BSc Mathematics for Computer Scientists 2: I. Foundations

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Spring semester 2026

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- THE MOST IMPORTANT example. If during the course the word "number" is used, it should be understood as "real number".

Numbers: Complex numbers

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- Every polynomial has a root in $\mathbb{C}$.
- If you know them and can calculate with them, that is very good. In this course, we do not use them.

- Addition, subtraction, multiplication, division (by a NON-0 number).

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• The precise description is technical. We accept that they can be carried out.

Numbers: Real numbers: Number line

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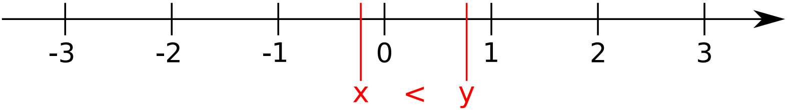

(Source: Wikipedia)

Numbers: Real numbers: Signs

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$-$	$-$	$-$	??
0	$-$	0	$+$
$+$	??	$+$	$+$

mult	$-$	0	$+$
$-$	$+$	0	$-$
0	0	0	0
$+$	$-$	0	$+$

div	$-$	0	$+$
$-$	$+$	nd	$-$
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Notation ($a, b \in \mathbb{R}$)

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Numbers: Real numbers: Absolute value

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$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

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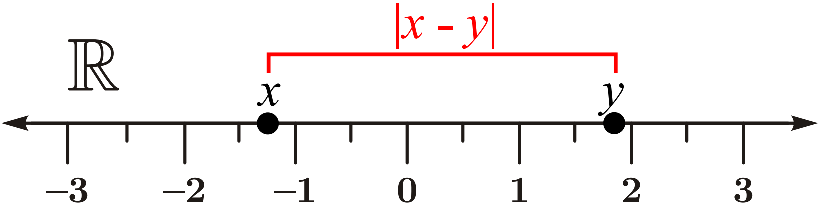

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Break

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Our goal is to introduce further important operations.

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Definition: Exponentiation, positive integer exponent k

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- Let k, ℓ be two arbitrary positive integers. Then

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Theorem

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$$\frac{\alpha^k}{\alpha^\ell} = \alpha^{k-\ell}.$$

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- $\alpha^{-3}? =? \frac{\alpha^2}{\alpha^5}$.

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$$\alpha^{\frac{1}{3}} \cdot \alpha^{\frac{1}{3}} \cdot \alpha^{\frac{1}{3}} \quad ?=? \quad \alpha^1.$$

Real numbers: exponentiation III

Let α be a POSITIVE number, i.e. $\alpha \in \mathbb{R}_{++}$. Furthermore let $k \in \mathbb{N}_+$.

Definition: Exponentiation, reciprocal exponent

Let $k \in \mathbb{N}_+$, then

$$\alpha^{\frac{1}{k}} = \sqrt[k]{\alpha}.$$

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Definition: Exponentiation, rational exponent

Let $r \in \mathbb{Q}$. Write r as $r = \frac{\ell}{k}$ $(k \in \mathbb{N}_+, \ell \in \mathbb{Z})$

$$\alpha^r = \sqrt[k]{\alpha^\ell}.$$

Real numbers: exponentiation IV

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That is, if α is a POSITIVE number ($\alpha \in \mathbb{R}_{++}$), $r = \frac{\ell}{k} = \frac{\ell'}{k'}$ ($k, k' \in \mathbb{N}_+$, $\ell, \ell' \in \mathbb{Z}$), then

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A technical remark:

Statement

Let $k \in \mathbb{N}_+$. Consider the following equation:

$$x^k = \alpha \qquad (\mathcal{E})$$

(1) IF k is odd, then $(\mathcal{E})$ has exactly one root; (2) IF k is even, then $(\mathcal{E})$ has exactly one non-negative root in the case $\alpha \in \mathbb{R}_+$, while it has no root if $\alpha < 0$. Moreover, the set of roots is closed under sign change.

Real numbers: exponentiation V

Theorem

Let α be a positive real number, i.e. $\alpha \in \mathbb{R}_{++}$. ASSUME $\alpha > 1$. Let $\beta \in \mathbb{R}$ be an arbitrary exponent.

(a) If
$$\ell_1 < \ell_2 < \ell_3 < \ldots < \beta < \ldots < u_3 < u_2 < u_1,$$
then
$$\alpha^{\ell_1} < \alpha^{\ell_2} < \alpha^{\ell_3} < \ldots < \alpha^{u_3} < \alpha^{u_2} < \alpha^{u_1}.$$

(b) If
$$\ell_1 < \ell_2 < \ell_3 < \ldots < \beta < \ldots < u_3 < u_2 < u_1,$$
$0 < u_i - \ell_i$ can be arbitrarily small, then
$$\alpha^{\ell_1} < \alpha^{\ell_2} < \alpha^{\ell_3} < \ldots < \alpha^{u_3} < \alpha^{u_2} < \alpha^{u_1},$$
and moreover $\alpha^{u_i} - \alpha^{\ell_i}$ can also be arbitrarily small.

Real numbers: exponentiation VI

Consequence (we ASSUME $\alpha > 1$)

If
$$\ell_1 < \ell_2 < \ell_3 < \ldots < \beta,$$
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Real numbers: exponentiation: the definition (we ASSUME $\alpha > 1$)

(a) If
$$\ell_1 < \ell_2 < \ell_3 < \ldots < \beta < \ldots < u_3 < u_2 < u_1,$$
$0 < u_i - \ell_i$ can be arbitrarily small, then α^β is the unique real number such that
$$\alpha^{\ell_1} < \alpha^{\ell_2} < \alpha^{\ell_3} < \ldots < \alpha^\beta < \ldots < \alpha^{u_3} < \alpha^{u_2} < \alpha^{u_1}.$$
(b) The value defined above does not depend on the choice of
$$\ell_1 < \ell_2 < \ell_3 < \ldots < \beta < \ldots < u_3 < u_2 < u_1.$$

Real numbers: exponentiation: identities

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Theorem: Identities of exponentiation

Let $\alpha, \beta \in \mathbb{R}_{++}$, $x, y \in \mathbb{R}$.

(a)
$$a^x \cdot a^y = a^{x+y},$$

(b)
$$\frac{a^x}{a^y} = a^{x-y},$$

(c)
$$(a^x)^y = a^{x \cdot y},$$

(d)
$$a^x \cdot b^x = (ab)^x,$$

(e)
$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x,$$

Let $\alpha \in \mathbb{R}_{++}$, $1 \neq a \in \mathbb{R}_{++}$.

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Theorem

Every positive number α can be written in the form a^x, where $0 < a \neq 1$ is a real number.

Real numbers: the logarithm

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$\log_a \alpha$ is the uniquely defined exponent x such that $\alpha = a^x$.

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That is, $\log_a \alpha$ "thinks of" the number α as a power of a and "encodes" the number by the exponent.

Real numbers: logarithm: identities

Theorem: Identities of the logarithm function

Let $\alpha, \beta \in \mathbb{R}_{++}$, $1 \neq a \in \mathbb{R}$.

(a)

$$\log_a(X \cdot Y) = \log_a X + \log_a Y \qquad // \quad a^x \cdot a^y = a^{x+y},$$

(b)

$$\log_a\left(\frac{X}{Y}\right) = \log_a X - \log_a Y \qquad // \quad a^x \cdot a^y = a^{x+y},$$

(c)

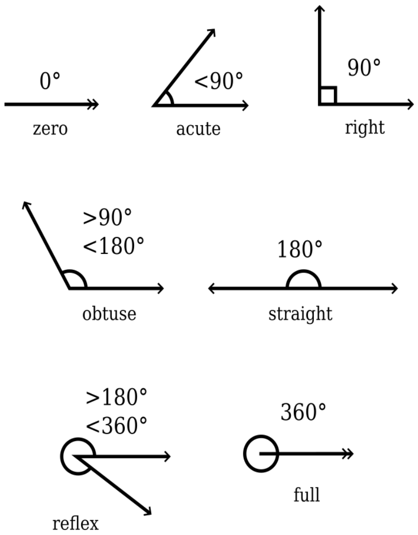
$$\log_a(X^y) = y \log_a X \qquad // \quad (a^x)^y = a^{x \cdot y},$$

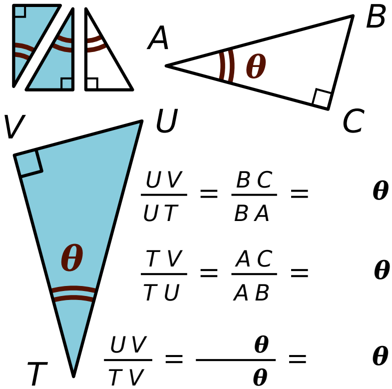
(d)

$$\log_B X = \frac{\log_a X}{\log_a B}, \quad \log_a B \log_B x = \log_a X \quad // \quad (a^b)^x = a^{bx}.$$

... and now for something completely different ...

Angles: Geometric angles and their types





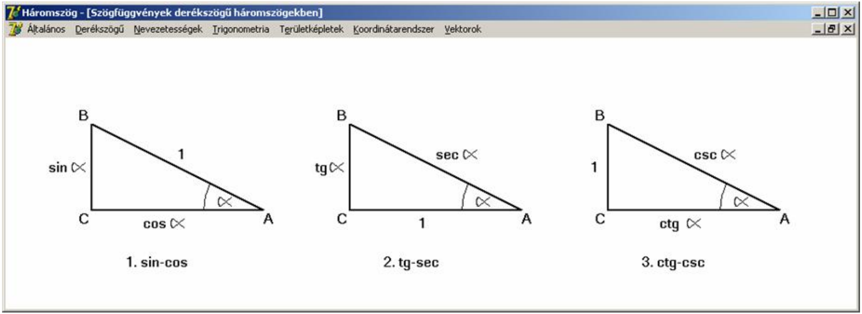
$$\frac{UV}{UT} = \frac{BC}{BA} = \qquad \theta$$

$$\frac{TV}{TU} = \frac{AC}{AB} = \qquad \theta$$

$$\frac{UV}{TV} = \frac{\theta}{\theta} = \qquad \theta$$

Source:
https://en.wikipedia.org/wiki/Trigonometricfunctions♯/media/File:Acade

Angles: Trigonometric functions of acute angles (continued)



Source: https://gorbem.hu/MT/Haromszog6elemei/image010.jpg

Angles: Special triangles, special trigonometric functions

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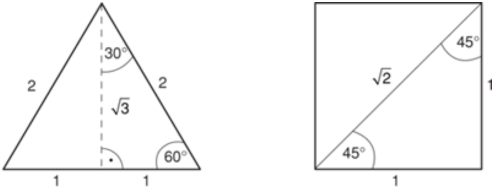

a) b)

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tg	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
ctg	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

Source:
https://mersz.hu/object/trigonometriaiharomszogekestablazat.png/1702

Angles: Real numbers

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Rotational angles and their measurement in radians:

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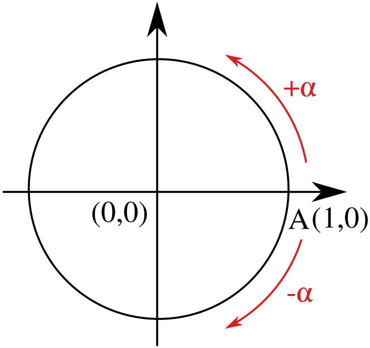
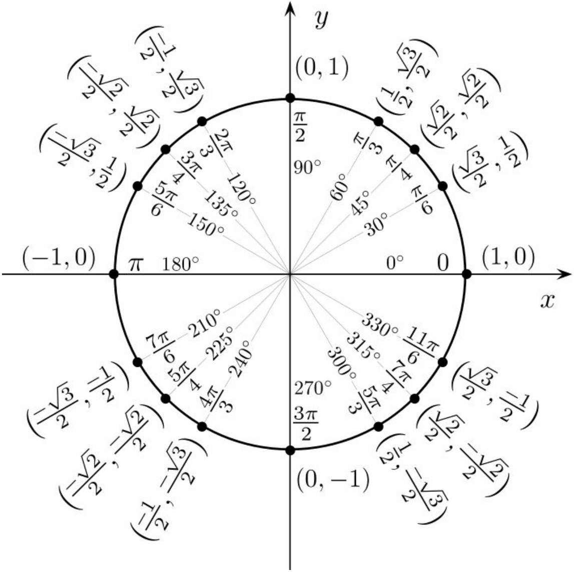


Angles: Special angles



Source: https://okosodjal.webnode.hu/news/nevezetes-szogek/

Angles: Special trigonometric values

Angles: Special trigonometric values

	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
ctg	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$-$

Source: https://gorbem.hu/MT/Haromszog6elemei/image026.gif

Trigonometric functions

(a)
$$\sin:\mathbb{R}\to\mathbb{R}, x\mapsto \sin x,$$

(b)
$$\cos:\mathbb{R}\to\mathbb{R}, x\mapsto \cos x,$$

(c)
$$\tan:\mathbb{R}\to\mathbb{R}, x\mapsto \tan x=\frac{\sin x}{\cos x},$$

(d)
$$\cot:\mathbb{R}\to\mathbb{R}, x\mapsto \cot x=\frac{\cos x}{\sin x},$$

(e)
$$\sec:\mathbb{R}\to\mathbb{R}, x\mapsto \sec x=\frac{1}{\cos x},$$

(f)
$$\csc:\mathbb{R}\to\mathbb{R}, x\mapsto \csc x=\frac{1}{\sin x}.$$

Properties of trigonometric functions

(a)
$$\sin^2 x + \cos^2 x = 1,$$

(b)
$$\sin 2x = 2 \sin x \cos x,$$

(c)
$$\cos 2x = 1 - 2\cos^2 x,$$

(d)
$$\sin(x + k \cdot 2\pi) = \sin x, \quad \cos(x + k \cdot 2\pi) = \cos x \quad (k \in \mathbb{Z}),$$

(e)
$$\tan(x + k \cdot \pi) = \tan x, \quad \cot(x + k \cdot 2\pi) = \cot x \quad (k \in \mathbb{Z}).$$

(f)
$$\sin(-x) = \sin x, \quad \sin(\frac{\pi}{2} - x) = \cos x, \quad \sin(\pi - x) = \sin x,$$
$$\cos(-x) = \cos x, \quad \cos(\frac{\pi}{2} - x) = \sin x, \quad \cos(\pi - x) = -\cos x.$$

Break

Functions

Functions

Definition

By a function $f : D \to C$ we mean an assignment/rule/formula that assigns to EVERY element $x \in D$ a UNIQUELY DETERMINED element of C, which is denoted by $f(x)$. The value $f(x)$ is called the image of x. Here $x \in D$ is an input, while $f(x) \in C$ is a value or image.

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Definition

A function $f : \mathrm{dom}(f) \to \text{co-dom}(f)$ is called a real function if $\mathrm{dom}(f), \text{co-dom}(f) \subset \mathbb{R}$.

Properties of functions

Properties of functions

Definition: surjective function

A function $f : \mathrm{dom}(f) \to \text{co-dom}(f)$ is called onto, or surjective, if $\text{co-dom}(f) = \mathrm{im}(f)$, that is, if for any possible value ($y \in \text{co-dom}(f)$) there exists an input ($x \in \mathrm{dom}(f)$) where the function takes the chosen value ($y = f(x)$).

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A function $f : \mathrm{dom}(f) \to \text{co-dom}(f)$ is called one-to-one, or injective, if for any attained value ($y \in \mathrm{im}(f)$) there exists exactly one input ($x \in \mathrm{dom}(f)$) where the function takes the chosen value ($y = f(x)$). Alternatively: f is injective if for any possible value ($y \in \text{co-dom}(f)$) there exists at most one input ($x \in \mathrm{dom}(f)$) where the function takes the chosen value ($y = f(x)$).

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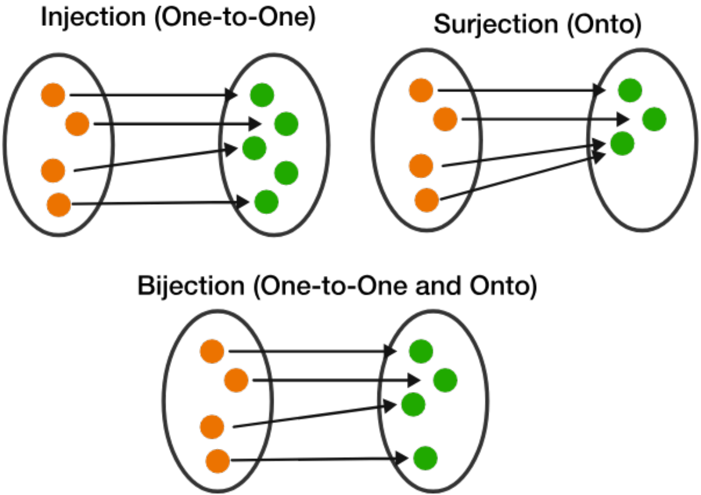
Definition: bijection function

A function is called a bijection (or a pairing mapping), if it is injective and surjective. That is, for any possible value ($y \in \text{co-dom}(f)$) there exists exactly one place/input ($x \in \mathrm{dom}(f)$) where the function takes the chosen value ($y = f(x)$).

Representation of functions: Domain defined on a finite set



Source: https://brilliant.org/wiki/bijection-injection-and-surjection/

Composite functions, function composition

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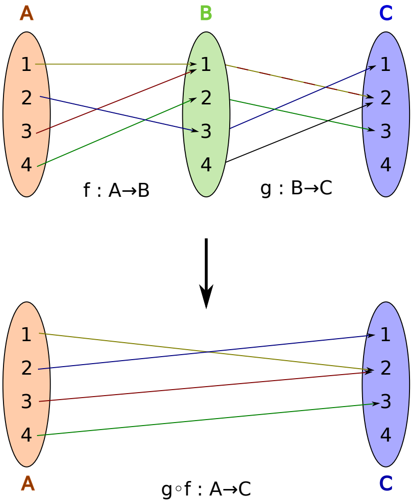

Source: https://en.wikipedia.org/wiki/Function_composition

Diagram of transformations defined on a finite set

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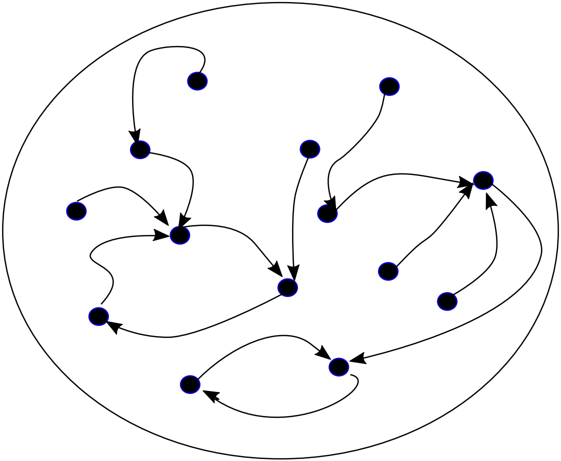

Diagram of bijective transformations defined on a finite set: Permutations

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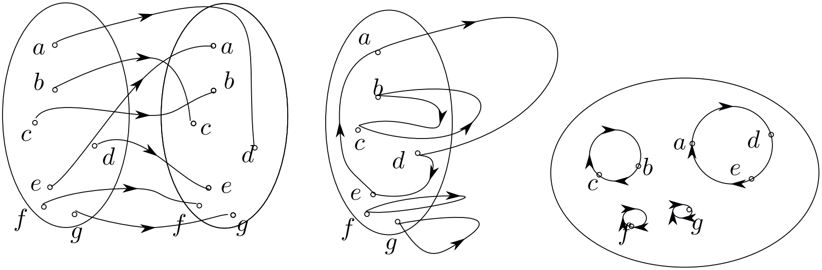

Inverse of bijective transformations

Definiton

Let $f : \mathrm{dom}(f) \to \text{co-com}(f)$ be a bijective function. Then $f^{(-1)} : \text{co-dom}(f) \to \mathrm{dom}(f)$ is the inverse function of f, that is defined as $f^{(-1)}(y) = x$ iff $f(x) = y$.

Examples of inverse functions

Examples of inverse functions

Observation

$\sin x : \mathbb{R} \to \mathbb{R}$, $\cos x : \mathbb{R} \to \mathbb{R}$, $\tan x : \mathbb{R} - \{\frac{2k+1}{2} \cdot \pi : k \in \mathbb{Z}\} \to \mathbb{R}$ are not bijective.

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Claim

(1) $\sin_{\text{bij}} x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is bijective.

(2) $\cos_{\text{bij}} x : [0, \pi] \to [-1, 1]$ is bijective.

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Definition

$\arcsin x$, $\arccos x$ and $\arctan x$ are the inverse function of $\sin_{\mathrm{bij}}$, $\cos_{\mathrm{bij}}$ and $\tan_{\mathrm{bij}}$, resp.

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Definition

(1) $\sqrt{x}$ is the inverse of x^2_{bij},

(2) $\log_2 x$ is the inverse of 2^x_{bij}.

Graph of real functions

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Definition

The graph of a function $f : \text{dom}(f) \to \text{co-dom}(f)$ is

$$graph(f) = \{(x, f(x)) : x \in \text{dom}(f) \subset \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

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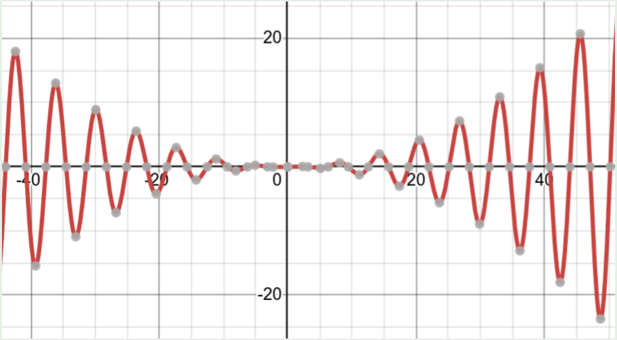
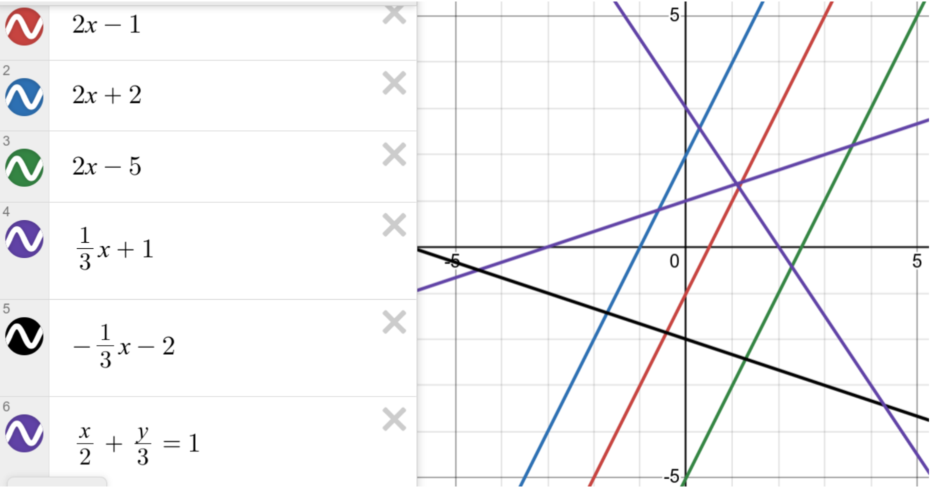
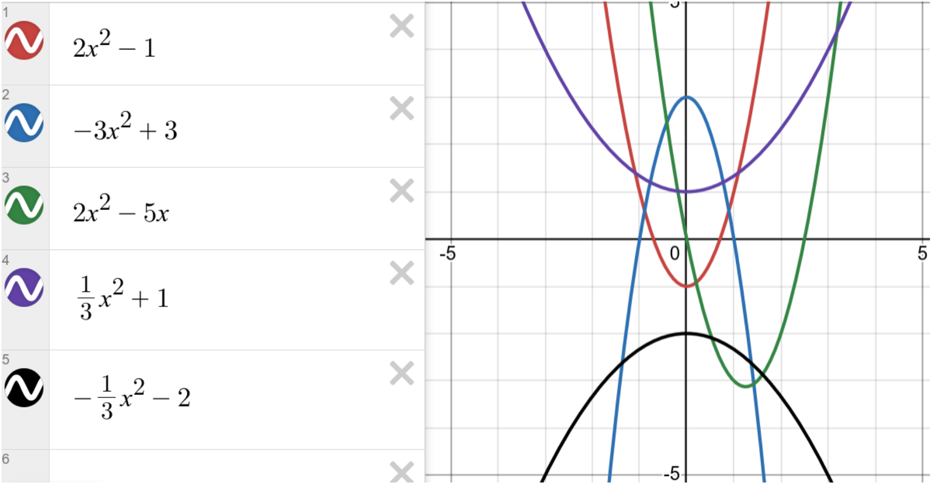
Example: $\frac{1}{100}x^2 \sin x$ (Source: https://www.desmos.com/calculator)



Linear functions



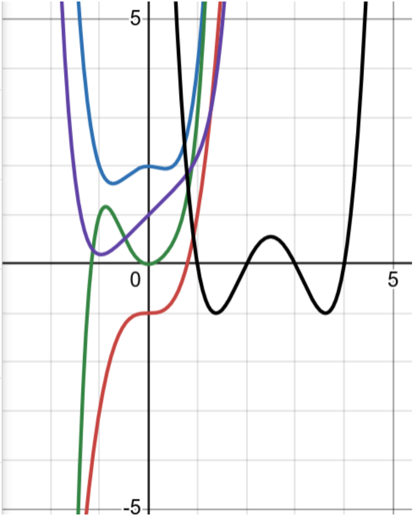
Quadratic functions



Power functions

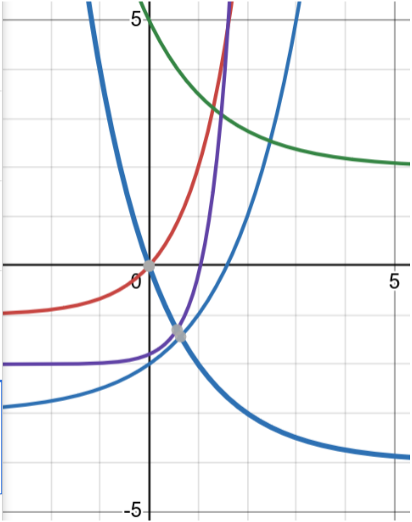
Polynomial functions

1. $2x^3 - 1$
2. $2x^4 + x^3 - x^2 + 2$
3. $2x^5 - x^3 + 2x^2$
4. $\frac{1}{5}x^6 + x + 1$
5. $(x-1)(x-2)(x-3)(x-4)$
6.

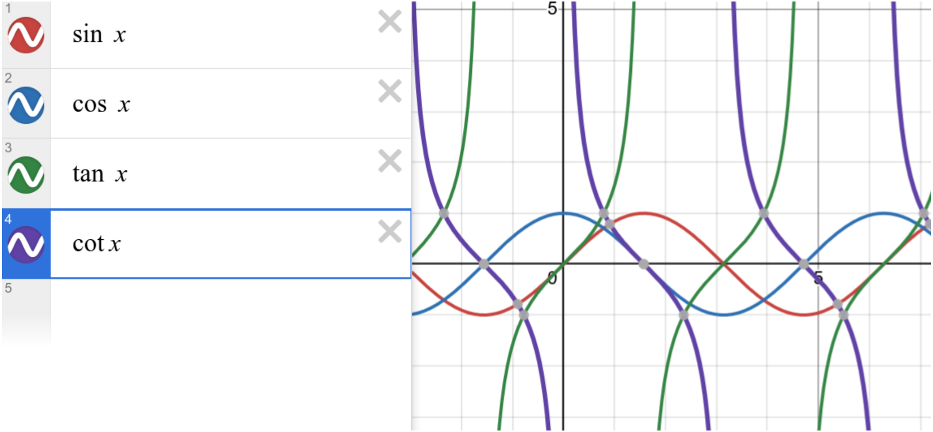


Exponential functions

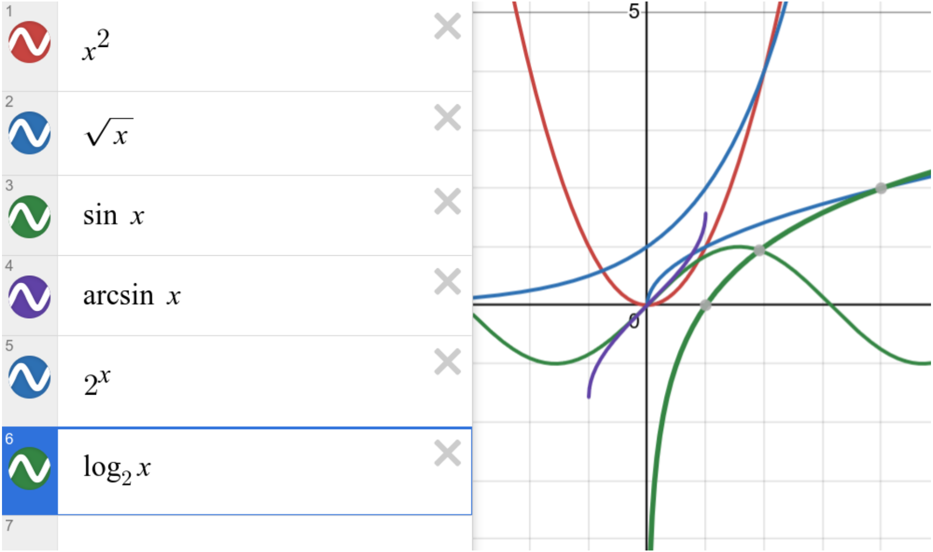
1. $3^x - 1$
2. $2^x - 3$
3. $3 \cdot 2^{-x} + 2$
4. $\frac{1}{5} 3^{2x} - 1 - 1$
5. $\left(\frac{1}{2}\right)^{(x-2)} - 4$
6.



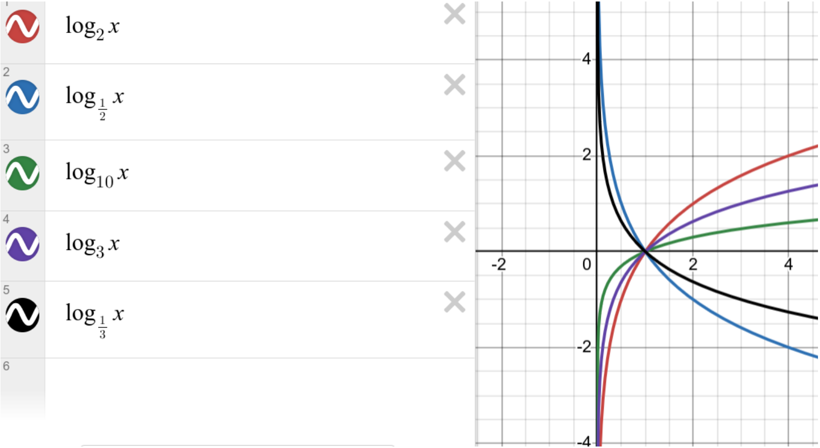
Trigonometric functions



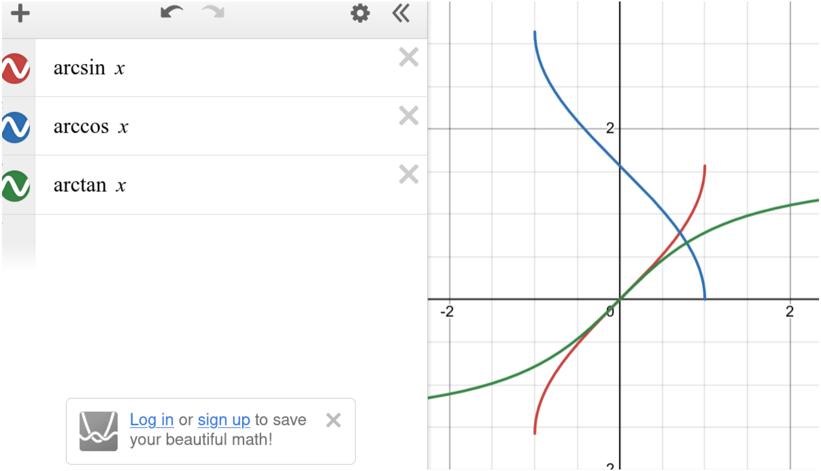
Inverse functions



Logarithmic functions

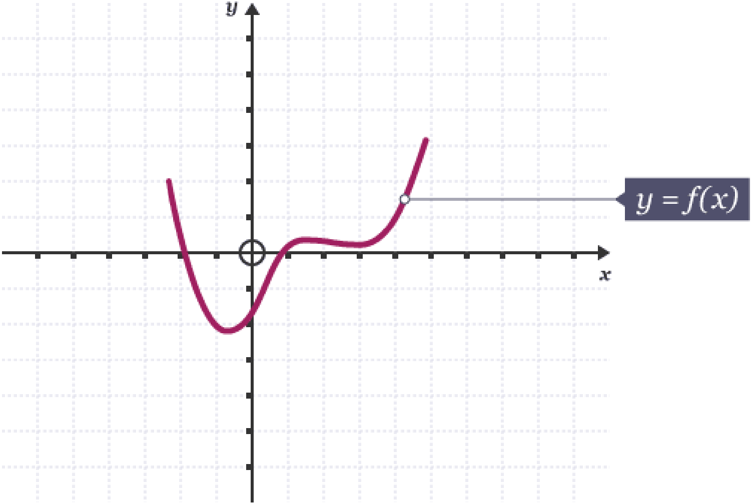


Inverse trigonometric functions



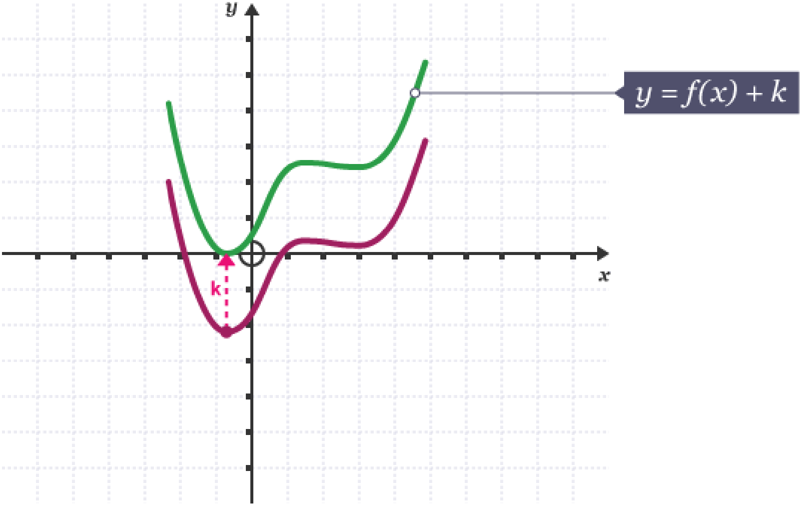
Break

Graph of a given function



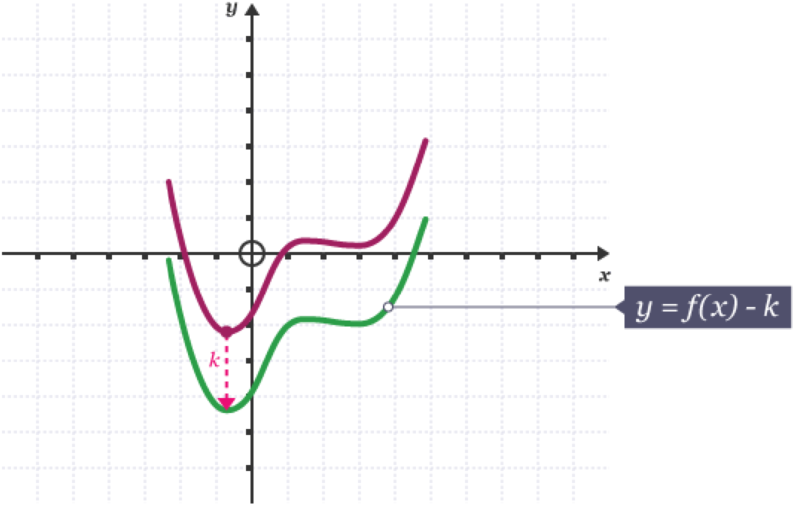
Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function I

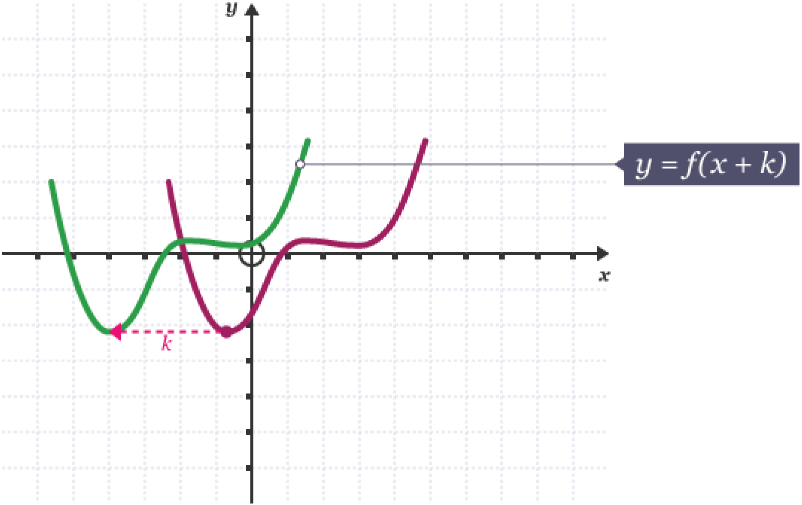


Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function II

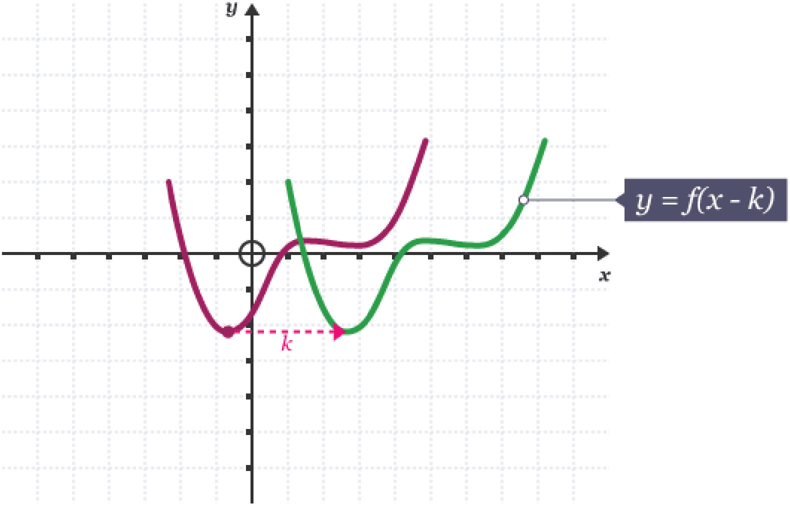


Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1



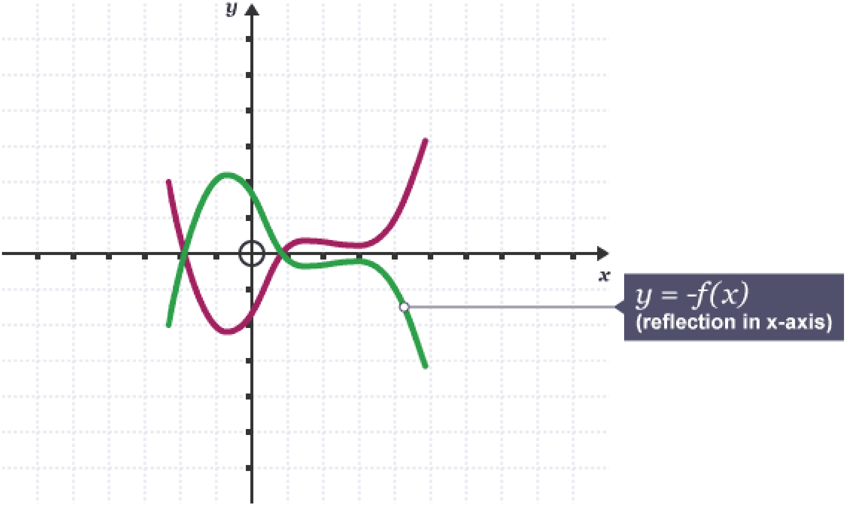
Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function IV



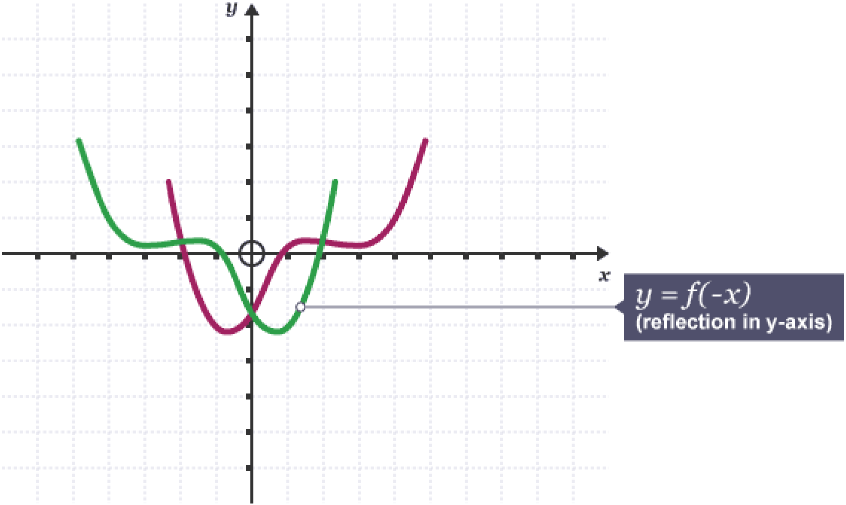
Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function V



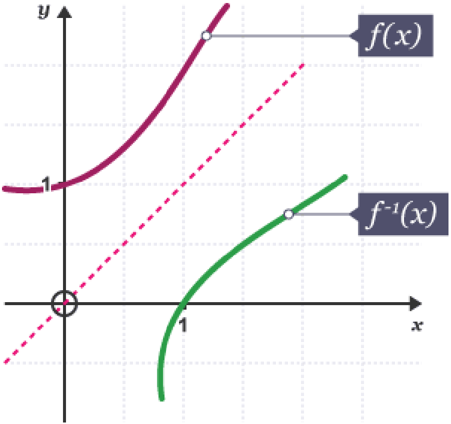
Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function VI



Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

Drawing the graph of a given function VII



Source: https://www.bbc.co.uk/bitesize/guides/zc6hhyc/revision/1

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- **monotonically decreasing** iff for all $x, y \in \mathbb{R}$ with $x < y$ we have $f(x) \ge f(y)$,
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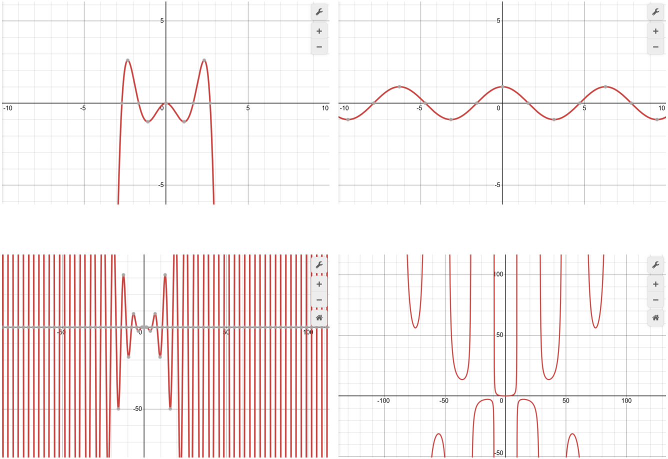
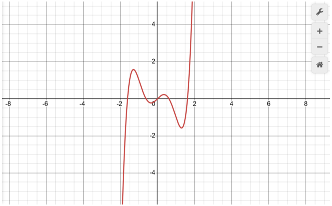
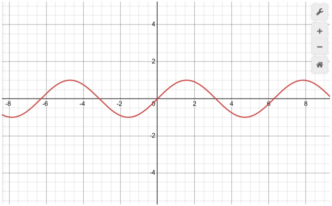
- **even** iff for all $x \in \mathbb{R}$ we have $f(x) = f(-x)$,
- **odd** iff for all $x \in \mathbb{R}$ we have $f(x) = -f(-x)$,
- **monotonically increasing** iff for all $x, y \in \mathbb{R}$ with $x < y$ we have $f(x) \leq f(y)$,
- **strictly monotonically increasing** iff for all $x, y \in \mathbb{R}$ with $x < y$ we have $f(x) < f(y)$,
- **monotonically decreasing** iff for all $x, y \in \mathbb{R}$ with $x < y$ we have $f(x) \geq f(y)$,
- **strictly monotonically decreasing** iff for all $x, y \in \mathbb{R}$ with $x < y$ we have $f(x) > f(y)$,
- **periodic with period** P iff for all $x \in \mathbb{R}$ we have $f(x+P) = f(x)$.

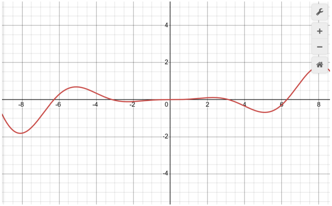
Reading the graph of a real function: Even functions

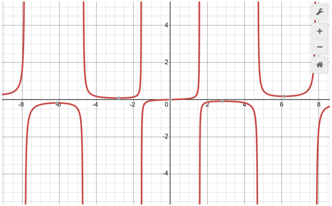


Reading the graph of a real function: Odd functions









Further properties of functions

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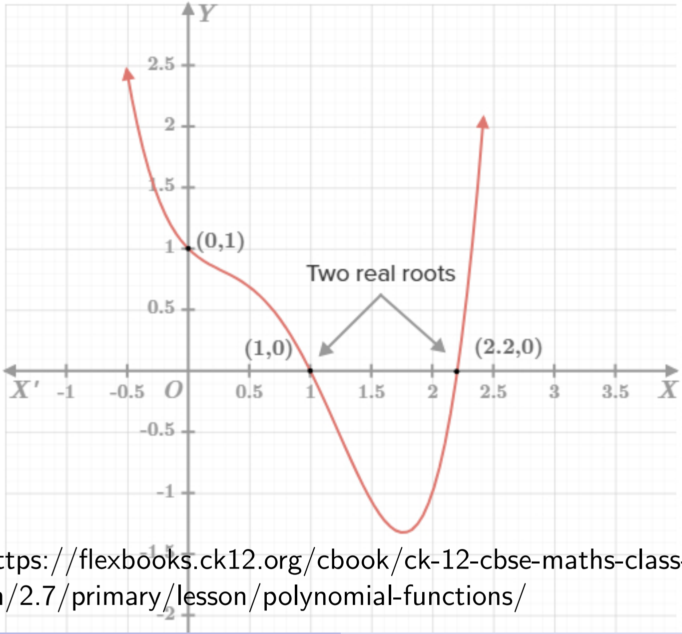
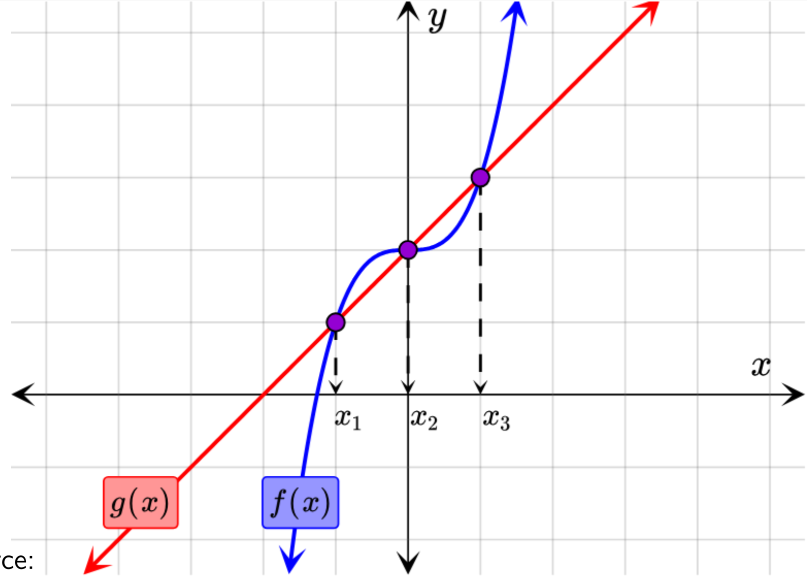
Note that f is concave iff $-f$ is convex.

Zeros of functions

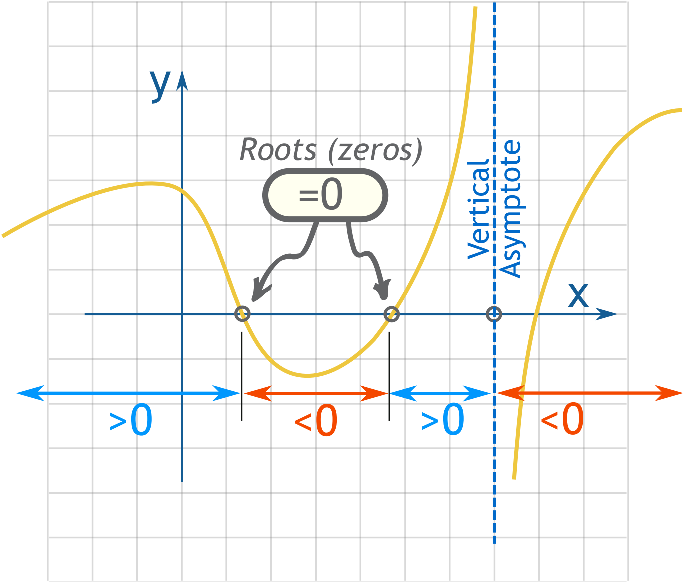


Source: https://flexbooks.ck12.org/cbook/ck-12-cbse-maths-class-11/section/2.7/primary/lesson/polynomial-functions/

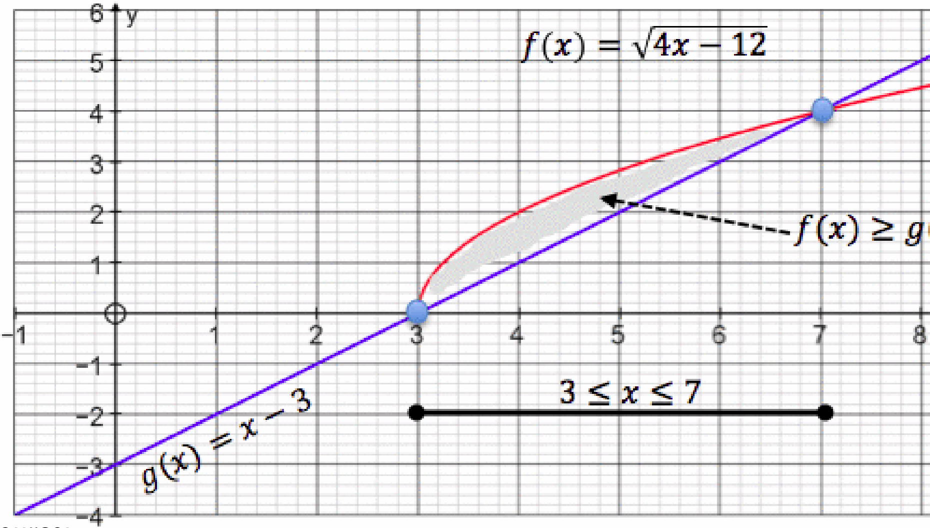


Source:
https://mathleaks.com/study/kb/method/solvinganequationgraphically

Sign of functions



Source:
https://www.mathsisfun.com/algebra/inequality-rational-solving.html



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Thank you for your attention!