BSc Mathematics for Computer Scientists 2: IV. Matrices and Linear Systems

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Matrices: Connection to the systems of linear equations

- Let $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$, $a_{2,1}, a_{2,2}, \ldots, a_{2,n}$, ..., $a_{k,1}, a_{k,2}, \ldots, a_{k,n}$ and $b_1, b_2, \ldots, b_k$ be given real numbers.

- Consider the system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n}x_n & = b_2 \\ \vdots & \\ a_{k,1}x_1 + a_{k,2}x_2 + a_{k,3}x_3 + \ldots + a_{k,n}x_n & = b_k \end{cases}$$

of linear equations.

- The matrix of the system is the matrix A: the entry in the intersection of the i-th row and j-th column is the coefficient of the j-th variable in the i-th equation.
That is, $A_{i,j} = a_{i,j}$.

- We note that writing the matrix requires fixing an ordering of the equations/constrains and of the variables.

System of linear equations: Matrix Form

• In vector/matrix notation our equation becomes the following.
Let $A \in \mathbb{R}^{k \times n}$ be a real matrix of size $k \times n$ and $b \in \mathbb{R}^k$.

$$Ax = b, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Linear Systems: On the Ordering of Equations and Variables

- In an application this ordering is often artificial.

- An alternative possibility is the following.
Let $\mathcal{E}$ be the set of equations and $\mathcal{X}$ the set of variables. The matrix A is a function $\mathcal{E} \times \mathcal{X} \to \mathbb{R}$.

- For $e \in \mathcal{E}$ and $x \in \mathcal{X}$ ($|E| = k$, $|X| = n$), the value $A(e, x)$ is the coefficient of the variable x in the equation e.

- A common notation for the set of functions $E \times X \to \mathbb{R}$ is $\mathbb{R}^{E \times X}$.
With this terminology, the matrix of the system is $A \in \mathbb{R}^{E \times X}$.

- In the case of k linear equations/constrains and n variables we have

$$\mathbb{R}^{\mathcal{E} \times \mathcal{X}} \simeq \mathbb{R}^{k \times n}.$$

Linear Systems: On the Ordering of Equations and Variables: Example

$$A \in \mathbb{R}^{k\times n}: \begin{pmatrix} 1 & -2 & -5 & 4 \\ 0 & 1 & 0 & -3 \\ -2 & 3 & 0 & -1 \end{pmatrix}$$

$$A \in \mathbb{R}^{\mathcal{E}\times\mathcal{X}}: \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_4 & x_3 \end{array} \\ \begin{pmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & -3 & 0 \\ -2 & 3 & -1 & 0 \end{pmatrix} \end{array}$$

Break

Gaussian Elimination: Reminder

During Gaussian elimination the set of solutions of a system does not change if we apply the following so-called elementary row operations:

- **Multiplying an equation by a scalar:** Multiply both sides of any equation by a nonzero real number.
- **Swapping two equations:** Interchanging the order of two equations.
- **Adding a multiple of one equation to another equation:** Add a scalar multiple of one equation (possibly negative) to another equation, while the first equation remains unchanged.

Remarks: Elementary row operations do not change the number of equations. Using elementary operations we can eliminate variables. There is no need to change the ordering of the variables.

Equivalent Form and Reduced Row Echelon Form

By applying elementary row operations we obtain systems that are equivalent to the original system (they have the same solution set).

Our goal is to reach the reduced row echelon form (RREF), where:

- the leading coefficient (pivot) of each nonzero row is 1,
- each pivot variable appears only in its own equation,
- the pivot of each row is to the right of the pivot in the previous row.

After this, the variables corresponding to pivots (dependent variables) can be expressed from the equations. The remaining variables (free variables) can be assigned arbitrary values. In this way all solutions of the system can be obtained.

Linear Systems: Gaussian Elimination: Example

$$\left.\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 && (e_1)\\ x_1 - 2x_2 + 3x_3 &= -3 && (e_2)\\ 2x_1 + 10x_2 + 12x_3 &= 2 && (e_3)\end{aligned}\right\} \equiv \begin{pmatrix}1 & 3 & 5\\ 1 & -2 & 3\\ 2 & 10 & 12\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 && (e_1)\\ -5x_2 - 2x_3 &= -3 && (e_2 - e_1)\\ 4x_2 + 2x_3 &= 2 && (e_3 - 2e_1)\end{aligned}\right\} \equiv \begin{pmatrix}1 & 3 & 5\\ 0 & -5 & -2\\ 0 & 4 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 5x_3 + 3x_2 &= 0 && (e_1)\\ -2x_3 - 5x_2 &= -3 && (e_2 - e_1)\\ 2x_3 + 4x_2 &= 2 && (e_3 - 2e_1)\end{aligned}\right\} \equiv \begin{pmatrix}1 & 5 & 3\\ 0 & -2 & -5\\ 0 & 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_3\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 5x_3 + 3x_2 &= 0 && (e_1)\\ -2x_3 - 5x_2 &= -3 && (e_2 - e_1)\\ -x_2 &= 2 && \end{aligned}\right\} \equiv \begin{pmatrix}1 & 3 & 5\\ 0 & -2 & -5\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_3\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ -3\\ -1\end{pmatrix}$$

Linear Systems: Gaussian Elimination: Example (continued)

$$\begin{pmatrix}1 & 3 & 5\\ 1 & -2 & 3\\ 2 & 10 & 12\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 5\\ 1 & -2 & 3\\ 2 & 10 & 12\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\left[\begin{pmatrix}1 & 3 & 5\\ 0 & -5 & -2\\ 0 & 4 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right]=\begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 5 & 3\\ 0 & -2 & -5\\ 0 & 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_3\\ x_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}0\\ -3\\ 2\end{pmatrix}$$

Gaussian Elimination: Matrix Formalization

$$Ax = b, \qquad A \in \mathbb{R}^{k\times n}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b \in \mathbb{R}^k \quad (SLE).$$

Reminder

Gaussian elimination: $Ax = b \to Rx = \widetilde{b}$ using elementary row operations (i.e. without changing the order of the variables).

• I have presented two variants. In the first case R is in row echelon form.
In the second case R is in reduced row echelon form.

• In both cases we may keep the ordering of the variables unchanged. During the algorithm it is sufficient to work with the augmented matrix.

$$Ax = b, \qquad A \in \mathbb{R}^{k\times n}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b \in \mathbb{R}^k \quad (SLE).$$

Definition

Coefficient matrix and augmented matrix of (SLE):

$$A, \qquad (A|b) \in \mathbb{R}^{k\times(n+1)}.$$

Theorem

For every matrix $A \in \mathbb{R}^{k \times n}$ there exists an invertible matrix $T \in \mathbb{R}^{k \times k}$ such that the matrix $TA \in \mathbb{R}^{k \times n}$ is in row echelon form.

Theorem

For every matrix $A \in \mathbb{R}^{k \times n}$ there exists an invertible matrix $\widehat{T} \in \mathbb{R}^{k \times k}$ such that the matrix $\widehat{T}A \in \mathbb{R}^{k \times n}$ is in reduced row echelon form.

Corollary

For every matrix $A \in \mathbb{R}^{n \times n}$ there exists an invertible matrix $\widehat{T} \in \mathbb{R}^{n \times n}$ such that $\widehat{T}A \in \mathbb{R}^{n \times n}$ is an upper triangular matrix.

Gaussian Elimination: Matrix Formalization: Theorems: Proofs

• Initial system of equations: $Ax = b \quad (\mathcal{E})$. $(\mathcal{E})$ consists of k equations and contains n unknowns, that is $A \in \mathbb{R}^{k\times n}$.

• Multiplying the i-th equation by λ:

$$Ax = b \quad \Rightarrow \quad \Lambda_i(\lambda)Ax = \Lambda_i(\lambda)b,$$

where $\Lambda_i(\lambda) \in \mathbb{R}^{k\times k}$ is a diagonal matrix with 1's on the main diagonal, except at position i where the entry is λ.

• Adding the j-th equation to the i-th:

$$Ax = b \quad \Rightarrow \quad S_{i,j}Ax = S_{i,j}b,$$

where $S_{i,j}$ has 1's on the main diagonal, the (i, j) entry is also 1, while all other entries are 0.

Gaussian Elimination: Matrix Formalization: Theorems: Proofs: Examples

- Our system of equations is $Ax = b$, where $A \in \mathbb{R}^{k\times n}$.

Example

$k = 4$, $n = 5$. Multiplying the 3rd equation by -2:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Ax = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} b$$

Example

$k = 4$, $n = 5$. Adding three times the second equation to the fourth:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} Ax = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} b$$

Example

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Example

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 3 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -3 & 0 & 1\end{pmatrix}$$

Theorem

Let (LSE) be a system of linear equations. Assume that $k = n$, that is, the number of equations equals the number of variables. (LSE) has a unique solution if one (equivalently all) of the following conditions holds:

(1) Its row echelon equivalent contains no free variables.

(2) Its reduced row echelon equivalent contains no free variables.

(3) In its reduced row echelon form the i-th equation ($i = 1, 2, \ldots, n$) is $x_i = \alpha_i$ for suitable numbers α_i.

(4) There are no real scalars $\varphi_1, \ldots, \varphi_n$, not all zero, such that $\varphi_1(e_1) + \varphi_2(e_2) + \ldots + \varphi_n(e_n)$ eliminates all variables on the left-hand side.

(5) The matrix A of the system is invertible.

Gaussian Elimination: Numbers

Theorem

In both versions of Gaussian elimination:

(1) if the coefficients are integers, then every number appearing during the algorithm is rational,

(2) if the coefficients are real, then every number appearing during the algorithm is real.

Theorem

If a linear system with integer coefficients has a unique solution, then the solution is rational.
If a linear system with integer coefficients is solvable, then its solution set contains a rational solution.

Theorem

If $A \in \mathbb{Z}^{n\times n}$ is invertible, then $A^{-1} \in \mathbb{Q}^{n\times n}$.

Gaussian Elimination: Alternative Theorem

Theorem

Let $(\mathcal{E})$ be a linear system containing k equations and n unknowns $(x_1, x_2, \ldots, x_n)$. Then exactly one of the following two alternatives holds:

(1) The system has a solution $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$.

(2) There exist real scalars $\varphi_1, \varphi_2, \ldots, \varphi_k$ such that $\varphi_1(e_1) + \varphi_2(e_2) + \ldots + \varphi_k(e_k)$ eliminates all variables on the left-hand side, while the right-hand side becomes a nonzero number.

Furthermore, in case (1) Gaussian elimination describes the complete nonempty solution set, while with a suitable extension the algorithm can compute appropriate multipliers in case (2).

Theorem

Consider the linear system $Ax = b$ ($A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$). Exactly one of the following two conditions holds:

(i) There exists a solution $x_0 \in \mathbb{R}^n$ such that $Ax_0 = b$,

(ii) There exist real scalars $p = (p_i)_{i=1}^k$ such that

$$p^\mathsf{T} A = 0^\mathsf{T} (\in \mathbb{R}^n) \text{ and } p^\mathsf{T} b = 1.$$

Alternative Theorem: Interpretation

Geometric interpretation

Consider the linear system $Ax = b$ with $A \in \mathbb{R}^{k\times n}$ and $b \in \mathbb{R}^k$.

(i) If there exists x such that $Ax = b$, then b is a linear combination of the columns of A, that is, b lies in the column space of A.

(ii) Otherwise there exists a vector $p \in \mathbb{R}^k$ such that

$$p^{\mathsf{T}} A = 0^{\mathsf{T}} \quad \text{but} \quad p^{\mathsf{T}} b \neq 0.$$

This means that p is orthogonal to every column of A, but not orthogonal to b.

Hence either b lies in the column space of A, or it can be separated from it by such a vector p.

Alternative Theorem: Geometric Picture

Idea

Let $A \in \mathbb{R}^{k\times n}$ and consider the system

$$Ax = b.$$

- The vectors $A_1, A_2, \ldots, A_n$ (columns of A) span a subspace of $\mathbb{R}^k$:

$$\mathrm{col}(A) = \langle A_1, \ldots, A_n \rangle_{lin}.$$

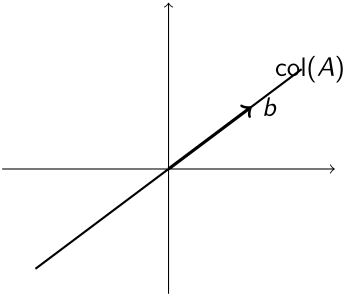
- The equation $Ax = b$ asks whether the vector b lies in this subspace.
- If $b \in \mathrm{col}(A)$, then $b = Ax$ for some x and the system has a solution.
- If $b \notin \mathrm{col}(A)$, then b and the column space of A can be seperated.
- There exists a vector $p \in \mathbb{R}^k$ such that

$$p^{\mathsf{T}} A = 0^{\mathsf{T}} \quad \text{but} \quad p^{\mathsf{T}} b \neq 0,$$

$$\text{i.e.} \qquad p^{\mathsf{T}} A_i = 0^{\mathsf{T}} \quad (i = 1, 2, \ldots, n) \quad \text{but} \quad p^{\mathsf{T}} b \neq 0.$$

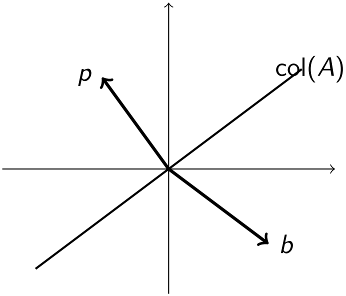
Alternative Theorem: Geometric Illustration

$b \in \mathrm{col}(A)$



$Ax = b$ has a solution

$b \notin \mathrm{col}(A)$



$p^{\mathsf{T}} A = 0, \; p^{\mathsf{T}} b \neq 0$

Interpretation

Either b lies in the column space of A (and the system has a solution), or b and the column space of A are seperated, i.e. there exists a vector p orthogonal to col(A) but not orthogonal to b.

Break

Summary

Reminder

Let $Ax = b$ be a linear system of equations with the same number of equations as unknowns ($A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$). The system has a unique solution if and only if A is invertible, that is, A^{-1} exists.

Summary: Example

$$2x = 6 \qquad \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}2\\ 0\\ -3\end{pmatrix}$$

$$2x = 6 \quad / : 2 \qquad \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}2\\ 0\\ -3\end{pmatrix} \quad / \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}^{-1} \cdot$$

$$1x = 3 \qquad \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}^{-1} \cdot \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$= \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}^{-1} \cdot \begin{pmatrix}2\\ 0\\ -3\end{pmatrix}$$

$$x = 3 \qquad \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}^{-1} \cdot \begin{pmatrix}2\\ 0\\ -3\end{pmatrix}$$

Summary: The Algorithm

Algorithm

Input/given: $Ax = b$ ($A \in \mathbb{R}^{n\times n}$ invertible, $b \in \mathbb{R}^n$)
Output/compute: The unique solution.

(Inversion): Compute the matrix A^{-1}.

("Division:") Compute the vector $A^{-1}b$ and report it as the final result.

How can the inverse of an invertible matrix be computed???

Computing the Inverse: Two Approaches

- Use Gaussian elimination to transform the matrix into reduced row echelon form. The result is the identity matrix. Keep track of the matrix we multiply by. The final result is the inverse matrix.

- Use Gaussian elimination to solve the systems

$$Ax = e_i, \qquad (i = 1, 2, \ldots, n)$$

The solution vectors give the columns of the inverse matrix. The systems of equations can be solved in PARALLEL.

Computing the Inverse: An Algorithm

Algorithm: Gauss–Jordan

Input/given: matrix A ($A \in \mathbb{R}^{n\times n}$ invertible)
Output/compute: the matrix A^{-1}.

(Start): Write the matrix $(A|I)$ ($I \in \mathbb{R}^{n\times n}$ identity matrix).

(Gauss): Transform the left matrix [using Gaussian elimination/elementary row operations] [into reduced row echelon form/into the identity matrix].
// The row operations also affect the right side.

Report the right-hand matrix of the final matrix as the final result.

Computing the Inverse: Example

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Break

Definition

Let $A \in \mathbb{R}^{n\times n}$ be a square real matrix.
The number $\det A$ is a real number associated with A, called the *determinant* of the matrix A.

We will give the precise technical definition later. Instead, we first describe some properties that we expect the determinant to satisfy. These properties will guide our definition.

Determinant Property (D1)

Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Let A' be obtained from A by swapping rows i and j. Then

$$\det A' = -\det A.$$

Consequence

If a matrix $A \in \mathbb{R}^{n\times n}$ has two identical rows, then

$$\det A = 0.$$

Determinant Property (D2)

Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Let A' be obtained from A by multiplying row i by λ. Then

$$\det A' = \lambda \det A.$$

Consequence

If one row of the matrix A consists entirely of zeros, then

$$\det A = 0.$$

Determinant: Properties

Determinant Property (D3)

Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Let A' be obtained from A by adding λ times row j to row i. Then

$$\det A' = \det A.$$

Consequence

If two rows of A are proportional (one row is a scalar multiple of the other), then

$$\det A = 0.$$

Determinant: Properties

Determinant Property (D4)

$$\det I_n = 1.$$

Consequence

Let D be a diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then

$$\det D = \alpha_1 \alpha_2 \cdots \alpha_n.$$

Consequence

If D is an upper or lower triangular matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\det D = \alpha_1 \alpha_2 \cdots \alpha_n.$$

Determinant: Computing with Gaussian Elimination

- Assume that for every n there exists a function

$$\det = det_n : \mathbb{R}^{n\times n} \to \mathbb{R}$$

satisfying properties $(D1) - (D4)$.

- Then for a given matrix A, the value $\det A$ can be computed using Gaussian elimination (i.e. elementary row operations).

- Two strategies are possible:
(1) Use only row operations that do not change the determinant.
(2) Allow row operations that change the determinant, but keep track of their effect.

- Possible target forms:
(1) Transform the matrix to *upper triangular form*.
(2) Transform the matrix to *diagonal form*.

Determinant: Algorithm via Gaussian Elimination

Algorithm

// Input: matrix A ($A \in \mathbb{R}^{n\times n}$), Output: the value $\det A$.

(Initialization) Set $A_{\text{curr}} := A$, let $\psi = 1$.
// ψ keeps track of the ratio between the determinant of the original matrix A and the determinant of the current matrix: A_{curr}.

(Gaussian elimination)
Apply elementary row operations to transform A into reduced row echelon form. After each row operation, update the value of ψ accordingly.

(Determinant computation)
The final matrix obtained from Gaussian elimination is an upper triangular matrix with 1's on the diagonal, possibly followed by zeros.
Thus $\det A_{\text{curr}}$ is either 1 or 0.

(Output)
Return $\psi \cdot \det A_{\text{curr}}$ as the determinant of A.

Determinant: Gaussian Elimination Example

Determinant: Small Matrices

Definition: Determinant for $n = 2$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Definition: Determinant for $n = 3$

$$\det\begin{pmatrix} a & b & c \\ A & B & C \\ \alpha & \beta & \gamma \end{pmatrix} = aB\gamma + \alpha bC + A\beta c - a\beta C - Ab\gamma - \alpha Bc.$$

Example: 4×4 determinant

$$\det\begin{pmatrix} a & b & c & d \\ A & B & C & D \\ \alpha & \beta & \gamma & \delta \\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} \end{pmatrix} = ????$$

Determinant: Recursion

We describe the functions $det_n = det : \mathbb{R}^{n\times n} \to \mathbb{R}$ recursively with respect to n.

Definition: Recursive description of the determinant

The case $n = 1$ has already been discussed. Assume now that $n = k + 1$ and that the function $det_k = det : \mathbb{R}^{k\times k} \to \mathbb{R}$ is known. Then

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,k+1} \end{pmatrix}$$

can be computed by the following formula.

Determinant: Recursion

Definition (continued)

$$a_{1,1}\det\begin{pmatrix} a_{2,2} & a_{2,3} & \dots & a_{2,k+1} \\ a_{3,2} & a_{3,2} & \dots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,2} & a_{k+1,3} & \dots & a_{k+1,k+1} \end{pmatrix}$$

$$-a_{1,2}\det\begin{pmatrix} a_{2,1} & a_{2,3} & \dots & a_{2,k+1} \\ a_{3,1} & a_{3,2} & \dots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,3} & \dots & a_{k+1,k+1} \end{pmatrix}$$

$$\pm \ldots + (-1)^k a_{1,k+1}\det\begin{pmatrix} a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ a_{3,1} & a_{3,2} & \dots & a_{3,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,k} \end{pmatrix}.$$

Determinant: Recursion

Exercise

Verify that for $n = 2$ and $n = 3$ the above recursive definition gives the same formulas as the ones we introduced earlier.

Exercise

Verify that the recursively defined functions $det_n = det : \mathbb{R}^{n \times n} \to \mathbb{R}$ satisfy properties (D1)–(D4).

Determinant: Formula

Definition: Sign of a permutation

Let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a bijection/permutation. Let ℓ denote the number of cycles in the cycle decomposition of π. Define $\mathrm{sign}\pi$ to be 1 if $n \equiv \ell \pmod 2$, and -1 otherwise.

Definition

Let $A = (a_{i,j})_{i=1,j=1}^{n,n} \in \mathbb{R}^{n\times n}$.

$$\det A = \sum_{\pi:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\} \text{ bijection}} \mathrm{sign}\pi\, a_{1\pi 1} \cdot a_{2\pi 2} \cdot \ldots \cdot a_{n\pi n}.$$

Exercise

Verify that the determinant defined above satisfies properties (D1)–(D4).

Determinant: Row/Column Duality

Row/Column duality of the determinant

Properties (1)–(3) remain valid if we replace rows by columns.

Determinant: Theorems

Theorem

(i)
$$\det A^{\mathsf{T}} = \det A,$$

(ii)
$$\det(AB) = \det A \det B.$$

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a real matrix. The following statements are equivalent:

(1) A is invertible,

(2) $\det A \neq 0$.

Determinant: Linear Systems

Theorem: Cramer's Rule

Let $Ax = b$ be a linear system, where $A = (a_{ij})_{i=1,j=1}^{n,n} \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, and assume that $\det A \neq 0$. Then the system has a unique solution, given by the formula $(i = 1, 2, \ldots, n)$:

$$x_i = \frac{1}{\det A} \det \begin{pmatrix} a_{11} & \ldots & a_{1,i-1} & b_1 & a_{1,i+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2,i-1} & b_2 & a_{2,i+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \ldots & a_{n-1,i-1} & b_{n-1} & a_{n,i+1} & \ldots & a_{n-1,n} \\ a_{n1} & \ldots & a_{n,i-1} & b_n & a_{n,i+1} & \ldots & a_{nn} \end{pmatrix}.$$

Theorem

Let A be a square matrix, $A = (a_{ij})_{i=1,j=1}^{n,n} \in \mathbb{R}^{n\times n}$, and assume $\det A \neq 0$. Then A^{-1} exists and its entries are given by

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{cof}(A)^{\mathsf{T}} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix},$$

where M_{ij} is the matrix obtained from A by deleting the i-th row and j-th column, and

$$C_{ij} = (-1)^{i+j}\det(M_{ij})$$

is the (i,j)-th cofactor.
Thus the matrix appearing in the formula is the transpose of the cofactor matrix.

Determinant: Geometric Meaning

Theorem

$$\det \begin{pmatrix} a & b & c \\ A & B & C \\ \alpha & \beta & \gamma \end{pmatrix}$$

Its absolute value equals the volume of the parallelepiped spanned by the vectors (a, A, α), (b, B, β) and (c, C, γ).

Its sign indicates whether the three vectors form a right-handed or left-handed system.

In summary: the determinant equals the *signed volume* of the parallelepiped spanned by the three column vectors of the matrix.

Thank you for your attention!