Not doors (quote attributed to James Munkres)

"Theorem"

A door is either open or closed.

Example

$]0,1]$: neither open nor closed.

Example

$\mathbb{R}$: both open and closed.

Real sets: Compact sets

Theorem

For $S \subset \mathbb{R}$, the following are equivalent:

(i) Every open cover has a finite subcover,

(ii) S is bounded and closed,

(iii) Every infinite subset of S has an accumulation point in S,

(iv) Every sequence in S has a convergent subsequence with limit in S.

Definition

A set $S \subset \mathbb{R}$ is called compact if it satisfies the above properties.

Infinite sums: The question

Reminder

Let $a_0, a_o, a_O \in \mathbb{R}$. Then
$a_0+a_o+a_O = a_0+a_O+a_o = a_o+a_0+a_O = a_o+a_O+a_0 = a_O+a_0+a_o = a_O+a_o+a_0$.

We may say that the above common value is the sum of the numbers $\langle a_0, a_o, a_O\rangle$ (order does not matter). The same holds for any finite collection of numbers.

Definition

Let I be a finite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). Then

$$\sum_{i: i\in I} a_i$$

is a well-defined sum.

What happens if I is infinite?

We assume that infinite means countably infinite.

Infinite sums: Positive numbers

- Let I be a countably infinite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). Assume that

$$a_i \geq 0, \text{ for all } i \in I.$$

Definition

If $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } I\right\}$ is bounded above, then the sum of the numbers a_i ($i \in I$) is defined as

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } I\right\}.$$

- It is easy to see that the definition still works if $\{j \in I : a_j < 0\}$ is a finite set. Similarly, sums of negative numbers can be defined (using infimum instead of supremum, under appropriate boundedness conditions).

Infinite sums: General case

- Let I be a countably infinite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). To treat the general case, we may assume that $N = \{j \in I : a_j < 0\}$ and $P = \{j \in I : a_j > 0\}$ are both infinite sets.

Definition

If $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } N\right\}$ is bounded below, and $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } P\right\}$ is bounded above, then the sum of the numbers a_i $(i \in I)$ is defined by

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in R} a_i : \ R \subset P \text{ finite}\right\} + \inf\left\{\sum_{i\in R} a_i : \ R \subset N \text{ finite}\right\}$$

- Note that $\sum_{i\in I} a_i$ is defined exactly when $\sum_{i\in I} |a_i|$ is defined.

Infinite sums: Examples

Example

$$\sum\{\frac{1}{i(i+1)} : i \in \mathbb{N}_+\} = 1$$

- Note that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$.

Example

$$\sum\{\frac{1}{i^2} : i \in \mathbb{N}_+\} \text{ exists.}$$

- By a deep theorem of Euler, the above sum equals $\frac{\pi^2}{6}$.

Example

$$\sum\{\frac{1}{i!} : i \in \mathbb{N}_+\} \text{ exists.}$$

- The real number defined above is denoted by e.

Infinite sums: Theorems

Theorem

Assume that $0 \leq a_i \leq b_i$ holds for all $i \in I$.
If $\sum_{i:i\in I} b_i$ exists, then $\sum_{i:i\in I} a_i$ also exists,
furthermore $\sum_{i:i\in I} b_i \leq \sum_{i:i\in I} a_i$.

Theorem

Assume that $0 \leq a_i, b_i$ holds for all $i \in I$.
If $\sum_{i:i\in I}(a_i + b_i)$ exists, then

$$\sum_{i:i\in I}(a_i + b_i) = \sum_{i:i\in I} a_i + \sum_{i:i\in I} b_i.$$

Extended $\mathbb{R}$: Order

Definition: $\widehat{\mathbb{R}}$

$$\widehat{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

Order on $\widehat{\mathbb{R}}$

For all real numbers $r < r^+$ we have

$$-\infty < r < r^+ < \infty.$$

Note that $0 < \infty$, while $-\infty < 0$. Thus ∞ is a positive value, while $-\infty$ is a negative value.

The usual order properties of real numbers, such as transitivity, remain valid.

Extended $\mathbb{R}$: Balls

Definition: Balls in $\widehat{\mathbb{R}}$

$$B^o_{1/N}(\infty) = \{r \in \mathbb{R} : r > \frac{1}{1/N} = N\} \cup \{\infty\}.$$

- Note that when the center is ∞, we used $\frac{1}{N}$ as the radius. This is justified by the following observation.

Observation

If $o \in \widehat{\mathbb{R}}$ and $0 < r' < r$, then

$$B_{r'}(o) \subset B_r(o).$$

- If we say that x "comes from a sufficiently small neighborhood (ball) of ∞", then we mean that x is "greater than some sufficiently large N".

Extended $\mathbb{R}$: Supremum

Theorem

Every non-empty set $S \subset \widehat{\mathbb{R}}$ has a supremum.

- If $S \subset \mathbb{R}$ is bounded above, then this is already known.
- If $S \subset \mathbb{R} \cup \{-\infty\}$ is not bounded above, then it has only one upper bound: ∞. Thus $\sup S = \infty$.
- If $\infty \in S$, then its only upper bound is ∞.

Example

$$\sup\{1, 2, 3, \ldots\} = \infty,$$

$$\sup\{-\infty\} = -\infty,$$

$$\sup\{\infty, -1, -2, -3, \ldots\} = \infty.$$

$$\sup\{-1, -2, -3, \ldots\} = -1.$$

Extended $\mathbb{R}$: Accumulation points

The notion of accumulation point can also be defined in $\widehat{\mathbb{R}}$.

Theorem

Every infinite set $S \subset \mathbb{R}$ has an accumulation point.

- If $S \cap \mathbb{R}$ is bounded, then we already know this.

- If $S \cap \mathbb{R}$ is not bounded above, then ∞ is an accumulation point. Why? If $S \cap \mathbb{R}$ is not bounded below, then $-\infty$ is an accumulation point.

Example

$[a, \infty[$ is a closed subset of $\mathbb{R}$.

$[a, \infty[$ is NOT closed in $\widehat{\mathbb{R}}$: ∞ is an accumulation point, but $\infty \notin [a, \infty[$.

Extended $\mathbb{R}$: Arithmetic

Definition: Addition in $\widehat{\mathbb{R}}$

Let $r, r' \in \mathbb{R}$. Then

- the sum of r and r' is $r + r'$.
- $r + \infty = \infty + r = \infty$.
- $r + (-\infty) = -\infty + r = -\infty$.
- $-\infty + \infty$ is NOT DEFINED.

Definition: Multiplication in $\widehat{\mathbb{R}}$

Let $r, r' \in \mathbb{R}$. Then

- the product of r and r' is rr'.
- $r\infty = \infty r = \infty$ and $r(-\infty) = (-\infty)r = -\infty$ if $r > 0$.
- $r\infty = \infty r = -\infty$ and $r(-\infty) = (-\infty)r = \infty$ if $r < 0$.
- $\infty(-\infty) = -\infty\infty = -\infty$ and $\infty\infty = (-\infty)(-\infty) = \infty$.
- 0∞, $0(-\infty)$ are NOT DEFINED.

Extended $\mathbb{R}$: Arithmetic

Definition: Division in $\widehat{\mathbb{R}}$

Let $r, r' \in \mathbb{R}$. Then

- the quotient $\frac{r}{r'}$ exists iff $r' \neq 0$.
- $\frac{r}{\infty} = \frac{r}{-\infty} = 0$.
- $\frac{\infty}{r} = \infty$ and $\frac{-\infty}{r} = -\infty$ if $r > 0$.
- $\frac{\infty}{r} = -\infty$ and $\frac{-\infty}{r} = \infty$ if $r < 0$.
- $\frac{r}{0}$, $\frac{\infty}{0}$, $\frac{-\infty}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, $\frac{-\infty}{-\infty}$ are NOT DEFINED.

Extended $\mathbb{R}$: Infinite sums

Definition

The sum of a countable collection $\{a_i : i \in I\}$ consisting of non-negative numbers is

$$\sup\left\{\sum_{i\in R} a_i : R \subset I \text{ finite}\right\}.$$

Example

$$\sum\left\{\frac{1}{n} : n \in \mathbb{N}_+\right\} = \infty.$$

- Note that

$$\frac{1}{3}+\frac{1}{4}, \quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8},$$
$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$$

are all greater than $\frac{1}{2}$.

Break

Our Functions

In what follows, we study functions $f : \text{dom } f \to \mathbb{R}$, where $\text{dom } f \subset \mathbb{R}$. We highlight one particularly important special case.

Definition

A function $f : \text{dom } f \to \mathbb{R}$ is called a sequence if $\text{dom} = \mathbb{N}$.
We often deviate from function terminology: instead of $f(i)$ we write f_i for the value of the function at $i \in \text{dom} = \mathbb{N}$.

- We also speak of a sequence if $\text{dom} = \mathbb{N}_+$ or $\text{dom} = 2\mathbb{N} = \{n \in \mathbb{N} : \ n \text{ is even}\}$

Notation

$$L = L_f = L_{\widehat{\mathbb{R}}}(\text{dom } f) = \{\text{accumulation points of dom } f \text{ in } \widehat{\mathbb{R}}\}$$

- Note that for sequences $L = \{\infty\}$.

Limit of Functions: Definition

Definition

Let $f : \mathrm{dom}\, f \to \mathbb{R}$ and $h \in L_f \subset \widehat{\mathbb{R}}$. We write

$$\lim_{x \to h} f(x) = \ell \in \widehat{\mathbb{R}},$$

if for every ball B centered at ℓ there exists a ball B_0 centered at h such that for all $x \in (B_0 - \{h\}) \cap \mathrm{dom}\, f$ we have $f(x) \in B$.

- Naturally, the balls in the definition are taken in $\widehat{\mathbb{R}}$.
- The ball B is under a "for all" quantifier. If we decrease its radius, the set becomes smaller. Thus decreasing the radius makes the condition harder. The correct intuition: choose B with a very small radius...
- Usually $h \notin \mathrm{dom}\, f$. Even if $h \in \mathrm{dom}\, f$, x only "approaches h". The value at h itself does not matter in the definition of the limit.

Limit of Functions: Definition

- Proving that $\lim_{x\to h} f(x) = \ell \in \widehat{\mathbb{R}}$ can be viewed as a game.
- There are two players: Prover and Refuter. The Prover states the claim: gives f and $h, \ell \in \widehat{\mathbb{R}}$. Then the Refuter chooses a ball B centered at ℓ. Next, the Prover chooses a ball B_0 centered at h. Finally, the Refuter picks a point from $(B_0 - \{h\}) \cap \operatorname{dom} f$. The game ends.
- Who wins? An independent Student decides: they compute $f(x)$. If $f(x) \in B$, the Prover wins. Otherwise, the Refuter wins.
- The Refuter aims to win, so acts adversarially. Thus they choose B with very small radius. The Refuter's second step is almost unnecessary: if for all $x \in B_0 \cap \operatorname{dom} f$ we have $f(x) \in B$, then they lose; otherwise they pick a winning x.
- If the Prover has a winning strategy, the equality is true. If the Refuter can win, the equality is false.

Limits: Examples

Example

$\lim_{n\to\infty} \frac{1}{n} = 0, \qquad \lim_{x\to\infty} \frac{1}{x} = 0.$

- Let B be any ball centered at 0. Let ε be its radius: an arbitrarily small positive number. Then $f(x) \in B$ means $-\varepsilon < f(x) < \varepsilon$.
- Let $B_0 =]\frac{1}{\varepsilon}, \infty]$. "The Prover wins".

Example

$\lim_{n\to\infty} n^k = \infty \quad (k > 0).$

- Let B be any ball centered at ∞. Let $\frac{1}{N}$ be the radius. Then $f(x) \in B$ means $N < f(x)$.
- Let $B_0 =]\sqrt[k]{N}, \infty]$. "The Prover wins".

Limits: Examples

Example

$\lim_{x\to 2}\frac{x^2-4}{x-2}=4.$

- $\frac{x^2-4}{x-2}=x+2.$
- Let B be any ball centered at 4 with radius ε.
- Let $B_0=]2-\varepsilon, 2+\varepsilon[$.

Example

$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2}.$

- $$\frac{\sqrt{1+x}-1}{x}=\frac{1}{\sqrt{1+x}+1}.$$
- Substitute $x=0.1,\ 0.01,\ 0.001,\ 0.0001,\ \ldots$

Limits: Examples

Example

$\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad (a > 0)$.

- Assume $a > 1$.
- Let B be a ball centered at 1 with radius ε.
- Let $B_0 =]\frac{\log a}{\log(1+\varepsilon)}, \infty]$.

Example

$\lim_{n\to\infty} \sqrt[n]{n} = 1$.

- Let $B_0 =]\max\{10, \frac{1}{\log^2(1+\varepsilon)}\}, \infty]$.

Limits: Examples

Example

$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.

- Let $B =]\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$.
- Let $B_0 =]-\tan(\frac{\pi}{2} - \varepsilon), \infty]$.

Example

$\lim_{x\to-\infty} e^x = 0$.

- Let $B =]-\varepsilon, \varepsilon[$
- Let $B_0 =]-\infty, \ln\varepsilon[$.

Limits: Examples

Example

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Reminder

$$\begin{aligned} e &= \sup\left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}_+\right\} \\ &= \sup\left\{1+\frac{1}{1}+\frac{1}{2!}+\ldots+\frac{1}{n!} : n\in\mathbb{N}_+\right\} \\ &= \sum\left\{\frac{1}{n!} : n\in\mathbb{N}\right\} \end{aligned}$$

- $\left(1+\frac{1}{n}\right)^n$ is a monotone increasing sequence:

$$A\left(1, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}\right) = \frac{1+n(1+\frac{1}{n})}{n+1} > G\left(1, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}\right).$$

Break

Limits: Properties

If a function has a limit at a given point, then it is unique. More precisely:

Theorem

Suppose that $\lim_{x\to h} f(x)$ exists, and moreover $\lim_{x\to h} f(x) = \ell$ and $\lim_{x\to h} f(x) = k$ both hold. Then necessarily $\ell = k$.

Limits: One-sided limits

Let $a \in \mathbb{R}$. Suppose that a is an accumulation point of the sets $\operatorname{dom} f \cap]a, \infty[$ and $\operatorname{dom} f \cap]-\infty, a[$.

Definition

$$\lim_{x \to a+} f(x) = \lim_{x \to a} f^{a\to}(x), \text{ where } f^{a\to}(x) = f|\operatorname{dom} f \cap]a, \infty[.$$

$$\lim_{x \to a-} f(x) = \lim_{x \to a} f^{\to a}(x), \text{ where } f^{\to a}(x) = f|\operatorname{dom} f \cap]-\infty, a[.$$

Theorem

Suppose that $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ both exist.
Then $\lim_{x \to a} f(x)$ exists if and only if
$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$.

Limits: One-sided limits: Examples

Example

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

Example

$$\lim_{x\to 0-} \frac{1}{x} = -\infty.$$

Example

$$\lim_{x\to \frac{\pi}{2}-} \tan x = \infty.$$

Example

$$\lim_{x\to \frac{\pi}{2}+} \tan x = -\infty.$$

Limits: One-sided limits: Examples

Example

$\lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty$.

- Substitute $x = 0.9$ into the function.

Substitute $x = 0.99$ into the function.
Substitute $x = 0.999$ into the function.
Substitute $x = 0.9999$ into the function...

Example

$\lim_{x\to 1^+} \frac{1}{\sqrt{1-x^2}} = -\infty$.

- Substitute $x = 1.1$ into the function.

Substitute $x = 1.01$ into the function.
Substitute $x = 1.001$ into the function.
Substitute $x = 1.0001$ into the function...

Limits: Properties

Theorem

Suppose that $\lim_{x\to h} f(x)$ and $\lim_{x\to h} g(x)$ exist, and moreover $\lim_{x\to h} f(x) = \ell$ and $\lim_{x\to h} g(x) = k$. Then

(i) Assume that $\ell + k$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h}(f(x) + g(x)) = \lim_{x\to h} f(x) + \lim_{x\to h} g(x) = \ell + k.$$

(ii) Assume that $\ell \cdot k$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h}(f(x)g(x)) = \lim_{x\to h} f(x) \lim_{x\to h} g(x) = \ell k.$$

(iii) Assume that $\frac{\ell}{k}$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h} \frac{f(x)}{g(x)} = \frac{\lim_{x\to h} f(x)}{\lim_{x\to h} g(x)} = \frac{\ell}{k}.$$

Limits: Properties: Examples

Example

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

- We know that $\lim_{x\to\infty} 1 = 1$ and $\lim_{x\to\infty} x = \infty$.
- We know that $\frac{1}{\infty} = 0$ using the arithmetic of $\widehat{\mathbb{R}}$.

Example

$$\lim_{x\to\infty} x + \frac{1}{x} = \infty.$$

- We know that $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} \frac{1}{x} = 0$.
- We know that $\infty + 0 = \infty$ using the arithmetic of $\widehat{\mathbb{R}}$.

Limits: Properties: Examples

Example

$\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$.

- $$\frac{x^n-a^n}{x-a} = \frac{(x-a)(x^{n-1}+x^{n-2}a+\ldots+xa^{n-2}+a^{n-1})}{x-a}$$
  $$= x^{n-1}+x^{n-2}a+\ldots+xa^{n-2}+a^{n-1}.$$
- $\lim_{x\to a} x^k a^{n-1-k} = a^{n-1}$.

Limits: Properties: Examples

Example

$$\lim_{x\to\infty} x^2 - 3x + 5 = ?.$$

- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} -3x = -\infty$ and $\lim_{x\to\infty} 5 = 5$.
- We know that $\infty + (-\infty)$ is NOT defined in $\widehat{\mathbb{R}}$, so the theorem cannot be applied in this form.

Example

$$\lim_{x\to\infty} x^2 - 3x + 5 = \infty.$$

- We know that $x^2 - 3x + 5 = x^2(1 - \frac{3}{x} + \frac{5}{x^2})$.
- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x^2} = 0$.
- Thus $\infty(1 + (-3)0 + 5 \cdot 0) = \infty$ using $\widehat{\mathbb{R}}$ arithmetic.

Limits: Properties: Examples

Example

$$\lim_{x\to\infty} \frac{x^2-3x+5}{x^3-3x^2+2} = 0.$$

- We rewrite:

$$\frac{x^2-3x+5}{x^3-3x^2+2} = \frac{1}{x} \cdot \frac{1-3\frac{1}{x}+5\frac{1}{x^2}}{1-3\frac{1}{x}+2\frac{1}{x^3}}.$$

- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, $\lim_{x\to\infty} \frac{1}{x^3} = 0$.
- Thus the limit equals 0.

Example

$$\lim_{x\to\infty} \frac{x^5-3x+5}{x^3-3x^2+2} =?, \qquad \lim_{x\to\infty} \frac{x^3-3x+5}{x^3-3x^2+2} =?.$$

Try to solve these on your own.

Limits: Properties

Theorem: Squeeze Theorem

If the function $f(x)$ is squeezed between two other functions, $g(x)$ and $h(x)$, i.e. $g(x) \leq f(x) \leq h(x)$, and both have the same limit (L) at the point a, then the limit of $f(x)$ is also L.

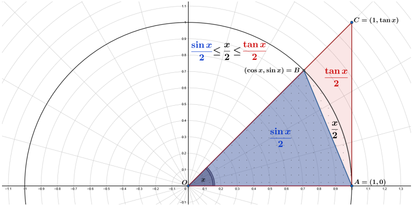
Theorem

Let a_n be a bounded above, monotonically increasing sequence/function. Then $\lim_{x\to\infty} f(x)$ exists.

Limits: Examples

Example

$\lim_{x\to 0} \frac{\sin x}{x} = 1.$



Source:
https://www.math-linux.com/mathematics/limits/article/proof-of-limit-of-sin-x-x-1-as-x-approaches-0

- After rearranging we obtain that for $0 < x < \frac{\pi}{2}$

$$\cos x < \frac{\sin x}{x} < 1.$$

Limits: Examples

Example

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

- We know that $1 = \cos^2\frac{x}{2} + \sin^2\frac{x}{2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$.

Thus $1 - \cos x = 2\sin^2\frac{x}{2}$. Hence

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2\frac{x}{2}}{x^2} = \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2.$$

- Let B be an arbitrary ball centered at $\frac{1}{2}$. Let ε be its radius.
- Let $B =]N_0, \infty]$. $N_0 = ?$

Solve inequalities.

Limits: Properties: Examples

Example

$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1.$

- $$1 < \frac{\sqrt{x^2+1}}{x} = \sqrt{1+\frac{1}{x^2}} < 1 + \frac{1}{2}\cdot\frac{1}{x^2}.$$

Example

$\lim_{x\to\infty} \frac{x^2+1}{2x+8} = \infty.$

$$\frac{1}{3}\cdot x = \frac{x^2}{3x} < \frac{x^2+1}{2x+8} < \frac{2x^2}{x} = 2x,$$

whenever $x \in [8, \infty[$.

Limits: Examples

Example

$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$

- $n! < n^{n-1} \cdot 1.$

Example

$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a > 1).$

- $$n! = n(n-1)(n-2)\ldots \frac{n}{2}(\frac{n}{2}-1)\ldots \cdot 2 \cdot 1 > \left(\frac{n}{2}\right)^{\frac{n}{2}} = \sqrt{\frac{n}{2}}^{n}.$$

Example

$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \quad (a > 1, k > 0).$

- $\frac{n^k}{a^n} = \left(\frac{\sqrt{n}}{(\sqrt[2k]{a})^n}\right)^{2k}$. Let $\sqrt[2k]{a} = 1 + h$. Bernoulli inequality: $(1+h)^n > 1 + hn$, if $h > -1$.

Limits: Examples

Example

$\lim_{n\to\infty} \frac{\log_\alpha n}{n^k} = 0 \quad (\alpha > 1, k > 0).$

- Let $\alpha = 1 + \epsilon$

$$\log_\alpha n < n^{k/2} \Leftrightarrow n < \alpha^{n^{k/2}} \Leftrightarrow$$

$$n < (1+\epsilon)^{n^{k/2}} = 1 + n^{k/2}\epsilon + \binom{n^{k/2}}{2}\epsilon^2 + \ldots + \binom{n^{k/2}}{4/k}\epsilon^{4/k} + \ldots$$

- In summary:

$$\log_\alpha n \lll n^k \lll a^n \lll n! \lll n^n,$$

where for positive sequences

$$f(n) \lll g(n) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty.$$

This is the end!

Thank you for your attention!