BSc Mathematics for Computer Scientists 2: XI. Riemann Integration

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Integration: Motivation: Area computation

Basic problem

Given a region in the plane. Determine its area.

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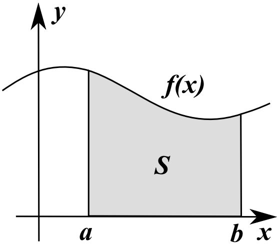

Integration: Motivation: Signed area

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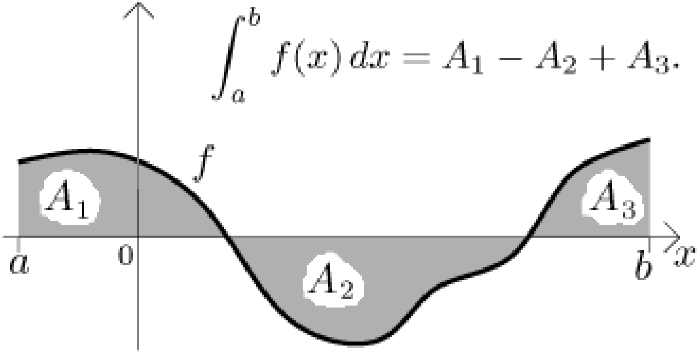
Generalization: Allow the function f to be arbitrary. The parts of the graph below the x-axis are counted with negative sign.



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• If the area corresponding to the graph of a function $f : [a, b] \to \mathbb{R}$ is well-defined, then we call the function integrable. The assigned area (which, according to the signed area convention, may also be negative) is denoted by

$$\int_a^b f(x)\,dx.$$

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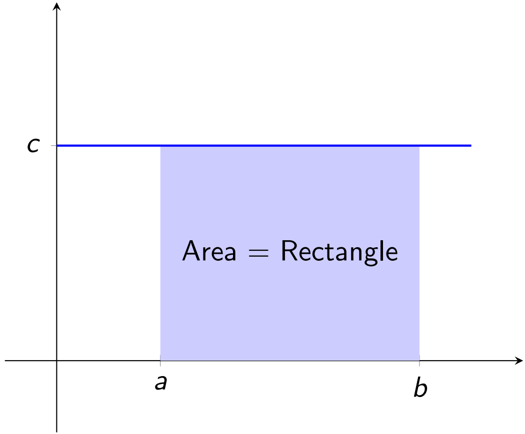
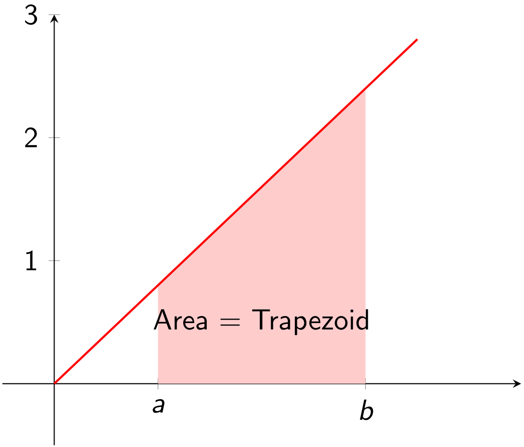
(Read as: "the integral from a to b of $f(x)$ dx".)
• To present the method mentioned above, we need to summarize some properties of area.

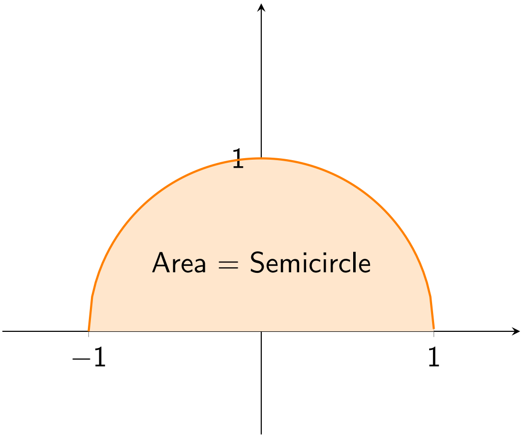
$$f(x) = c \quad \to \quad \int_a^b c\, dx = c(b-a)$$



$$f(x) = mx \quad \rightarrow \quad \int_a^b mx\, dx = \frac{mb^2}{2} - \frac{ma^2}{2}$$



$$f(x) = \sqrt{1-x^2} \quad \rightarrow \quad \int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{r^2\pi}{2} = \frac{\pi}{2}$$



Property (additivity)

Let T be a region with area, which is partitioned by lines into regions $T_1, T_2, \ldots, T_\ell$. Then

$$A(T) = A(T_1) + A(T_2) + \ldots + A(T_\ell).$$

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Property (monotonicity)

Let T be a region with area and R be a subregion of T with area. Then

$$A(R) \leq A(T).$$

Integration: Partitions

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Definition

A partition of the interval $[a, b]$ is a sequence $\beta = (t_i)_{i=0}^{\ell}$ such that

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Lemma

For any two partitions β, β' there exists a common refinement $\beta' \wedge \beta$ which is the coarsest among common refinements.

Integration: Lower/upper sums

• Let β be a partition of the interval $[a, b]$, which is the domain of the function f. For each subinterval I_i we define

$$m_i = \inf\{f(x) : x \in I_i\} \qquad \text{and} \qquad M_i = \sup\{f(x) : x \in I_i\}.$$

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The lower sum corresponding to the partition β is

$$L_\beta(f) = m_1(t_1-t_0)+m_2(t_2-t_1)+\ldots+m_\ell(t_\ell-t_{\ell-1}) = \sum_{i=1}^{\ell} m_i(t_i-t_{i-1}).$$

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The upper sum corresponding to the partition β is

$$U_\beta(f) = M_1(t_1 - t_0) + M_2(t_2 - t_1) + \ldots + M_\ell(t_\ell - t_{\ell-1}) = \sum_{i=1}^{\ell} M_i(t_i - t_{i-1}).$$

Observation

Let β be a partition of $[a, b]$ and β' be a refinement of it. Then

(i) $L_\beta(f) \leq L_{\beta'}(f)$,

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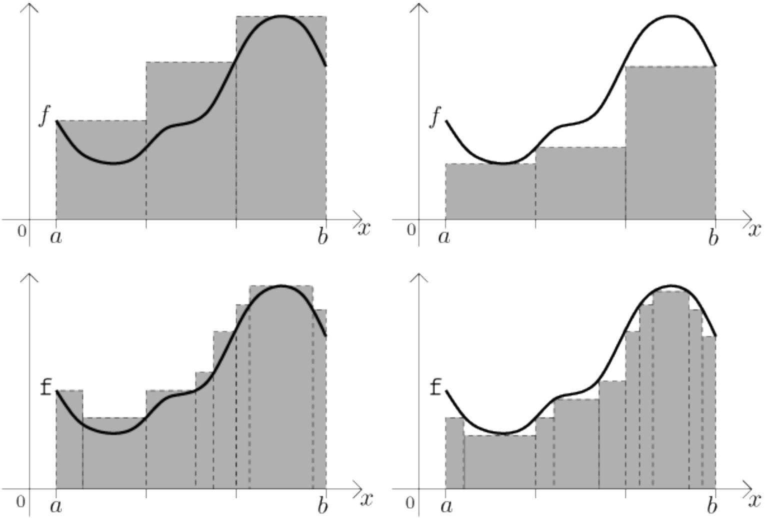
Proof. Consider their common refinement $\beta \wedge \gamma$.

Integration: Lower/Upper Sums: Geometry



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Let $f : [a, b] \to \mathbb{R}$ be a function. Then

$$\underline{\int_a^b} f(x)\,dx = \sup\{L_\beta(f) : \beta \text{ is a partition of } [a,b]\},$$

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$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

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Convention

If $a < b$, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Example

The constant function $f(x) = c$ is integrable on every interval $[a, b]$ and

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The linear function $f(x) = mx$ is integrable on every interval $[a, b]$ and

$$\int_a^b mx\,dx = \frac{m}{2}(b^2 - a^2).$$

Integration: Riemann Integrability: Negative Example

Example

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

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Then

$$\overline{\int_0^1} f(x)\,dx = 1 \qquad \text{and} \qquad \underline{\int_0^1} f(x)\,dx = 0.$$

Thus f is not Riemann integrable.

Break

Riemann Integral: Properties

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Theorem

If f is Riemann integrable on $[a, b]$ and on $[b, c]$, then it is Riemann integrable on $[a, c]$, and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

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Theorem

If f and g are Riemann integrable functions on $[a, b]$, then $f + g$ and λf are also integrable, and

(i)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(ii)

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

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(iii) If f is bounded, more precisely $\mu \leq f \leq M$, then $\mu(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

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Definition

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the average value of f is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Riemann Integrability: Sufficient Conditions

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

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If $f : [a, b] \to \mathbb{R}$ is bounded and piecewise continuous (with respect to some partition β), then f is Riemann integrable.

Riemann Integrability: Theoretical Foundations

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Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let $(\beta_k)_{k=1}^{\infty}$ be a sequence of partitions of $[a, b]$ such that $\|\beta_k\| \to 0$ as $k \to \infty$. For each partition β_k, let $(\xi_i^{(k)})$ be points such that $\xi_i^{(k)} \in I_i^{(k)}$, where $I_i^{(k)}$ is the i-th subinterval.
Then

$$\sum_{i=1}^{\ell_k} f(\xi_i^{(k)})(t_i^{(k)} - t_{i-1}^{(k)}) \to \int_a^b f(x)\,dx \quad \text{as } k \to \infty.$$

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$$\sum_{i=1}^{\ell_k} f(\xi_i^{(k)})(t_i^{(k)} - t_{i-1}^{(k)}) \to \int_a^b f(x)\,dx \quad \text{as } k \to \infty.$$

Proof:

$$L_{\beta_k}(f) \le \sum_{i=1}^{\ell_k} f(\xi_i^{(k)})(t_i^{(k)} - t_{i-1}^{(k)}) \le U_{\beta_k}(f)$$

for all k.

Break

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt.$$

(i) Then F is continuous.

(ii) If moreover f is continuous at x_0, then F is differentiable at x_0 and

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Definition

Let f be a function defined on an interval. A function F is called a primitive function (or antiderivative) of f if F is differentiable and $F' = f$.

Theorem

Let f be a function defined on an interval with primitive functions F_1 and F_2. Then $F_1 - F_2$ is a constant function.
Conversely, if F is a primitive function, then $F(x) + c$ is also a primitive function for any constant c.

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Notation

If f is a function defined on an interval and F is its primitive function, then we write

$$\int f = \int f(x)\,dx = F(x) + c = F + c,$$

where c denotes an arbitrary constant.

Example

Let $f :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be the constant function 1.

Let $F_1 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by $F_1(x) = x$.

Let $F_2 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by

$$F_2(x) = \begin{cases} x - 1, & \text{if } x < 0, \\ x + 1, & \text{if } x > 0. \end{cases}$$

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- Both F_1 and F_2 are primitive functions of f.
- $F_1 - F_2$ is not constant.

Riemann Integral: Primitive Function: Example

Example

Let $f :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be the constant function 1.

Let $F_1 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by $F_1(x) = x$.

Let $F_2 :]-\infty, 0[\cup]0, \infty[\to \mathbb{R}$ be defined by

$$F_2(x) = \begin{cases} x - 1, & \text{if } x < 0, \\ x + 1, & \text{if } x > 0. \end{cases}$$

- Both F_1 and F_2 are primitive functions of f.
- $F_1 - F_2$ is not constant.
- The correct interpretation: the domain of f consists of two intervals. Thus f is composed of functions defined on separate intervals, which can (and should) be treated independently.

Riemann Integral: Primitive Functions: Reminder

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Reminder

$(c)' = 0 \ (c \in \mathbb{R})$	$(x^\alpha)' = \alpha x^{\alpha-1} \ (\alpha \in \mathbb{R})$
$(e^x)' = e^x$	$(a^x)' = a^x \ln a \ (a > 0,\ a \neq 1)$
$(\ln x)' = \frac{1}{x}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\arctan x)' = \frac{1}{1+x^2}$	$(\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$
$(\mathrm{arcsec}\, x)' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$(\mathrm{arccsc}\, x)' = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$(\sinh x)' = \cosh x$	$(\cosh x)' = \sinh x$
$(\tanh x)' = \mathrm{sech}^2 x$	$(\coth x)' = -\mathrm{csch}^2 x$
$(\mathrm{sech}\, x)' = -\mathrm{sech}\, x \tanh x$	$(\mathrm{csch}\, x)' = -\mathrm{csch}\, x \coth x$
$(\mathrm{arsinh}\, x)' = \frac{1}{\sqrt{1+x^2}}$	$(\mathrm{arcosh}\, x)' = \frac{1}{\sqrt{x^2-1}} \ (x > 1)$
$(\mathrm{artanh}\, x)' = \frac{1}{1-x^2} \ (\lvert x\rvert < 1)$	$(\mathrm{arcoth}\, x)' = \frac{1}{1-x^2} \ (\lvert x\rvert > 1)$
$(\mathrm{arsech}\, x)' = -\frac{1}{x\sqrt{1-x^2}}$	$(\mathrm{arcsch}\, x)' = -\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

Example

$\int 0\ dx = c$

$\int e^x\ dx = e^x$

$\int \frac{1}{x}\ dx = \ln x + c \quad (x \in \mathbb{R}_{++})$

$\int \cos x\ dx = \sin x$

$\int \sec^2 x\ dx = \tan x + c$

$\int \sec x \tan x\ dx = \sec x + c$

$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + c$

$\int \frac{1}{1+x^2}\ dx = \arctan x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}}\ dx = \operatorname{arcsec} x + c$

$\int \cosh x\ dx = \sinh x + c$

$\int \operatorname{sech}^2 x\ dx = \tanh x + c$

$\int \operatorname{sech} x \tanh x\ dx = -\operatorname{sech} x + c$

$\int \frac{1}{\sqrt{1+x^2}}\ dx = \operatorname{arsinh} x + c$

$\int \frac{1}{1-x^2}\ dx = \operatorname{artanh} x + c \quad (|x| < 1)$

$\int \frac{1}{x\sqrt{1-x^2}}\ dx = -\operatorname{arsech} x + c$

$\int x^\alpha\ dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$

$\int a^x\ dx = \frac{1}{\ln a} a^x \quad (a > 0,\ a \neq 1)$

$\int \frac{1}{x}\ dx = \ln(-x) + c\ (x \in \mathbb{R}_{--})$

$\int \sin x\ dx = -\cos x + c$

$\int \csc^2 x\ dx = \cot x + c.$

$\int \csc x \cot x = -\csc x + c$

$\int \frac{1}{\sqrt{1-x^2}}\ dx = -\arccos x + c$

$\int \frac{1}{1+x^2}\ dx = -\operatorname{arccot} x + c$

$\int \frac{1}{|x|\sqrt{x^2-1}} = -\operatorname{arccsc} x + c$

$\int \sinh x\ dx = \cosh x + c$

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Riemann Integral: Primitive Function: A Remark

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• In the literature one often finds the formula $\int \frac{1}{x} dx = \ln|x| + c$, which merges the two cases. This is somewhat misleading, in my personal opinion.

Riemann Integral: Fundamental Theorem (Second)

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Theorem

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let F be a primitive (antiderivative) of f. Then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Riemann Integral: Fundamental Theorem (Second): Algorithmic View

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• The theorem suggests a two-phase algorithm for computing integrals:

(I) Find a primitive function of the integrand f.

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• The good news: this scheme works very often. However, finding a primitive may require technical skill.

• The bad news: there exist nice functions that cannot be expressed with a simple primitive in closed form. For example: e^{-x^2}.

Riemann Integral: Fundamental Theorem (Second): Consequences

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Theorem [Integration by Parts]

Let f and g be two Riemann integrable functions on $[a, b]$. Let F and G be their primitive functions. Then

(i)

$$\int f(x)G(x)\ dx = F(x)G(x) - \int F(x)g(x)\ dx,$$

(ii)

$$\int_a^b f(x)G(x)\ dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)\ dx,$$

• The theorem is somewhat strange. On the right-hand side we still have an integral, but evaluating the left-hand side also requires computing an integral / primitive function.

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Riemann Integral: Integration by Parts: Interpretation

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• Applying integration by parts (and sometimes even recognizing when to use it) requires intuition and mathematical insight.

Riemann Integral: Fundamental Theorem (Second): Consequences

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Theorem [Substitution Rule]

Let $\alpha : [a, b] \to [\alpha(a), \alpha(b)]$ be a strictly increasing, differentiable function whose derivative is Riemann integrable. Let $f : [\alpha(a), \alpha(b)] \to \mathbb{R}$ be a Riemann integrable function. Then $f(\alpha(x))\alpha'(x)$ is Riemann integrable on $[a, b]$, and moreover

(i)

$$\int f(\alpha(x))\alpha'(x)\, dx = \int f(u)\, du\Big|_{u=\alpha(x)},$$

(ii)

$$\int_a^b f(\alpha(x))\alpha'(x)\, dx = \int_{\alpha(a)}^{\alpha(b)} f(u)\, du.$$

Riemann Integral: Substitution: Interpretation

- Recognizing when to apply substitution is usually easier than recognizing integration by parts. We need to identify that the derivative of an essential (possibly repeating) subexpression appears in the formula. In such a case, it is worth introducing a new variable for that subexpression.

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• If we use the substitution $u = \alpha(x)$, then $\alpha'(x)\,dx$ must be replaced by du. If we formally write the derivative as $\frac{du}{dx} = \alpha'(x)$, then by formal rearrangement we obtain $du = \alpha'(x)\,dx$.

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◦ It is already apparent that integration is a more subtle task than differentiation. Intuition and clever algebraic manipulations and observations play a significant role. Effective integration requires a lot of practice. Further examples will help with this.

Example

$\int \frac{x^2}{x^3+1} \ dx$

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$$\int \frac{x^2}{x^3+1}\, dx \left[\begin{smallmatrix} u=x^3+1 \\ du=3x^2\ dx \end{smallmatrix}\right] = \int \frac{1}{u} \cdot \frac{1}{3}\, du = \frac{1}{3} \ln u + c = \frac{1}{3} \ln(x^3+1) + c.$$

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$$\begin{aligned} \int_0^1 \frac{x^2}{x^3+1}\, dx \left[\begin{smallmatrix} u=x^3+1 \\ du=3x^2\ dx \end{smallmatrix} \right] &= \int_1^2 \frac{1}{u} \cdot \frac{1}{3}\ du \\ &= \left[\frac{1}{3} \ln u \right]_1^2 = \frac{1}{3} \ln 2 - \frac{1}{3} \ln 1 \approx 0.231. \end{aligned}$$

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$$\int \sin^4 x \cos x \; dx \left[{u=\sin x \atop du=\cos x \; dx} \right] = \int u^4 \, du = \frac{u^5}{5} + c = \frac{\sin^5 x}{5} + c.$$

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$$\int_0^{\pi/2} \sin^4 x \cos x \; dx \left[{}^{u=\sin x}_{du=\cos x \; dx}\right] = \int_0^1 u^4 \, du = \left[\frac{u^5}{5}\right]_0^1 = \frac{1}{5}.$$

Riemann Integration: Examples: Substitution

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$$\begin{aligned}\int x\sqrt{x^2-4}\ dx \left[{}^{u=x^2-4}_{du=2x\ dx}\right] &= \int \sqrt{u}\cdot\frac{1}{2}\ du=\frac{1}{2}\int u^{1/2}\ du\\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+c=\frac{1}{3}(x^2-4)^{3/2}+c.\end{aligned}$$

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$$\int_2^3 x\sqrt{x^2-4}\ dx \left[{u=x^2-4 \atop du=2x\ dx}\right]=\frac{1}{2}\int_0^5 u^{1/2}\ du=\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_0^5=\frac{1}{3}\cdot 5^{3/2}\approx 3.726.$$

Example

$\int \ln x \; dx.$

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• Here the integrand is not even a product. We use the idea of viewing $\ln x$ as the product $1 \cdot \ln x$. We know that $(\ln x)' = \frac{1}{x}$, while the antiderivative of 1 is x. Integration by part (in this form) is "a good deal".

$$\int \ln x \; dx \begin{bmatrix} u{=}\ln x, \; u'{=}\frac{1}{x} \\ v'{=}1, \; v{=}x \end{bmatrix} = x \ln x - \int \frac{1}{x} \cdot x \; dx = x \ln x - x + c.$$

$$\begin{aligned} \int_1^e \ln x \; dx \begin{bmatrix} u{=}\ln x, \; u'{=}\frac{1}{x} \\ v'{=}1, \; v{=}x \end{bmatrix} &= [x \ln x]_1^e - \int_1^e 1 \; dx \\ &= (e \ln e - 1 \ln 1) - [x]_1^e = e - (e-1) = 1. \end{aligned}$$

Riemann integration: Examples: Integration by parts

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$$\begin{aligned}\int (2x-3)e^x \, dx \left[\begin{smallmatrix} u=2x-3, \; u'=2 \\ v'=e^x, \; v=e^x \end{smallmatrix}\right] &= (2x-3)e^x - \int 2e^x \, dx \\ &= (2x-3)e^x - 2e^x + c = (2x-5)e^x + c.\end{aligned}$$

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$$\begin{aligned}\int_0^1 (2x-3)e^x \ dx \begin{bmatrix} u=2x-3,\ u'=2 \\ v'=e^x,\ v=e^x \end{bmatrix} &= [(2x-3)e^x]_0^1 - \int_0^1 2e^x \, dx \\ &= (-e+3) - (2e-2) = 5-3e \approx -3.15\end{aligned}$$

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$$\begin{aligned}\int x\sin(2x)\ dx \left[{u=x,\ u'=1 \atop v'=\sin 2x,\ v=-\frac{\cos 2x}{2}}\right] &= -\frac{x\cos 2x}{2}-\int -\frac{\cos 2x}{2}\ dx\\ &= -\frac{x\cos(2x)}{2}+\frac{\sin(2x)}{4}+c.\end{aligned}$$

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$$= -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + c.$$

$$\int_0^{\pi/2} x\sin(2x)\ dx \left[{u=x,\ u'=1 \atop v'=\sin 2x,\ v=-\frac{\cos 2x}{2}}\right] = \left[-\frac{x\cos 2x}{2}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos 2x}{2}\ dx$$
$$= \frac{\pi}{4} + 0 = \frac{\pi}{4} \approx 0.785.$$

Riemann integration: Examples: Substitution and integration by parts

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- Integration by parts now leads to the solution of the problem.
- The remaining details are left to the students.

Riemann integration: Examples: Substitution and integration by parts

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Eliminating denominators yields: $5x - 3 = A(x+1) + B(x-3)$. Substituting $x = 3$ yields $12 = 4A$, hence $A = 3$. Substituting $x = -1$ gives $-8 = -4B$, hence $B = 2$.

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$$\begin{aligned}\int \frac{5x-3}{x^2-2x-3}\ dx &= \int \frac{3}{x-3}\ dx+\int \frac{2}{x+1}\ dx\\ &= 3\ln(x-3)+2\ln(x+1)+c.\end{aligned}$$

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$$\begin{aligned} \int \frac{x^2+1}{x(x-1)^2}\ dx &= \int \left(\frac{1}{x} + \frac{2}{(x-1)^2}\right)\ dx \\ &= \ln x - \frac{2}{x-1} + c. \end{aligned}$$

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$$\begin{aligned} \int \frac{2x^2+x+4}{x^3+4x}\ dx &= \int \frac{1}{x}\ dx + \int \frac{x+1}{x^2+4}\ dx \\ &= \ln x + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c. \end{aligned}$$

Thank you for your attention!