BSc Mathematics for Computer Scientists 2: VII Continuity of real functions

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2026 Spring semester

Reminder: Limit

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Let f : dom $f \to \mathbb{R}$ and $h \in L_f \subset \widehat{\mathbb{R}}$. We write

$$\lim_{x \to h} f(x) = \ell \in \widehat{\mathbb{R}},$$

if for every ball B centered at ℓ there exists a ball B_0 centered at h such that for $x \in (B_0 - \{h\}) \cap \mathrm{dom}\, f$ we have $f(x) \in B$.

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Reminder: Limit of sequences

Let $f : \mathbb{N} \to \mathbb{R}$. We write

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if for every ball B centered at ℓ there exists a ball $B_0 =]N, \infty[$ such that for $x \in B_0 \cap \text{dom } f$ we have $f(x) \in B$.

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Definition

The sequence $f : \mathbb{N} \to \mathbb{R}$ is oscillatory if it is divergent but not properly divergent.

Definition: Cauchy sequence

A sequence $a : \mathbb{N} \to \mathbb{R}$ is a Cauchy sequence if for every positive ε there exists a ball B centered at ∞ such that for all $n, m \in B$ we have $|a_n - a_m| < \varepsilon$.

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Theorem: Cauchy theorem

The following two properties of sequences are equivalent:

(i) The sequence $s : \mathbb{N} \to \mathbb{R}$ is convergent,

(ii) The sequence $s : \mathbb{N} \to \mathbb{R}$ is a Cauchy sequence.

Limit: Convergence: Theorems

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The following two properties of sequences are equivalent:

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The above equivalence also holds if we consider rational sequences $s : \mathbb{N} \to \mathbb{Q}$. Naturally, the limits of the sequences are real numbers. This theorem can serve as a foundation for the mathematically rigorous construction of the real numbers based on the rational numbers.

Break

Functions: Continuity

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Definition: Local continuity without limits

The real function $f : \text{dom } f \to \mathbb{R}$ is continuous at the point $x_0 \in \text{dom } f$ if for every ball B centered at $f(x_0)$ there exists a ball B_0 centered at x_0 such that for all $x \in B_0 \cap \text{dom } f$ we have $f(x) \in B$.

Functions: Continuity

Definition: Local continuity with limits

The function $f : \text{dom } f(\subset \mathbb{R}) \to \mathbb{R}$ is continuous at the point $x_0 \in \text{dom } f \cap L_f$ if:

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• If $x_0 \in \text{dom } f$ but $x_0 \notin L_f$, then we say that x_0 is an isolated point of the domain.

At isolated points, the function f is considered continuous!

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$$\lim_{n \to \infty} f(s_n) = f(x_0).$$

Functions: Continuity: The global concept

Definition: Global continuity

The function $f : \text{dom } f(\subset \mathbb{R}) \to \mathbb{R}$ is continuous (on its domain) if it is continuous at every point $x_0 \in \text{dom } f$.

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Theorem

Assume that $a \in \mathbb{R}$ is not in the domain of f. Let
$f^{\to a} : \text{dom } f (\subset \mathbb{R}) \cap]-\infty, a[\to \mathbb{R}, \quad x \mapsto f(x)$, and
$f^{a\to} : \text{dom } f (\subset \mathbb{R}) \cap]a, \infty[\to \mathbb{R}, \quad x \mapsto f(x)$.
Then f is continuous if and only if both $f^{\to a}$ and $f^{a\to}$ are continuous.

Functions: Continuity: Examples

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Example: Constant function

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Let k be a nonzero integer. $p_k : \mathbb{R} - \{0\} \to \mathbb{R}, \quad x \mapsto x^k$ is continuous.

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Let k be a nonzero integer. $p_k : \mathbb{R} - \{0\} \to \mathbb{R}, \quad x \mapsto x^k$ is continuous.

It is useful to think of (for example) $p_{-1}(x) = \frac{1}{x}$ as the combination of two continuous functions: $p^-_{-1} :]-\infty, 0[\to \mathbb{R}, \quad x \mapsto \frac{1}{x}$ and $p^+_{-1} :]0, \infty[\to \mathbb{R}, \quad x \mapsto \frac{1}{x}$. The domains of p^-_{-1} and p^+_{-1} are intervals, in contrast to the domain of p_{-1}.

Functions: Continuity: Examples

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Example: Power function

For $\alpha > 0$: $p_\alpha : \mathbb{R}_+ = [0, \infty[\to \mathbb{R}, \quad x \mapsto x^\alpha$ is continuous.

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Example: Exponential function

For $a > 0$: $exp_a : \mathbb{R}_+ \to \mathbb{R}, \quad x \mapsto a^x$ is continuous.

Functions: Continuity: Examples: Trigonometry

Example: Sine function

$\sin : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin x$ is continuous.

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Example: Tangent function

$\tan : \mathbb{R} - \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} \to \mathbb{R}, \quad x \mapsto \tan x$ is continuous.

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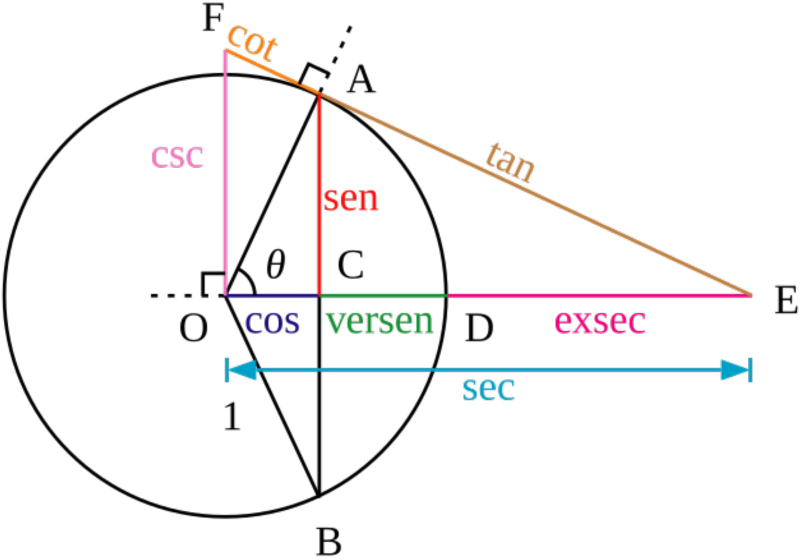
Again, we can think of the tangent function as being composed of functions $\tan_k :]\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi[\to \mathbb{R}, \quad x \mapsto \tan x \ (k \in \mathbb{Z})$, each of which is continuous. The domains of the $\tan_k$ functions are intervals.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Continuous functions: Operational properties

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Theorem

If f and g are continuous at a point x_0, then the following are also continuous at x_0:

(i) Sum, difference and product: $(f \pm g)$ and $(f \cdot g)$.

(ii) Quotient: f/g is continuous if $g(x_0) \neq 0$.

(iii) Composition: If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

(iv) Absolute value: $|f|$ is continuous.

(v) Inverse: If moreover f is strictly increasing in a neighborhood of x_0, then its inverse is also continuous at $f(x_0)$.

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Naturally, under global continuity assumptions, global continuity results can be derived from this theorem.

Example

$\arcsin x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a continuous function.

Functions: Continuity: Further examples

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$\ln x = \log_e x : \mathbb{R}_{++} =]0, \infty[\to \mathbb{R}$ is a continuous function.

Functions: Continuity: Further examples

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Example

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is a continuous function.

Example

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right): \quad]-\infty, \infty[\to \mathbb{R}$ is a continuous function.

Functions: Continuity: Further examples

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$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right) : \quad [1, \infty[\to \mathbb{R}$ is a continuous function.

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Example

$\operatorname{artanh} x = \frac{1}{2} \ln \frac{1-x}{1+x} : \quad]-1, 1[\to \mathbb{R}$ is a continuous function.

Break

Continuous functions on compact intervals: Properties

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Theorem [Weierstrass Theorem]

A continuous function defined on a compact interval is bounded, and moreover it attains its maximum and minimum on this interval.

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Theorem [Weierstrass Theorem for compact sublevel sets]

Let f be a continuous function on its domain dom f. If for some $\tau \in \mathbb{R}$ the set

$$\{x \in \text{dom } f : f(x) \leq \tau\}$$

is a nonempty compact set, then f attains its minimum.

Continuous functions on compact intervals: Properties

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Theorem: Bolzano Theorem

If a continuous function on $[a, b] \subset \operatorname{dom} f$ takes values of opposite sign at the endpoints ($f(a) \cdot f(b) < 0$), then it has at least one zero in the interior of the interval.

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Corollary

If $p \in \mathbb{R}[x]$ is a real polynomial of odd degree, then p has a real root.

$$x^3 - 2x^2 + 5x - 3 = x^3 \left(1 - 2\frac{1}{x} + 5\frac{1}{x^2} - 3\frac{1}{x^3}\right)$$

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$1 - 2\frac{1}{x} + 5\frac{1}{x^2} - 3\frac{1}{x^3} \to 1$, as $x \to \infty$.

Thus $f(N) > 0$, $f(-M) < 0$.

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Theorem: Darboux Theorem (Intermediate Value Theorem)

A continuous function on $[a, b] \subset \operatorname{dom} f$ takes every value between $f(a)$ and $f(b)$ at least once.

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The continuous image of an interval is also an interval.

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The continuous image of a compact interval is also a compact interval.

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Corollary

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Corollary

The continuous image of a compact interval is also a compact interval.

Example

Let $f(x) = \tan^2 x$ be a continuous function on the interval $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then the image of the bounded, open interval I, namely $f(I) = \{f(i) : i \in I\}$, is the unbounded, closed interval $[0, \infty[$.

Continuous functions on compact intervals: Properties

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Definition

A function is uniformly continuous if for every positive ε there exists a positive $\delta \in \mathbb{R}$ such that for any two points in the domain ($a, b \in \operatorname{dom} f$ with $|a - b| < \delta$), the distance of the function values is less than ε ($|f(a) - f(b)| \leq \varepsilon$).

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Theorem: Heine Theorem

Every continuous function on a compact interval is uniformly continuous.

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Theorem: Heine Theorem

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Example

$f :]0,1] \to \mathbb{R}, \quad x \mapsto \frac{1}{x}$ is continuous, but not uniformly continuous.

Break

Discontinuity: Example

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Example

Let $f(x) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \lfloor x \rfloor$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \mathbb{Z}$.

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- 2nd statement: If $x_0 \notin \mathbb{Z}$, then f is continuous.

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- 1st statement: If $x_0 \in \mathbb{Z}$, then f is not continuous. Use the definition.
- 2nd statement: If $x_0 \notin \mathbb{Z}$, then f is continuous. Continuity at a point is a local property.

Discontinuity: Example

Example

Let $f(x) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \lfloor x \rfloor$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \mathbb{Z}$.

- 1st statement: If $x_0 \in \mathbb{Z}$, then f is not continuous. Use the definition.
- 2nd statement: If $x_0 \notin \mathbb{Z}$, then f is continuous. Continuity at a point is a local property.

Theorem

Let f and g be two real functions. Assume that $f|_{B_\varepsilon(x_0)} = g|_{B_\varepsilon(x_0)}$, i.e., f and g coincide in a neighborhood of x_0.

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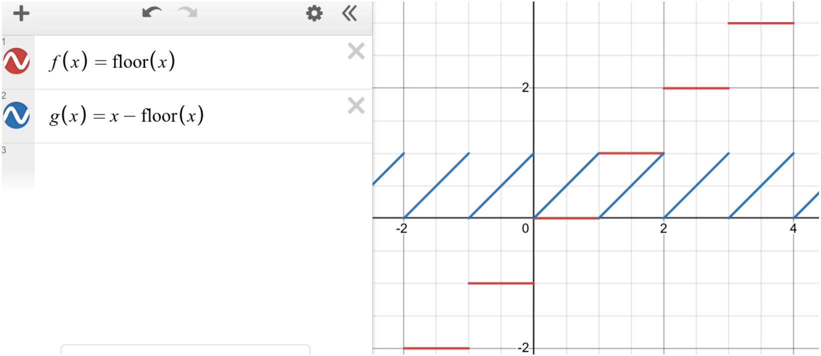
Let $f(x) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \{x\}$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $x_0 \in \mathbb{Z}$.

Discontinuity: Illustration



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Let $f(x): \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin\frac{1}{x}$. Then f is continuous at $x_0 \in \mathbb{R} - \{0\}$.

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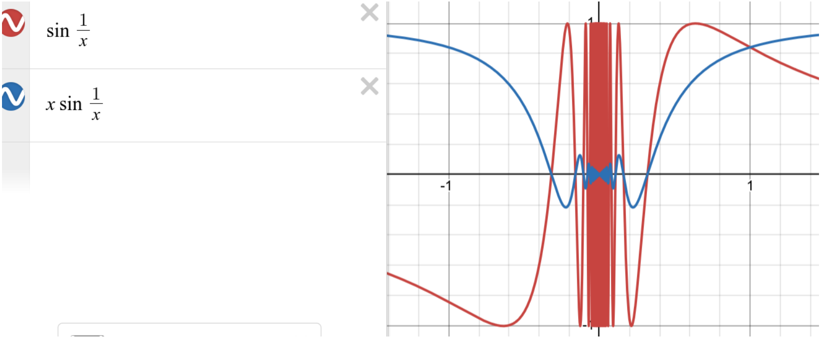
Example

Let $f(x) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Then f is continuous at $x_0 \in \mathbb{R}$.

Discontinuity: Illustration



Discontinuity: Example

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Let $f(x) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}.$

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Statement

For every center $c \in \mathbb{R}$ and every radius $\varepsilon > 0$, the ball $B_\varepsilon(c)$ contains both rational and irrational numbers.

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Example [Thomae function / popcorn function / Riemann function]

Let

$$f(x):\mathbb{R}\to\mathbb{R},\quad x\mapsto\begin{cases}0, & \text{if } x \text{ is irrational}\\ \frac{1}{q}, & \text{if } x=\frac{p}{q} \text{ is rational, } \gcd(p,q)=1\end{cases}.$$

f is continuous at x_0 if and only if x_0 is irrational.

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- 2nd statement: If x_0 is irrational, then f is continuous.

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f is continuous at x_0 if and only if x_0 is irrational.

• 1st statement: If x_0 is rational, then f is not continuous. Use the definition.

• 2nd statement: If x_0 is irrational, then f is continuous. Draw a picture.

Discontinuity: Illustration

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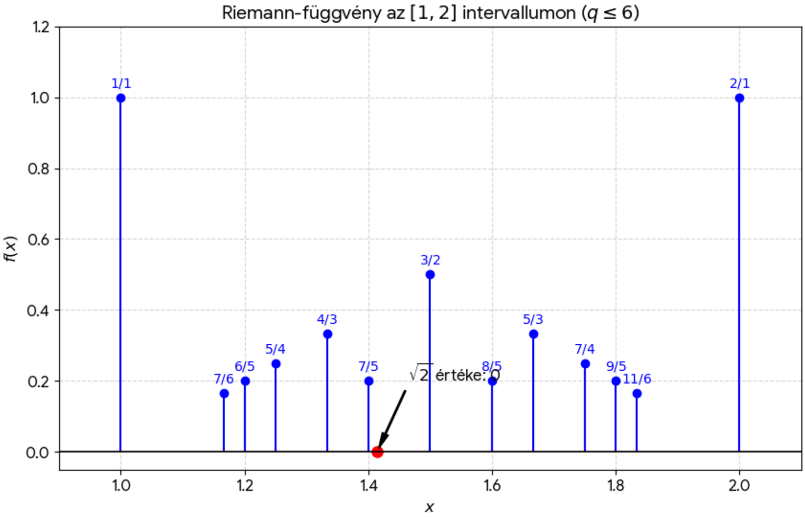

Discontinuity: Points of discontinuity

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A point x_0 is a point of discontinuity of f if $x_0 \in \operatorname{dom} f$ and f is not continuous at x_0.

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Theorem

There is no function $f : \mathbb{R} \to \mathbb{R}$ such that $D_f = \mathbb{I} = \mathbb{R} - \mathbb{Q}$.

Discontinuity: Classification of discontinuities

Definition

Let $x_0 \in D_f$ be a point of discontinuity.
x_0 is a discontinuity of the first kind if $\lim_{x \to x_0-} f(x)$ and $\lim_{x \to x_0+} f(x)$ both exist.
Otherwise, x_0 is of the second kind.

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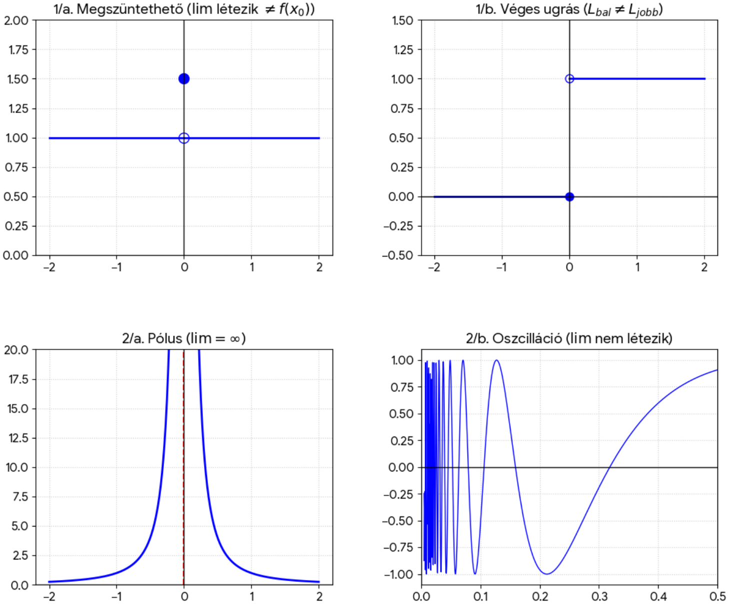
A discontinuity of the second kind is called a pole if $\lim_{x\to x_0-} f(x)$ and $\lim_{x\to x_0+} f(x)$ are equal and their common value is ∞ or $-\infty$.
Otherwise, it is called an oscillatory discontinuity.



1/a. Megszüntethető (lim létezik ≠ $f(x_0)$)
1/b. Véges ugrás ($L_{bal} \neq L_{jobb}$)
2/a. Pólus (lim = ∞)
2/b. Oszcilláció (lim nem létezik)

Break

Continuity: Limits

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$$\lim_{x \to 2} e^{x^2+x} = ?$$

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$\lim_{x\to 2} (\sin^2(x^2+x) + \cos x^3) = f(2) = \sin^2 6 + \cos 8$.

Thank you for your attention!