

BSc Mathematics for Computer Scientists 2: IX. Applications of Differential Calculus: Function Analysis, L'Hospital's Rule

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Reminder: Basic Derivatives

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Reminder

$$(c)' = 0 \quad (c \in \mathbb{R})$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\sin x)' = \cos x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(\sinh x)' = \cosh x$$

$$(\tanh x)' = \operatorname{sech}^2 x$$

$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$

$$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}}$$

$$(\operatorname{artanh} x)' = \frac{1}{1-x^2} \quad (|x| < 1)$$

$$(\operatorname{arsech} x)' = -\frac{1}{x\sqrt{1-x^2}}$$

$$(x^\alpha)' = \alpha x^{\alpha-1} \quad (\alpha \in \mathbb{R})$$

$$(a^x)' = a^x \ln a \quad (a > 0, a \neq 1)$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\cos x)' = -\sin x$$

$$(\cot x)' = -\operatorname{csc}^2 x$$

$$(\operatorname{csc} x)' = -\operatorname{csc} x \cot x$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$(\operatorname{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$(\cosh x)' = \sinh x$$

$$(\operatorname{coth} x)' = -\operatorname{csch}^2 x$$

$$(\operatorname{csch} x)' = -\operatorname{csch} x \operatorname{coth} x$$

$$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$$

$$(\operatorname{arcoth} x)' = \frac{1}{1-x^2} \quad (|x| > 1)$$

$$(\operatorname{arsch} x)' = -\frac{1}{|x|\sqrt{1+x^2}}$$

Function Analysis: Theorems

Theorem

Let f be a differentiable function defined on an interval.

- (i) If $f'(x) \geq 0$ ($x \in \text{dom } f$), then f is monotone increasing.
- (ii) If $f'(x) > 0$ ($x \in \text{dom } f$), then f is strictly increasing.

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Theorem

Let f be a differentiable function defined on an interval.

- (i) If $f'(x) \leq 0$ ($x \in \text{dom } f$), then f is monotone decreasing.
- (ii) If $f'(x) < 0$ ($x \in \text{dom } f$), then f is strictly decreasing.

Function Analysis: Theorems (continued)

Theorem

Let f be a differentiable function defined on an interval.

- (i) If $f''(x) \geq 0$ ($x \in \text{dom } f$), then f is convex.
- (ii) If $f''(x) > 0$ ($x \in \text{dom } f$), then f is strictly convex.

Function Analysis: Theorems (continued)

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Let f be a differentiable function defined on an open interval. If f has a local extremum at x_0 , then $f'(x_0) = 0$.

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Example

$$f(x) = x^3$$

Function Analysis: Theorems (continued)

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Theorem

Let f be a differentiable function defined on an open interval, containing x_0 . If $f'(x_0) = 0$ and $f''(x_0) > 0$ then has a local minimum point at x_0 .

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Function Analysis: Theorems (continued)

Theorem

Let f be a differentiable function defined on an open interval, containing x_0 . If $f'(x_0) = 0$ and for a suitable $\delta > 0$ $f'(x) < 0$ if $x \in]x_0 - \delta, x_0[$, furthermore $f'(x) > 0$ if $x \in]x_0, x_0 + \delta[$ then f has a local minimum point at x_0 .

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Function Analysis: Example 1

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- In its definition, only the division is problematic. Thus it is undefined exactly at $x = -2$ and $x = 2$ (since we would divide by 0).
- Domain: $] -\infty, -2 [\cup] -2, 2 [\cup] 2, \infty [$.

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- The numerator is positive if $x \in]-\infty, -1 [\cup] 1, \infty [$. The numerator is negative if $x \in]-1, 1 [$. The denominator is positive if $x \in]-\infty, -2 [\cup] 2, \infty [$. The denominator is negative if $x \in]-2, 2 [$.

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- The function is positive on $] -\infty, -2 [\cup] -1, 1 [\cup] 2, \infty [$. The function is negative on $] -2, -1 [\cup] 1, 2 [$.

Function Analysis: Example 1: Limits

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- $\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^2 - 4} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{1}{x^2}}{1 - \frac{4}{x^2}} = 1.$

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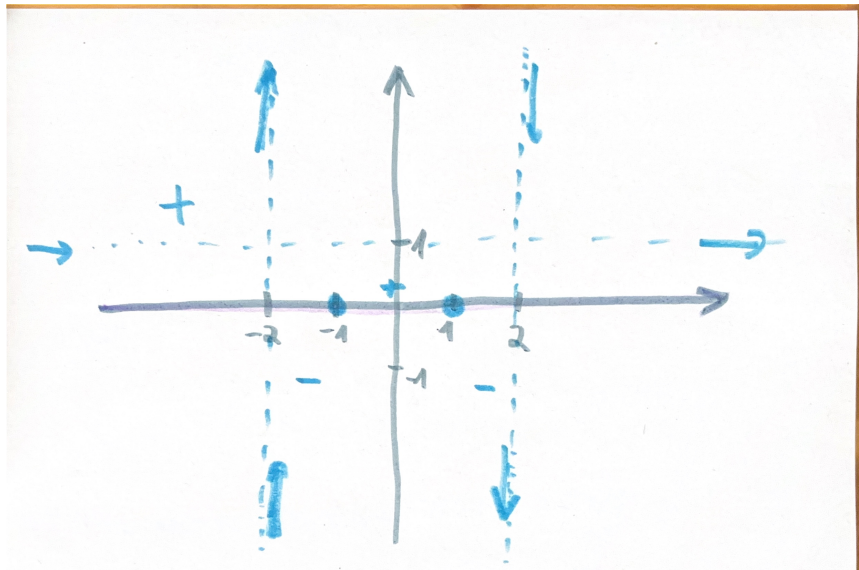
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Function Analysis: Example 1: First Summary



Function Analysis: Example 1: Differentiation

First derivative (quotient rule):

$$\begin{aligned}f'(x) &= \frac{2x(x^2 - 4) - (x^2 - 1)2x}{(x^2 - 4)^2} \\&= \frac{2x[(x^2 - 4) - (x^2 - 1)]}{(x^2 - 4)^2} \\&= \frac{2x(-3)}{(x^2 - 4)^2} = \frac{-6x}{(x^2 - 4)^2}\end{aligned}$$

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First derivative ($(g^{-1})' = -g^{-1} \cdot g'$):

$$f'(x) = \left(\frac{x^2 - 1}{x^2 - 4}\right)' = \left(1 + \frac{3}{x^2 - 4}\right)' = 3 \cdot (-1) \frac{1}{(x^2 - 4)^2} \cdot 2x.$$

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The denominator is positive for all x in the domain, thus the sign of $f'(x)$ is the same as the sign of $-6x$.

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- On the interval $] 2, \infty [$, $f'(x) < 0$: f is strictly decreasing.

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- In a neighborhood of $x = 0$, f is increasing from the left and decreasing from the right.
- There is a local maximum at $x = 0$.
- The maximal value on the interval $] - 2, 2[$ is

$$f(0) = \frac{-1}{-4} = \frac{1}{4}.$$

Function Analysis: Example 1: Range

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- On the interval $(-\infty, -2)$, f is increasing and continuous, $\lim_{x \rightarrow -\infty} f = 1^+$, $\lim_{x \rightarrow -2^-} f = +\infty$. The set of values taken here is $]1, \infty[$.

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- On the interval $(-2, 0)$, f is continuous and increasing, $\lim_{x \rightarrow -2^+} f = -\infty$, $f(0) = 1/4$. The set of values taken here is $(-\infty, 1/4]$.

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- On the interval $(2, \infty)$, f is continuous and decreasing, $\lim_{x \rightarrow 2^+} f = +\infty$, $\lim_{x \rightarrow \infty} f = 1^+$. The set of values taken here is $(1, \infty)$.
- Summarizing:

$$\text{im}_f =]-\infty, \frac{1}{4}] \cup]1, \infty[$$

Function Analysis: Example 1: Second Derivative

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Second derivative:

$$f''(x) = \frac{d}{dx} \left(\frac{-6x}{(x^2 - 4)^2} \right)$$

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We apply the quotient rule again:

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- Since $3x^2 + 4 > 0$ for all x , the sign of $f''(x)$ is determined by $(x^2 - 4)^3$, which has the same sign as $x^2 - 4$ (raising to the third power preserves the sign).

Function Analysis: Example 1: Convexity

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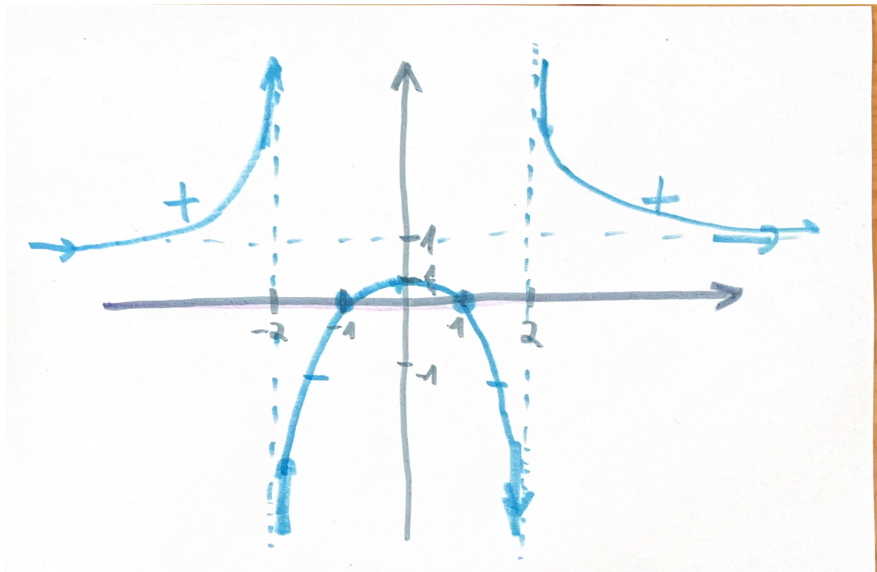
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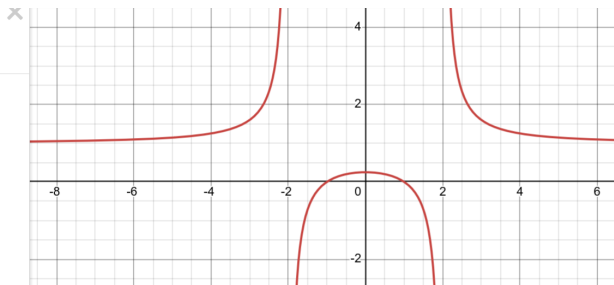
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- On the interval $] 2, \infty [$, $x^2 - 4 > 0$, hence $f''(x) > 0$. Thus on $] 2, \infty [$, f is strictly convex.

Function Analysis: Example 1: Final Summary



Function Analysis: Example 1: The Exact Graph

$$\frac{x^2 - 1}{x^2 - 4}$$



Function Analysis: Example 2: Domain, Signs, Roots

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- Domain: $] -\infty, -1 [\cup] -1, 1 [\cup] 1, \infty [$.
- The function value is the reciprocal of a positive number. Thus it is always positive. Hence it is never 0, it has no roots.

Function Analysis: Example 2: Limits

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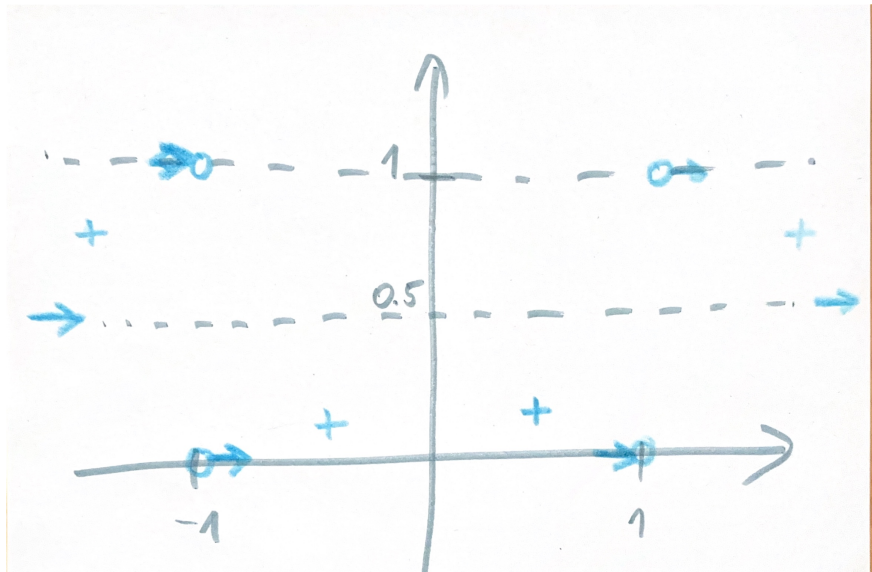
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Function Analysis: Example 2: First Summary



Function Analysis: Example 2: Differentiation

Function Analysis: Example 2: Differentiation

First derivative (quotient / chain rule): Let $k(x) = \frac{1}{1-x^2}$. Then $g(x) = (1 + e^{k(x)})^{-1}$. First derivative:

$$\begin{aligned}g'(x) &= -\frac{1}{(1 + e^{k(x)})^2} \cdot e^{k(x)} \cdot k'(x) = -\frac{1}{(1 + e^{\frac{1}{1-x^2}})^2} \cdot e^{\frac{1}{1-x^2}} \cdot k'(x) \\&= -\frac{1}{(1 + e^{\frac{1}{1-x^2}})^2} \cdot e^{\frac{1}{1-x^2}} \cdot (-1) \frac{1}{(1-x^2)^2} (-2x) \\&= -\frac{2x \cdot e^{\frac{1}{1-x^2}}}{(1 + e^{\frac{1}{1-x^2}})^2 (1-x^2)^2}.\end{aligned}$$

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- Thus $\text{sgn}(g'(x)) = -\text{sgn}(x)$.

Function Analysis: Example 2: Differentiation, Monotonicity

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- In a neighborhood of $x = 0$, f is increasing from the left and decreasing from the right.
- Thus, there is a local maximum at $x = 0$.
- The maximal value on the interval $] - 1, 1[$ is

$$f(0) = \frac{1}{1} = \frac{1}{1 + e}.$$

Function Analysis: Example 2: Differentiation, Range

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- On the interval $(-\infty, -1)$, f is increasing and continuous, $\lim_{x \rightarrow -\infty} f = \frac{1}{2}^+$, $\lim_{x \rightarrow -1^-} f = 1$. The set of values taken here is $]\frac{1}{2}, 1[$.

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- On the interval $(0, 1)$, f is continuous and decreasing, $f(0) = \frac{1}{1+e}$, $\lim_{x \rightarrow 1^-} f = 0$. The set of values taken here is $]0, \frac{1}{1+e}]$.

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- On the interval $(1, \infty)$, f is continuous and decreasing, $\lim_{x \rightarrow 1^+} f = 1$, $\lim_{x \rightarrow \infty} f = \frac{1}{2}^+$. The set of values taken here is $(\frac{1}{2}, 1)$.
- Summarizing:

$$\text{im}_f =]0, \frac{1}{1+e}] \cup]\frac{1}{2}, 1[.$$

Function Analysis: Example 2: Second Derivative

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The second derivative:

$$g''(x) = -\frac{e^k(1+e^k)}{(1-x^2)^4(1+e^k)^4} G(x),$$

where $k = \frac{1}{1-x^2}$ and

$$G(x) = 2 + 8x^2 - 6x^4 + (2 - 6x^4)e^k.$$

The denominator is positive, thus $\operatorname{sgn}(f'') = -\operatorname{sgn}(G)$.

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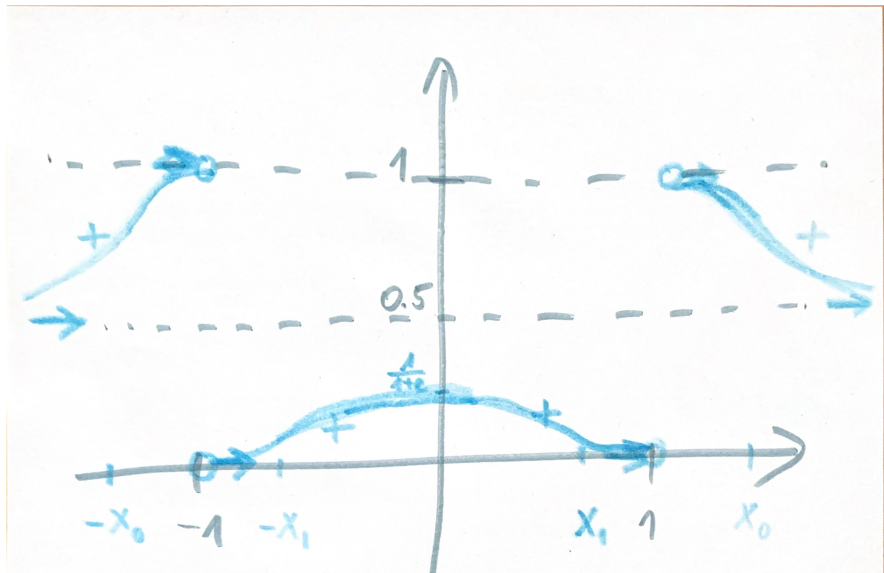
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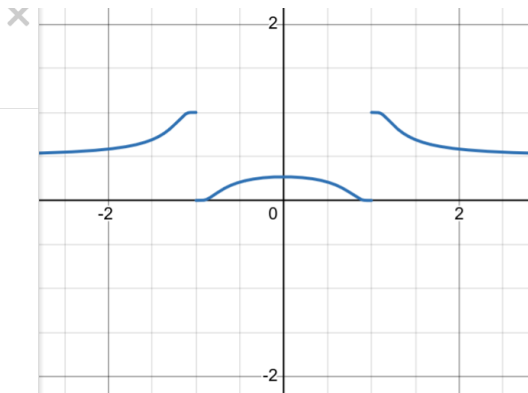
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- On $(1, x_1)$: $G < 0 \Rightarrow f'' < 0 \Rightarrow f$ is concave.
- On (x_1, ∞) : $G > 0 \Rightarrow f'' > 0 \Rightarrow f$ is convex.
- The inflection points are of the form $\pm x_1$ and $\pm x_2$, where $0 < x_1 < 1$ and $x_0 > 1$.

Function Analysis: Example 2: Final Summary



Function Analysis: Example 2: The Exact Graph

$$\frac{1}{1 + e^{\frac{1}{1-x^2}}}$$



Break



Rolle's Theorem

Theorem [Rolle's Theorem]

Let f be continuous on the closed interval $[a, b]$, differentiable on the open interval $]a, b[$, and suppose that $f(a) = f(b)$. Then there exists at least one $\xi \in]a, b[$ such that

$$f'(\xi) = 0.$$

Mean Value Theorem (Lagrange)

Mean Value Theorem (Lagrange)

Theorem [Lagrange's Theorem]

Let f be continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists $\xi \in]a, b[$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

(Equivalently: $f(b) - f(a) = f'(\xi)(b - a)$.)

Cauchy's Mean Value Theorem

Cauchy's Mean Value Theorem

Theorem [Cauchy's Theorem]

Let f and g be continuous on $[a, b]$, differentiable on $]a, b[$, and assume that $g'(x) \neq 0$ for all $x \in]a, b[$. Then there exists $\xi \in]a, b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Remark: From the assumptions it follows that $g(b) \neq g(a)$.

Darboux's Theorem: Intermediate Value Property of Derivatives

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Theorem [Darboux's Theorem]

Let f be differentiable on an interval. Then f' (not necessarily continuous) has the intermediate value property: if $f'(a) < k < f'(b)$, then there exists a point c between a and b such that $f'(c) = k$.

Relationships

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- Rolle's theorem is a special case of Lagrange's theorem ($f(a) = f(b)$).
- Lagrange's theorem is a special case of Cauchy's theorem ($g(x) = x$).
- The proof of Cauchy's theorem is based on Rolle's theorem. (Apply Rolle's theorem to the auxiliary function $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$.)

L'Hospital's Rule: Theorem

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Theorem [L'Hospital's Rule]

Let f and g be differentiable functions in a neighborhood of a (possibly excluding a itself), and assume:

1. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$.
2. $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists $\widehat{\mathbb{R}}$.
3. $g'(x) \neq 0$ in a ball centered at a (possibly excluding a).

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

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Proof: Application of Cauchy's mean value theorem.

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Then

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Proof: Application of Cauchy's mean value theorem.

Remark: The rule also holds for one-sided limits and for $a = \pm\infty$ (with appropriate reparametrization).

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Example ($\frac{0}{0}$ form)

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = ?$$

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$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{(\sin x)'}{(x)'} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1.$$

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Example ($\frac{0}{0}$ form)

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = ?$$

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L'Hospital's Rule: Example

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Example ($\frac{\infty}{\infty}$ form)

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = ?$$

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Example ($\frac{\infty}{\infty}$ form)

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$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0.$$

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Example ($0 \cdot (-\infty)$ form \rightarrow rewrite as $\frac{\infty}{\infty}$ or $\frac{0}{0}$)

$$\lim_{x \rightarrow 0^+} x \ln x = ?$$

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Rewrite: $x \ln x = \frac{\ln x}{1/x}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \left(\frac{-\infty}{\infty} \right) \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

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$$\frac{1}{\sin x} - \frac{1}{x} = ?$$

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Rewrite:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}.$$

L'Hospital's Rule: Example

Example ($0 \cdot (-\infty)$ form \rightarrow rewrite $\rightarrow \frac{\infty}{\infty}$ or $\frac{0}{0}$)

$$\frac{1}{\sin x} - \frac{1}{x} = ?$$

Rewrite:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}.$$

Now $\frac{0}{0}$ form:

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x}.$$

Still $\frac{0}{0}$, apply L'Hospital again:

$$= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

L'Hospital's Rule: Example

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Example (0^0 form \rightarrow logarithm \rightarrow $\frac{0}{0}$ form)

$$\lim_{x \rightarrow 0^+} x^x = ?$$

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$$\lim_{x \rightarrow 0^+} x^x = ?$$

Let $y = x^x$, $\ln y = x \ln x$. From a previous example $\lim_{x \rightarrow 0^+} x \ln x = 0$. Since $\ln y \rightarrow 0$, we get $y = e^{\ln y} \rightarrow e^0 = 1$.
Thus

$$\lim_{x \rightarrow 0^+} x^x = 1.$$

L'Hospital's Rule: Example

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Example (∞^0 form)

$$\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\cos x} = ?$$

L'Hospital's Rule: Example

Example (∞^0 form)

$$\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\cos x} = ?$$

Let $y = (\tan x)^{\cos x}$, $\ln y = \cos x \cdot \ln(\tan x)$. Since $\cos x \rightarrow 0^+$, $\ln(\tan x) \rightarrow +\infty$, we have the form $0 \cdot \infty$.

Rewrite:

$$\ln y = \frac{\ln(\tan x)}{1/\cos x} = \frac{\ln(\tan x)}{\sec x}.$$

This is $\frac{\infty}{\infty}$. Apply L'Hospital:

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec x \tan x} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1/\cos x}{\sin^2 x / \cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

Thus $\ln y \rightarrow 0 \Rightarrow y \rightarrow 1$.

L'Hospital's Rule: Counterexample

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Counterexample

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = ?$$

L'Hospital's Rule: Counterexample

Counterexample

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = ?$$

- Numerator: $f(x) = x + \sin x$, denominator: $g(x) = x$.

L'Hospital's Rule: Counterexample

Counterexample

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = ?$$

- Numerator: $f(x) = x + \sin x$, denominator: $g(x) = x$.
- As $x \rightarrow \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.

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- As $x \rightarrow \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.
- Both f and g are differentiable for $x > 0$.

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- Both f and g are differentiable for $x > 0$.
- $g'(x) = 1 \neq 0$.

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- As $x \rightarrow \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.
- Both f and g are differentiable for $x > 0$.
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- Ratio of derivatives: $\frac{f'(x)}{g'(x)} = \frac{1 + \cos x}{1} = 1 + \cos x$.

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- Numerator: $f(x) = x + \sin x$, denominator: $g(x) = x$.
- As $x \rightarrow \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.
- Both f and g are differentiable for $x > 0$.
- $g'(x) = 1 \neq 0$.
- Ratio of derivatives: $\frac{f'(x)}{g'(x)} = \frac{1 + \cos x}{1} = 1 + \cos x$.
- $\lim_{x \rightarrow \infty} (1 + \cos x)$ DOES NOT EXIST, since $\cos x$ oscillates between -1 and 1 .

L'Hospital's Rule: Counterexample

Counterexample

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = ?$$

- Numerator: $f(x) = x + \sin x$, denominator: $g(x) = x$.
- As $x \rightarrow \infty$: $\lim f(x) = \infty$, $\lim g(x) = \infty$. So we have $\frac{\infty}{\infty}$ form.
- Both f and g are differentiable for $x > 0$.
- $g'(x) = 1 \neq 0$.
- Ratio of derivatives: $\frac{f'(x)}{g'(x)} = \frac{1 + \cos x}{1} = 1 + \cos x$.
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- L'Hospital's rule requires that $\lim \frac{f'}{g'}$ exists (finite or infinite). This condition is not satisfied here.

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$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{\sin x}{x} \right) = 1 + 0 = 1.$$

This is the end!

Thank you for your attention!