BSc Mathematics for Computer Scientists 2: VI. Foundations of Analysis

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Real sets: Bounded sets: Definitions

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Definition: Bounded set

A set $S \subset \mathbb{R}$ is bounded if there exists $B \in \mathbb{R}$ such that for all $s \in S$, $|s| \leq B$.

Real sets: Bounded sets: Examples

Notation

$B_r(c) = \{x \in \mathbb{R} : |x - c| \leq r\}$ is the closed ball with center c and radius r.

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Lemma

A set K is bounded if and only if it is contained in some ball.

Real sets: Upper bounded sets: Maximum

Reminder

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What happens for infinite sets?

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$\{x \in \mathbb{Q} : x^2 \leq 2\} = \{x \in \mathbb{Q} : x^2 < 2\}$: Does it have a maximum? No.

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More precisely, there exists $u \in \mathbb{R}$ such that: (1) u is an upper bound of S, (2) for every upper bound U, $u \leq U$.

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For a non-empty set $S \subset \mathbb{R}$,

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The existence of the supremum is a fundamental property of $\mathbb{R}$. It does not hold in $\mathbb{Q}$.

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The existence of the infimum is an easy consequence of the existence of supremum: $\inf S = -\sup(-S)$.

Real sets: Nested intervals

Let $I_i = [a_i, b_i]$ be a nested sequence of closed intervals:

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Proof idea:

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$$\Rightarrow \sup\{a_i\} \in I_i \text{ for all } i.$$

Real sets: Infimum/Supremum: Examples

Example

$$\begin{aligned}\sup\{x \in \mathbb{R} : x^2 \leq 2\} &= \sup\{x \in \mathbb{R} : x^2 < 2\} \\ &= \sup\{x \in \mathbb{Q} : x^2 < 2\} = \sqrt{2}.\end{aligned}$$

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$$\inf\left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} = \inf\left\{\frac{1}{n} : n \in \mathbb{N}_+\right\} = 0.$$

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Theorem

$$\sup\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} : n \in \mathbb{N}_+\right\} = 2.$$

Real sets: Infimum/Supremum: Examples

Theorem

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Definition

Their common value is denoted by e.

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A point b is an interior point of S if there exists $\varepsilon > 0$ such that $B^o_\varepsilon(b) \subset S$.

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Example

$[a, b]$ is NOT open, since a is not an interior point.

Real sets: Accumulation points, closed sets

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A point ℓ is an accumulation point of S if for every $\varepsilon > 0$, $B_\varepsilon(\ell) \cap S$ is infinite.

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A set S is closed if it contains all its accumulation points.

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$]a, b[$ is NOT closed, since a is an accumulation point but not in the set.

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Theorem

Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

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$]0, 1]$: neither open nor closed.

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$]0,1]$: neither open nor closed.

Example

$\mathbb{R}$: both open and closed.

Real sets: Compact sets

Theorem

For $S \subset \mathbb{R}$, the following are equivalent:

(i) Every open cover has a finite subcover,

(ii) S is bounded and closed,

(iii) Every infinite subset of S has an accumulation point in S,

(iv) Every sequence in S has a convergent subsequence with limit in S.

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Definition

A set $S \subset \mathbb{R}$ is called compact if it satisfies the above properties.

Infinite sums: The question

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We may say that the above common value is the sum of the numbers $\langle a_0, a_o, a_O\rangle$ (order does not matter).

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Definition

Let I be a finite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). Then

$$\sum_{i:i\in I} a_i$$

is a well-defined sum.

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What happens if I is infinite?

We assume that infinite means countably infinite.

- Let I be a countably infinite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). Assume that

$$a_i \geq 0, \text{ for all } i \in I.$$

Infinite sums: Positive numbers

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Definition

If $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } I\right\}$ is bounded above, then the sum of the numbers a_i $(i \in I)$ is defined as

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } I\right\}.$$

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• It is easy to see that the definition still works if $\{j \in I : a_j < 0\}$ is a finite set. Similarly, sums of negative numbers can be defined (using infimum instead of supremum, under appropriate boundedness conditions).

Infinite sums: General case

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- Let I be a countably infinite index set. $a_i : i \in I$ is a system of numbers (repetitions are allowed). To treat the general case, we may assume that $N = \{j \in I : a_j < 0\}$ and $P = \{j \in I : a_j > 0\}$ are both infinite sets.

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Definition

If $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } N\right\}$ is bounded below, and $\left\{\sum_{i\in R} a_i : \ R \text{ is a finite subset of } P\right\}$ is bounded above, then the sum of the numbers a_i $(i \in I)$ is defined by

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in R} a_i : \ R \subset P \text{ finite}\right\} + \inf\left\{\sum_{i\in R} a_i : \ R \subset N \text{ finite}\right\}$$

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- Note that $\sum_{i\in I} a_i$ is defined exactly when $\sum_{i\in I} |a_i|$ is defined.

Infinite sums: Examples

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$$\sum\{\frac{1}{i^2} : i \in \mathbb{N}_+\} \text{ exists.}$$

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If $\sum_{i:i\in I} b_i$ exists, then $\sum_{i:i\in I} a_i$ also exists,

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Theorem

Assume that $0 \leq a_i, b_i$ holds for all $i \in I$.
If $\sum_{i:i\in I}(a_i + b_i)$ exists, then

$$\sum_{i:i\in I}(a_i + b_i) = \sum_{i:i\in I} a_i + \sum_{i:i\in I} b_i.$$

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The usual order properties of real numbers, such as transitivity, remain valid.

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$$B^o_{1/N}(\infty) = \{r \in \mathbb{R} : r > \frac{1}{1/N} = N\} \cup \{\infty\}.$$

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If $o \in \widehat{\mathbb{R}}$ and $0 < r' < r$, then

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- If we say that x "comes from a sufficiently small neighborhood (ball) of ∞", then we mean that x is "greater than some sufficiently large N".

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Example

$$\sup\{1, 2, 3, \dots\} = \infty,$$

$$\sup\{-\infty\} = -\infty,$$

$$\sup\{\infty, -1, -2, -3, \dots\} = \infty.$$

$$\sup\{-1, -2, -3, \dots\} = -1.$$

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$[a, \infty[$ is a closed subset of $\mathbb{R}$.

$[a, \infty[$ is NOT closed in $\widehat{\mathbb{R}}$: ∞ is an accumulation point, but $\infty \notin [a, \infty[$.

Extended $\mathbb{R}$: Arithmetic

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Definition: Addition in $\widehat{\mathbb{R}}$

Let $r, r' \in \mathbb{R}$. Then

- the sum of r and r' is $r + r'$.
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- Note that

$$\frac{1}{3}+\frac{1}{4},\quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8},$$
$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$$

are all greater than $\frac{1}{2}$.

Break

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- Note that for sequences $L = \{\infty\}$.

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Let $f : \mathrm{dom}\, f \to \mathbb{R}$ and $h \in L_f \subset \widehat{\mathbb{R}}$. We write

$$\lim_{x \to h} f(x) = \ell \in \widehat{\mathbb{R}},$$

if for every ball B centered at ℓ there exists a ball B_0 centered at h such that for all $x \in (B_0 - \{h\}) \cap \mathrm{dom}\, f$ we have $f(x) \in B$.

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• Usually $h \notin \text{dom } f$. Even if $h \in \text{dom } f$, x only "approaches h". The value at h itself does not matter in the definition of the limit.

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- If the Prover has a winning strategy, the equality is true. If the Refuter can win, the equality is false.

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$\lim_{n\to\infty} n^k = \infty \quad (k > 0).$

- Let B be any ball centered at ∞. Let $\frac{1}{N}$ be the radius. Then $f(x) \in B$ means $N < f(x)$.

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$\lim_{n\to\infty} \frac{1}{n} = 0, \qquad \lim_{x\to\infty} \frac{1}{x} = 0.$

• Let B be any ball centered at 0. Let ε be its radius: an arbitrarily small positive number. Then $f(x) \in B$ means $-\varepsilon < f(x) < \varepsilon$.
• Let $B_0 =]\frac{1}{\varepsilon}, \infty]$. "The Prover wins".

Example

$\lim_{n\to\infty} n^k = \infty \quad (k > 0).$

• Let B be any ball centered at ∞. Let $\frac{1}{N}$ be the radius. Then $f(x) \in B$ means $N < f(x)$.
• Let $B_0 =]\sqrt[k]{N}, \infty]$.

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Limits: Examples

Example

$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$

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Limits: Examples

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$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$

- $\frac{x^2-4}{x-2} = x + 2.$
- Let B be any ball centered at 4 with radius ε.
- Let $B_0 =]2 - \varepsilon, 2 + \varepsilon[$.

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$\lim_{x\to 2}\frac{x^2-4}{x-2}=4.$

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$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2}.$

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$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$.

- $$\frac{\sqrt{1+x}-1}{x} = \frac{1}{\sqrt{1+x}+1}.$$

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$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}.$

- $$\frac{\sqrt{1+x}-1}{x} = \frac{1}{\sqrt{1+x}+1}.$$
- Substitute $x = 0.1,\ 0.01,\ 0.001,\ 0.0001,\ \ldots$

Limits: Examples

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- Assume $a > 1$.
- Let B be a ball centered at 1 with radius ε.
- Let $B_0 =]\frac{\log a}{\log(1+\varepsilon)}, \infty]$.

Limits: Examples

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$\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad (a > 0)$.

- Assume $a > 1$.
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Example

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

- Let $B_0 =]\max\{10, \frac{1}{\log^2(1+\varepsilon)}\}, \infty]$.

Limits: Examples

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$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.

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- Let $B =]\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon[$.
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Example

$\lim_{x\to -\infty} e^x = 0$.

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Example

$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.

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Example

$\lim_{x\to-\infty} e^x = 0$.

- Let $B =] - \varepsilon, \varepsilon[$
- Let $B_0 =] - \infty, \ln \varepsilon[$.

Limits: Examples

Example

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Limits: Examples

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$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$

Reminder

$$\begin{aligned} e &= \sup\left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}_+\right\} \\ &= \sup\left\{1+\frac{1}{1}+\frac{1}{2!}+\ldots+\frac{1}{n!} : n\in\mathbb{N}_+\right\} \\ &= \sum\left\{\frac{1}{n!} : n\in\mathbb{N}\right\} \end{aligned}$$

Limits: Examples

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Reminder

$$e = \sup\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}_+\right\}$$
$$= \sup\left\{1+\frac{1}{1}+\frac{1}{2!}+\ldots+\frac{1}{n!} : n \in \mathbb{N}_+\right\}$$
$$= \sum\left\{\frac{1}{n!} : n \in \mathbb{N}\right\}$$

- $\left(1+\frac{1}{n}\right)^n$ is a monotone increasing sequence:

$$A\left(1, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}\right) = \frac{1+n(1+\frac{1}{n})}{n+1} > G\left(1, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}\right).$$

Break

Limits: Properties

If a function has a limit at a given point, then it is unique. More precisely:

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Theorem

Suppose that $\lim_{x\to h} f(x)$ exists, and moreover $\lim_{x\to h} f(x) = \ell$ and $\lim_{x\to h} f(x) = k$ both hold. Then necessarily $\ell = k$.

Limits: One-sided limits

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Let $a \in \mathbb{R}$. Suppose that a is an accumulation point of the sets $\mathrm{dom}\ f \cap]a, \infty[$ and $\mathrm{dom}\ f \cap]-\infty, a[$.

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Definition

$$\lim_{x \to a+} f(x) = \lim_{x \to a} f^{a\to}(x), \text{ where } f^{a\to}(x) = f|\text{dom } f \cap]a, \infty[.$$

$$\lim_{x \to a-} f(x) = \lim_{x \to a} f^{\to a}(x), \text{ where } f^{\to a}(x) = f|\text{dom } f \cap]-\infty, a[.$$

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Theorem

Suppose that $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ both exist.
Then $\lim_{x \to a} f(x)$ exists if and only if
$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$.

Limits: One-sided limits: Examples

Example

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

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$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

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$$\lim_{x\to 0-} \frac{1}{x} = -\infty.$$

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Example

$$\lim_{x\to \frac{\pi}{2}-} \tan x = \infty.$$

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$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

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$$\lim_{x\to \frac{\pi}{2}-} \tan x = \infty.$$

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Limits: One-sided limits: Examples

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$\lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty.$

- Substitute $x = 0.9$ into the function.

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Limits: One-sided limits: Examples

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• Substitute $x = 0.9$ into the function.
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$\lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty.$

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$\lim_{x\to 1^+} \frac{1}{\sqrt{1-x^2}} = -\infty.$

Limits: One-sided limits: Examples

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$\lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty$.

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Example

$\lim_{x\to 1^+} \frac{1}{\sqrt{1-x^2}} = -\infty$.

- Substitute $x = 1.1$ into the function.

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- Substitute $x = 0.9$ into the function.

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Example

$\lim_{x\to 1^+} \frac{1}{\sqrt{1-x^2}} = -\infty.$

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Substitute $x = 1.01$ into the function.

Example

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$\lim_{x \to 1^+} \frac{1}{\sqrt{1-x^2}} = -\infty.$

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Limits: Properties

Limits: Properties

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Suppose that $\lim_{x\to h} f(x)$ and $\lim_{x\to h} g(x)$ exist, and moreover $\lim_{x\to h} f(x) = \ell$ and $\lim_{x\to h} g(x) = k$. Then

(i) Assume that $\ell + k$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h}(f(x)+g(x)) = \lim_{x\to h} f(x) + \lim_{x\to h} g(x) = \ell + k.$$

(ii) Assume that $\ell \cdot k$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h}(f(x)g(x)) = \lim_{x\to h} f(x) \lim_{x\to h} g(x) = \ell k.$$

(iii) Assume that $\frac{\ell}{k}$ is defined in $\widehat{\mathbb{R}}$. Then

$$\lim_{x\to h} \frac{f(x)}{g(x)} = \frac{\lim_{x\to h} f(x)}{\lim_{x\to h} g(x)} = \frac{\ell}{k}.$$

Limits: Properties: Examples

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$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

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Limits: Properties: Examples

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$\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$.

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$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$.

- $$\begin{aligned}\frac{x^n - a^n}{x-a} &= \frac{(x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x-a} \\ &= x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}.\end{aligned}$$

Example

$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$.

- $$\frac{x^n - a^n}{x-a} = \frac{(x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x-a}$$
  $$= x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}.$$
- $\lim_{x\to a} x^k a^{n-1-k} = a^{n-1}$.

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• We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} -3x = -\infty$ and $\lim_{x\to\infty} 5 = 5$.

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Example

$$\lim_{x\to\infty} x^2 - 3x + 5 = \infty.$$

• We know that $x^2 - 3x + 5 = x^2(1 - \frac{3}{x} + \frac{5}{x^2})$.

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• We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} -3x = -\infty$ and $\lim_{x\to\infty} 5 = 5$.
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• We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

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$$\lim_{x\to\infty} x^2 - 3x + 5 = \infty.$$

• We know that $x^2 - 3x + 5 = x^2(1 - \frac{3}{x} + \frac{5}{x^2})$.
• We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x^2} = 0$.
• Thus $\infty(1 + (-3)0 + 5 \cdot 0) = \infty$ using $\widehat{\mathbb{R}}$ arithmetic.

Limits: Properties: Examples

Example

$$\lim_{x\to\infty} \frac{x^2 - 3x + 5}{x^3 - 3x^2 + 2} = 0.$$

Example

$$\lim_{x\to\infty} \frac{x^2-3x+5}{x^3-3x^2+2} = 0.$$

- We rewrite:

$$\frac{x^2-3x+5}{x^3-3x^2+2} = \frac{1}{x} \cdot \frac{1-3\frac{1}{x}+5\frac{1}{x^2}}{1-3\frac{1}{x}+2\frac{1}{x^3}}.$$

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- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, $\lim_{x\to\infty} \frac{1}{x^3} = 0$.

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- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, $\lim_{x\to\infty} \frac{1}{x^3} = 0$.
- Thus the limit equals 0.

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- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, $\lim_{x\to\infty} \frac{1}{x^3} = 0$.
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Example

$$\lim_{x\to\infty} \frac{x^5-3x+5}{x^3-3x^2+2} =?, \qquad \lim_{x\to\infty} \frac{x^3-3x+5}{x^3-3x^2+2} =?.$$

Limits: Properties: Examples

Example

$$\lim_{x\to\infty} \frac{x^2-3x+5}{x^3-3x^2+2} = 0.$$

- We rewrite:

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- We know that $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} 1 = 1$, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$, $\lim_{x\to\infty} \frac{1}{x^3} = 0$.
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Example

$$\lim_{x\to\infty} \frac{x^5-3x+5}{x^3-3x^2+2} =?, \qquad \lim_{x\to\infty} \frac{x^3-3x+5}{x^3-3x^2+2} =?.$$

Try to solve these on your own.

Limits: Properties

Theorem: Squeeze Theorem

If the function $f(x)$ is squeezed between two other functions, $g(x)$ and $h(x)$, i.e. $g(x) \leq f(x) \leq h(x)$, and both have the same limit (L) at the point a, then the limit of $f(x)$ is also L.

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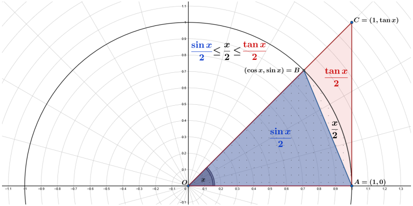
Theorem

Let a_n be a bounded above, monotonically increasing sequence/function. Then $\lim_{x \to \infty} f(x)$ exists.

Limits: Examples

Example

$\lim_{x\to 0} \frac{\sin x}{x} = 1.$



Source:
https://www.math-linux.com/mathematics/limits/article/proof-of-limit-of-sin-x-x-1-as-x-approaches-0

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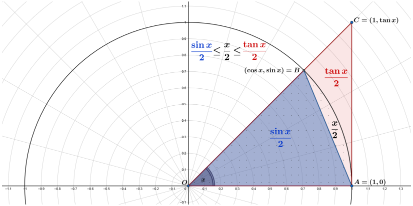

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- After rearranging we obtain that for $0 < x < \frac{\pi}{2}$

$$\cos x < \frac{\sin x}{x} < 1.$$

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Solve inequalities.

Limits: Properties: Examples

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$\lim_{x\to\infty} \frac{x^2+1}{2x+8} = \infty.$

$$\frac{1}{3}\cdot x = \frac{x^2}{3x} < \frac{x^2+1}{2x+8} < \frac{2x^2}{x} = 2x,$$

whenever $x \in [8,\infty[$.

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- $$n! = n(n-1)(n-2)\dots\frac{n}{2}(\frac{n}{2}-1)\dots\cdot 2\cdot 1 > \left(\frac{n}{2}\right)^{\frac{n}{2}} = \sqrt{\frac{n}{2}}^{\,n}.$$

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- $\frac{n^k}{a^n} = \left(\frac{\sqrt{n}}{(\sqrt[2k]{a})^n}\right)^{2k}$. Let $\sqrt[2k]{a} = 1 + h$. Bernoulli inequality: $(1+h)^n > 1 + hn$, if $h > -1$.

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- Let $\alpha = 1 + \epsilon$

$$\log_\alpha n < n^{k/2} \Leftrightarrow n < \alpha^{n^{k/2}} \Leftrightarrow$$

$$n < (1+\epsilon)^{n^{k/2}} = 1 + n^{k/2}\epsilon + \binom{n^{k/2}}{2}\epsilon^2 + \ldots + \binom{n^{k/2}}{4/k}\epsilon^{4/k} + \ldots$$

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- In summary:

$$\log_\alpha n \lll n^k \lll a^n \lll n! \lll n^n,$$

where for positive sequences

$$f(n) \lll g(n) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty.$$

Thank you for your attention!