BSc Mathematics for Computer Scientists 2: III. Matrices and Matrix arithmetic

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Tables of Numbers

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Clarification

Two tables are equal if and only if, first, their shapes coincide. That is, they have the same number of rows (with rows correspondingly matched) and the same number of columns (with columns correspondingly matched). Thus, the positions in the two matrices correspond. Second, the number in every corresponding position are the same in the two tables/matrices.

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• In summary: Specifying a matrix means giving its size and the element in each of its positions.

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Definition: Identity Matrix

The identity matrix $1 = I = 1_{\ell,\ell} = I_{\ell,\ell} \in \mathbb{R}^{\ell\times\ell}$ is the matrix whose (i,j) position contains 0 if $i \neq j$ and 1 if $i = j$.

Special Matrices (continued)

For a square matrix $M \in \mathbb{R}^{\ell\times\ell}$, the elements $M_{i,i}$ are called diagonal elements. The position of $M_{i,j}$ is said to be below the main diagonal if $i > j$. The position of $M_{i,j}$ is said to be above the main diagonal if $i < j$.

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A matrix $M \in \mathbb{R}^{\ell \times \ell}$ is lower triangular if all elements above the main diagonal are 0.

Definition: Upper Triangular Matrices

A matrix $M \in \mathbb{R}^{\ell \times \ell}$ is upper triangular if all elements below the main diagonal are 0.

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Definition

M is symmetric if $M^{\mathsf{T}} = M$. That is, M is square (why?) and $M_{i,j} = M_{j,i}$.

Definition

If we list numbers in an ordered sequence, the resulting mathematical object is called a vector. The elements of the sequence are called components/coordinates. If the length of the sequence v is n (the numbers listed by v are $v_1, v_2, \ldots, v_n$, where $v_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$), then we say the vector is n-dimensional. In the case of real numbers, the set of n-dimensional vectors is denoted by $\mathbb{R}^n$.

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Example: 4-Dimensional Zero Vectors

$$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrices: Vectors (continued)

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If we write the components in a column, then the result can be regarded as an $n \times 1$ matrix. In this case we speak of a column vector: $\mathbb{R}^n \simeq \mathbb{R}^{n\times 1}$.

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If we write the components in a column, then the result can be regarded as an $n \times 1$ matrix. In this case we speak of a column vector: $\mathbb{R}^n \simeq \mathbb{R}^{n\times 1}$.

Observation

If v is a row vector, then v^{T} is a column vector. If u is a column vector, then u^{T} is a row vector.

- A matrix $M \in \mathbb{R}^{n\times m}$ can be viewed as a collection of m column vectors:

$$M = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_m \\ | & | & & | \end{pmatrix},$$

where $c_i \in \mathbb{R}^n$.

Matrices: Vectors: Relationship

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- A matrix $M \in \mathbb{R}^{n\times m}$ can also be viewed as a collection of n row vectors:

$$M = \begin{pmatrix} \text{---} & r_1^{\mathsf{T}} & \text{---} \\ \text{---} & r_2^{\mathsf{T}} & \text{---} \\ & \vdots & \\ \text{---} & r_n^{\mathsf{T}} & \text{---} \end{pmatrix}$$

where $r_i \in \mathbb{R}^m$.

Break

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Definition

Let A and B be two matrices (that is, for suitable k, ℓ, m, n we have $A \in \mathbb{R}^{k\times\ell}$, $B \in \mathbb{R}^{m\times n}$). A and B can be added if and only if their shapes coincide ($k = m$ and $\ell = n$). In this case, the matrix $A + B$ has the same shape, and for every (i, j) position ($1 \leq i \leq k$, $1 \leq j \leq \ell$)

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Example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} + \begin{pmatrix} -1 & -4 & -7 & -11 \\ -2 & -5 & -8 & -12 \\ -3 & -6 & -9 & -13 \end{pmatrix} =$$

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Let A be a matrix (that is, for suitable k, ℓ we have $A \in \mathbb{R}^{k\times\ell}$. Then the matrix A can be multiplied by a scalar $\lambda \in \mathbb{R}$. The matrix λA has the same shape, and for every (i, j) position $(1 \leq i \leq k,\ 1 \leq j \leq \ell)$

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Example

$$2 \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 10 & 12 & 14 & 16 \\ 18 & 20 & 22 & 24 \end{pmatrix}$$

Systems of Vectors: Linear Combinations

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Definition

Given $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$, take $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. The expressions of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

are called linear combinations of the vectors $v_1, v_2, \ldots, v_n$. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ are called the coefficients of the linear combination.

Systems of Vectors: Linear Combinations (continued)

- Choosing a linear combination of the vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ means choosing the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. The choice $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ is always possible:

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Definition

The representation

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \cdot v_1 + 0 \cdot v_2 + \ldots + 0 \cdot v_n$$

is called the trivial representation of the zero vector (as a linear combination).

Definition

The subspace generated by the vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ is

$$[v_1, v_2, \ldots, v_n]_{\text{lin}} = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\} \subset \mathbb{R}^d.$$

Systems of Vectors: Generated Subspace

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Observation

A linear combination of linear combinations is again a linear combination.

Systems of Vectors: Linear Independence, Spanning, Basis in $\mathbb{R}^d$

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The vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ form a basis of $\mathbb{R}^d$ if they span the space and are linearly independent.

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The vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ span $\mathbb{R}^d$ if every vector in $\mathbb{R}^d$ can be written (in at least one way) as a linear combination of $v_1, v_2, \ldots, v_n$.

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Proposition

The vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ are linearly independent if and only if every vector in $\mathbb{R}^d$ that can be written as a linear combination of $v_1, v_2, \ldots, v_n$ has a unique representation.

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Definition

The vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ form a basis of $\mathbb{R}^d$ if every vector in $\mathbb{R}^d$ can be written in exactly one way as a linear combination of $v_1, v_2, \ldots, v_n$.

Systems of Vectors: Spanning and Basis in a Subspace

Definition

The vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^d$ span the subspace A if

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Definition

The vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ form a basis of the subspace A if they span the subspace and are linearly independent.

Systems of Vectors: Dimension

Theorem

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Definition

The common cardinality in the above theorem is called the dimension of the subspace A and is denoted by $\dim A$.

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$$e_i = (\overset{1}{\frown}\!\!\!\!\!\!\;0, \ldots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \ldots, \overset{d}{0}).$$

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Proposition

$H = \{x \in \mathbb{R}^d :$ the sum of its coordinates is $0\}$ is a subspace and $\dim H = d - 1$.

Definition

A set V is called a (real) vector space if addition and (real) scalar multiplication are defined on it and satisfy the following rules:

(1) For all $u, v, w \in V$: $(u+v)+w = u+(v+w)$ and $u+v = v+u$,

(2) There exists $0_V \in V$ such that for all $v \in V$, $v + 0_V = 0_V + v = v$, and for all $v \in V$ there exists v^- with $v + v^- = v^- + v = 0_V$.

(3) For all $\alpha, \beta \in \mathbb{R}$ and $v \in V$: $(\alpha+\beta)v = \alpha v + \beta v$, $(\alpha\beta)v = \alpha(\beta v)$, $1v = v$.

(4) For all $\alpha \in \mathbb{R}$ and $u, v \in V$: $\alpha(u+v) = \alpha u + \alpha v$.

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(1) For all $u, v, w \in V$: $(u+v)+w = u+(v+w)$ and $u+v = v+u$,

(2) There exists $0_V \in V$ such that for all $v \in V$, $v + 0_V = 0_V + v = v$, and for all $v \in V$ there exists v^- with $v + v^- = v^- + v = 0_V$.

(3) For all $\alpha, \beta \in \mathbb{R}$ and $v \in V$: $(\alpha+\beta)v = \alpha v + \beta v$, $(\alpha\beta)v = \alpha(\beta v)$, $1v = v$.

(4) For all $\alpha \in \mathbb{R}$ and $u, v \in V$: $\alpha(u+v) = \alpha u + \alpha v$.

Proposition

For all $v \in V$ and $s \in \mathbb{R}$ we have $0 \cdot v = 0_V$, $s \cdot 0_V = 0_V$, $(-1) \cdot v = v^-$ and $s \cdot v = 0_V$ implies that $s = 0$ or $v = 0_V$.

Abstract Vector Spaces: Examples

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Main example

$V = \mathbb{R}^d$ with the usual addition and scalar multiplication.

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- Periodic Functions with Constant Period:
$$\{f : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}\ f(x) = f(x + 2\pi)\}.$$

Abstract Vector Spaces: $\mathbb{R}^d$ as a Universal Example

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Theorem

For every vector $v \in V$ there exists exactly one coordinate vector $v_{\mathcal{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$ such that $v = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d$.

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This correspondence can be reversed: we can establish a bijection between coordinate vectors and vectors. The pairing is compatible with the operations ($\rightarrow$ isomorphism). The two structures are identical up to naming.

Vectors: Scalar Product

Reminder

Historically: $d = 2, 3$ geometric structures: plane and space. The standard basis (e_1, e_2, e_3) is a natural choice of pairwise orthogonal unit vectors.

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For $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 + \ldots + u_d v_d.$$

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Theorem: Cauchy–Schwarz Inequality

For $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$

$$\langle u, v \rangle \leq |u| \cdot |v|, \quad \text{that is} \quad \frac{\langle u, v \rangle}{|u| \cdot |v|} \in [-1, 1].$$

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• In the plane and in space these notions coincide with the usual geometric concepts. For $d = 4, 5, 6, \ldots$ these notions make it possible to introduce geometry and become familiar with it.

Hyperplanes

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Definition: Hyperplane in $\mathbb{R}^d$

Let $\nu \in \mathbb{R}^d - \{0\}$ be a nonzero vector and $v_0 \in \mathbb{R}^d$ a point. Then

$$\mathcal{H}_{\nu, v_0} = \{x \in \mathbb{R}^d : \langle \nu, x - v_0 \rangle = 0\}$$

is a hyperplane in $\mathbb{R}^d$.

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- For the hyperplane $\mathcal{H}_{\nu,v_0}$ defined above, the vectors lying in it and the point v_0 are orthogonal to the vector ν. $\mathcal{H}_{\nu,v_0}$ is the hyperplane through v_0 perpendicular to ν. ν is a normal vector of the hyperplane.

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- Let $\nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{R}^d - \{0\}$ be a nonzero sequence of coefficients. The solution set of

$$\nu_1 x_1 + \nu_2 x_2 + \ldots + \nu_d x_d = b$$

is geometrically a hyperplane in $\mathbb{R}^d$.

Break

Matrices: Multiplication

Definition

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(i) In this case the product matrix has size $n \times m$, that is, $AB \in \mathbb{R}^{n \times m}$.

(ii) Moreover, the (i, j)-entry is

$$(AB)_{i,j} = \sum_{s=1}^{\ell} A_{i,s} B_{s,j}.$$

Matrices: Multiplication: Example

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} =$$

$$\begin{pmatrix} 1\cdot 5+2\cdot 7 & 1\cdot 6+2\cdot 8 \\ 3\cdot 5+4\cdot 7 & 3\cdot 6+4\cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

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Matrices: Multiplication: Scalar Product of Vectors

Notice that the scalar product of vectors can also be interpreted as matrix multiplication.
If $u, v \in \mathbb{R}^d$ are defined as column vectors, then

$$\langle u, v \rangle = u^{\mathsf{T}} v.$$

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In algebraic arguments involving matrices we prefer this notation of the scalar product. The notation $\langle u, v\rangle$ emphasizes the geometric viewpoint.

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$$A = \begin{pmatrix} \text{---} & r_1^\mathsf{T} & \text{---} \\ \text{---} & r_2^\mathsf{T} & \text{---} \\ & \vdots & \\ \text{---} & r_n^\mathsf{T} & \text{---} \end{pmatrix} \qquad B = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_m \\ | & | & & | \end{pmatrix}$$

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- Then $(AB)_{i,j} = r_i^{\mathsf{T}} c_j$, that is, the (i,j)-entry of the product matrix is obtained by taking the scalar product of the i-th row of A and the j-th column of B.

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• The product matrix is a kind of "multiplication table" for the two vector systems obtained from the factors with respect to the scalar product.

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$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_\ell \\ \beta_1 & \beta_2 & \ldots & \beta_\ell \\ \vdots & \vdots & \ldots & \vdots \\ \omega_1 & \omega_2 & \ldots & \omega_\ell \end{pmatrix} \qquad B = \begin{pmatrix} \text{---} & s_1^T & \text{---} \\ \text{---} & s_2^T & \text{---} \\ & \vdots & \\ \text{---} & s_\ell^T & \text{---} \end{pmatrix}$$

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- Multiplicability $\equiv$ each row of the first factor contains as many entries as the second factor has rows.
- Start computing according to the definition:

$$(AB)_{1,1} = \alpha_1(s_1^\mathsf{T})_1 + \alpha_2(s_2^\mathsf{T})_1 + \ldots + \alpha_\ell(s_\ell^\mathsf{T})_1$$
$$(AB)_{1,2} = \alpha_1(s_1^\mathsf{T})_2 + \alpha_2(s_2^\mathsf{T})_2 + \ldots + \alpha_\ell(s_\ell^\mathsf{T})_2$$
$$\vdots$$
$$(AB)_{1,m} = \alpha_1(s_1^\mathsf{T})_m + \alpha_2(s_2^\mathsf{T})_m + \ldots + \alpha_\ell(s_\ell^\mathsf{T})_m$$

Matrix Multiplication: Viewpoint II (continued)

- Thus the first row of the product matrix is

$$\alpha_1 s_1^{\mathsf{T}} + \ldots + \alpha_\ell s_\ell^{\mathsf{T}}.$$

Matrix Multiplication: Viewpoint II (continued)

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- In summary

$$AB = \begin{pmatrix} — & \alpha_1 \mathsf{s}_1^{\mathsf{T}} + \ldots + \alpha_\ell \mathsf{s}_\ell^{\mathsf{T}} & — \\ — & \beta_1 \mathsf{s}_1^{\mathsf{T}} + \ldots + \beta_\ell \mathsf{s}_\ell^{\mathsf{T}} & — \\ & \vdots & \\ — & \omega_1 \mathsf{s}_1^{\mathsf{T}} + \ldots + \omega_\ell \mathsf{s}_\ell^{\mathsf{T}} & — \end{pmatrix}$$

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- In particular, each row of the product matrix is a linear combination of the vectors $s_1^{\mathsf{T}}, \ldots, s_\ell^{\mathsf{T}}$ (the rows of the second factor).

Matrix Multiplication: Viewpoint III

- Let $A \in \mathbb{R}^{n\times \ell}$ and $B \in \mathbb{R}^{\ell \times m}$ and decompose both into column vectors.

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$$A = \begin{pmatrix} | & | & & | \\ o_1 & o_2 & \ldots & o_\ell \\ | & | & & | \end{pmatrix} \qquad B = \begin{pmatrix} \alpha_1 & \beta_1 & \ldots & \omega_1 \\ \alpha_2 & \beta_2 & \ldots & \omega_2 \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_\ell & \beta_\ell & \ldots & \omega_\ell \end{pmatrix}$$

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$$A = \begin{pmatrix} | & | & & | \\ o_1 & o_2 & \ldots & o_\ell \\ | & | & & | \end{pmatrix} \qquad B = \begin{pmatrix} \alpha_1 & \beta_1 & \ldots & \omega_1 \\ \alpha_2 & \beta_2 & \ldots & \omega_2 \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_\ell & \beta_\ell & \ldots & \omega_\ell \end{pmatrix}$$

- Multiplicability $\equiv$ each column of the second factor contains as many entries as the first factor has columns.

Matrix Multiplication: Viewpoint III (continued)

Similarly, the product matrix is

$$AB = \left(\begin{array}{ccc} \Big| & \Big| & \\ \alpha_1 o_1 + \alpha_2 o_2 + \ldots + \alpha_\ell o_\ell & \beta_1 o_1 + \beta_2 o_2 + \ldots + \beta_\ell o_\ell & \\ \Big| & \Big| & \\ & & \Big| \\ & \ldots & \omega_1 o_1 + \omega_2 o_2 + \ldots + \omega_\ell o_\ell \\ & & \Big| \end{array} \right)$$

Matrix Multiplication: Viewpoint III (continued)

Similarly, the product matrix is

$$AB = \begin{pmatrix} \Big| & \Big| & & \Big| \\ \alpha_1 o_1 + \alpha_2 o_2 + \ldots + \alpha_\ell o_\ell & \beta_1 o_1 + \beta_2 o_2 + \ldots + \beta_\ell o_\ell & \ldots & \omega_1 o_1 + \omega_2 o_2 + \ldots + \omega_\ell o_\ell \\ \Big| & \Big| & & \Big| \end{pmatrix}$$

• In particular, each column of the product matrix is a linear combination of the column vectors $o_1, \ldots, o_\ell$ (the columns of the first factor).

Theorem

Assume the following matrix products are defined. Then

(a)

$$A(BC) = (AB)C,$$

(b)

$$A(B+C) = AB + AC, \qquad (A+B)C = AC + BC,$$

(c)

$$(\lambda A)B = \lambda(AB) = A(\lambda B),$$

(d) $A, B \in \mathbb{R}^{n\times n}$

$$(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}},$$

(d)* $A, B \in \mathbb{R}^{k\times n}$

$$\left(AB^{\mathsf{T}}\right)^{T} = BA^{\mathsf{T}}.$$

Matrices: Algebraic Rules: Non-Commutativity

Observation

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Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 24 & 10 \end{pmatrix} \neq \begin{pmatrix} 10 & 16 \\ 6 & 10 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a square real matrix. Let $0, I \in \mathbb{R}^{n\times n}$. Then

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$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrices: Algebraic Rules: Inverse

Definition

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. If there exists a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that

$$A^{-1}A = AA^{-1} = I,$$

then we say that A is invertible. For such matrices we denote the inverse by A^{-1}.

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Theorem

If $A, B \in \mathbb{R}^{n\times n}$ are invertible matrices, then AB, A^{T} and A^{-1} are also invertible, moreover

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}, \quad (A^{-1})^{-1} = A.$$

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Matrices: Algebraic Rules: Inverse: Questions

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- The answer: Yes. If A is invertible ($\rightarrow\ A^{-1}$) and B is also an inverse, then
$$AB = I \quad \Rightarrow \quad A^{-1}AB = A^{-1}I \quad \Rightarrow \quad B = A^{-1}.$$

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Problem

Given a matrix, how can we decide whether it is invertible? If it is invertible, how can we compute its inverse?

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$$\mathbb{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \ A \text{ is invertible}\}$$

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Theorem

Multiplication and inverse are defined in $\mathbb{GL}(n, \mathbb{R})$. Moreover, $\mathbb{GL}(n, \mathbb{R})$ is closed under transposition.

This is the end!

Thank you for your attention!