Notes

97.39 Fibonometry

Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. The analogous Lucas numbers may be defined either by $L_n = F_{n-1} + F_{n+1}$, or by starting the same recurrence with $L_0 = 2$ and $L_1 = 1$. We also have $5F_n = L_{n-1} + L_{n+1}$. The Fibonacci and Lucas numbers satisfy many further identities that can be found by an analogue of Osborn's famous rule for the identities between hyperbolic functions. George Osborn (not Osborne or Osbourne!) first published his rule 111 years ago in this journal [1]. We quote his statement:

> 'In any Trigonometrical formula for θ, 2θ, 3θ, or θ and ϕ, after changing sin to sinh, cos to cosh, etc., change the sign of any term that contains a product of sinhs.'
>
> [He should have told us to change the sign for each successive pair of sinhs, and so multiply by $(-1)^k$ any term that contains $2k$ or $2k + 1$ sinhs.]

Every trigonometric identity relating sums of products of sines and cosines also gives a corresponding identity in which sines and cosines are replaced by Fibonacci and Lucas numbers, respectively. Only the constants are changed. For instance, the trigonometric addition formulae

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

give the 'Fibonometric' ones

$$2F_{a+b} = F_aL_b + L_aF_b, \qquad 2L_{a+b} = L_aL_b + 5F_aF_b.$$

The precise rule is that an angle $\theta = p\alpha + q\beta + r\gamma + \ldots$ should be replaced by a subscript $n = pa + qb + rc + \ldots$, $\sin\theta$ by $\frac{1}{2}i^nF_n$, and $\cos\theta$ by $\frac{1}{2}i^nL_n$. We should then insert a factor of -5 (replacing Osborn's -1) for each successive pair of sines, and so multiply by $(-5)^k$ any term that contains $2k$ or $2k + 1$ sines.

Our Fibonometric addition formulae are derived from the trigonometric ones by applying this rule and multiplying by $4/i^{a+b}$. The special cases

$$2\sin\theta\cos\theta = \sin 2\theta, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta, \qquad \cos^2\theta + \sin^2\theta = 1$$

similarly convert to particularly useful Fibonometric ones

$$F_nL_n = F_{2n}, \qquad (L_n)^2 + 5(F_n)^2 = 2L_{2n}, \qquad (L_n)^2 - 5(F_n)^2 = (-1)^n4$$

on multiplication by $4/i^{2n} = (-1)^n4$. As another example

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

yields

$$\frac{i^{3n}}{2}F_{3n} = 3\frac{i^n}{2}F_n - (-5)\times 4\left(\frac{i^n}{2}F_n\right)^3$$

which after multiplying by $2 / i^{3n}$ simplifies to

$$F_{3n} = 5(F_n)^3 + 3(-1)^n F_n.$$

We suggest that the reader derives our starting formulae:

$$L_n = F_{n-1} + F_{n+1}, \qquad 5F_n = L_{n-1} + L_{n+1}$$

from the standard 'sin − sin' and 'cos − cos' formulae.

Note added February 1st 2013. We thank the editor for directing us to Barry Lewis's paper [2] (also in this journal!), which discusses relations between Fibonacci identities and trigonometric ones at greater length.

References

1. G. Osborn, Mnemonic for hyperbolic formulae, *Math. Gaz.*, **2** (July 1902) p. 189.
2. Barry Lewis, Trigonometry and Fibonacci Numbers, *Math. Gaz.*, **91** (July 2007) pp. 216- 226.

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97.40 Infinitely many proofs that there are infinitely many primes

Virtually every mathematician knows the classical and beautiful proof that there are infinitely many prime numbers which goes back to Euclid (circa 250 BC). In summary, the proof runs like this:

If the set of primes is finite, then there is a biggest prime p_n; consider the number $N = 2 \times 3 \times 5 \times \ldots \times p_n + 1$. Since $N > p_n$, N cannot be prime and so N has proper factors. Thus N has a prime factor (this step requires strong induction, unless the unique prime factorisation theorem has been assumed, a fact not always acknowledged in textbooks). But any prime divided into N leaves remainder 1, a contradiction, and the result is established.

We note that we could have equally well considered $M = 2 \times 3 \times 5 \times \ldots \times p_n - 1$ and have arrived at the same conclusion.

Those meeting Euclid's proof for the first time sometimes enthusiastically conclude that every member of the sequences $\{a_i\}$ and $\{b_i\}$ given by $a_i = 2 \times 3 \times 5 \times \ldots \times p_i + 1$ and $b_i = 2 \times 3 \times 5 \times \ldots \times p_i - 1$, where p_i is the ith prime, is a prime number. A little calculation shows that this is not the case. For example, $a_6 = 30031 = 59 \times 509$ and $b_4 = 209 = 11 \times 19$. All one can salvage is that neither a_i nor b_i is divisible by any primes less