The number of slim rectangular lattices

Gábor Czédli, Tamás Dékány, Gergő Gyenizse, and Júlia Kulin

Abstract. *Slim rectangular lattices* are special planar semimodular lattices introduced by G. Grätzer and E. Knapp in Acta Sci. Math. 75:29–48, 2009. They are finite semimodular lattices L such that the ordered set $\mathrm{Ji}\, L$ of join-irreducible elements of L is the cardinal sum of two nontrivial chains. After describing these lattices of a given length n by permutations, we determine their number, $|\mathrm{SRectL}(n)|$. Besides giving recursive formulas, which are effective up to about $n = 1000$, we also prove that $|\mathrm{SRectL}(n)|$ is asymptotically $(n-2)! \cdot e^2/2$. Similar results for *patch lattices*, which are special rectangular lattices introduced by G. Czédli and E. T. Schmidt in Order 30:689–721, 2013, and for slim rectangular lattice *diagrams* are also given.

1. Introduction

1.1. Target. The key definitions are given in Section 2. Unless otherwise stated, all lattices occurring in this paper are finite.

Slim rectangular lattices and, in particular, slim patch lattices are of particular importance, because each planar semimodular lattice can be obtained from them easily; see G. Grätzer and E. Knapp [19], G. Czédli and E. T. Schmidt [15], and G. Grätzer [18]. The present paper describes slim rectangular lattices by permutations. Using this description, we are going to enumerate slim rectangular lattices and slim patch lattices of a given length n. Also, we enumerate their diagrams. We give asymptotic formulas and recursive ones. By means of computer algebra, the recursive formulas lead to concrete numbers for $n \leq 1000$.

1.2. Outline. The rest of this section gives a brief historical overview of planar semimodular lattices, including slim rectangular and slim patch lattices. Section 2 recalls the main concepts and some tools we need from the theory of planar semimodular lattices; however, the reader is assumed to be familiar with the rudiments of lattice theory. In Section 3, we describe slim rectangular lattices by certain permutations, and we prove several auxiliary statements that could be of separate interest. We count these lattices of a given height n

Key words and phrases: Rectangular lattice, patch lattice, semimodularity, slim lattice, planar lattice, combinatorics of permutations.

2010 *Mathematics Subject Classification.* 06C10 **November 20, 2013**.

This research was supported by the European Union and co-funded by the European Social Fund under the project "Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences" of project number "TÁMOP-4.2.2.A-11/1/KONV-2012-0073", and by NFSR of Hungary (OTKA), grant numbers K83219 and K104251.

and their diagrams recursively in Section 4, and asymptotically for $n \to \infty$ in Section 5. Finally, Section 6 contains some concrete numerical values.

1.3. Historical overview. The concept of slim semimodular lattices and that of rectangular lattices appeared first in G. Grätzer and E. Knapp's pioneering papers [19] and [20]. These lattices are planar, and these two papers were soon followed by more than twenty others devoted to planar semimodular lattices. Slim semimodular lattices are natural tools in generalizing the classical Jordan–Hölder theorem for groups, see G. Czédli and E. T. Schmidt [13] and G. Grätzer and J. B. Nation [22]. Rectangular lattices play an important role in the finite congruence lattice representation problem, see G. Czédli [4] G. Grätzer and E. Knapp [20] and [21], and E. T. Schmidt [25]. We know from [19] that, to understand planar semimodular lattices, it suffices to describe the slim semimodular ones. By G. Czédli and E. T. Schmidt [15], slim semimodular lattices are obtained from slim patch lattices, which are special rectangular lattices, by means of successive (Hall–Dilworth) gluings; see also G. Grätzer [18] for another approach. These facts indicate that slim rectangular lattices and slim patch lattices are natural objects to study.

There are two known structure theorems for slim rectangular lattices: one is given in [15, Proposition 2.3], see also G. Czédli [5, Theorem 3.7] for a stronger version, while the other one is proved in G. Czédli and G. Grätzer [9, Corollary 3]. The idea of using permutations to describe slim semimodular lattices goes back to H. Abels [1], and it was fully developed in G. Czédli and E. T. Schmidt [16].

The enumeration of slim semimodular lattices and their planar diagrams started in G. Czédli, L. Ozsvárt and B. Udvari [11], and continued in G. Czédli, T. Dékány, L. Ozsvárt, N. Szakács and B. Udvari [8], and G. Czédli [6]. There are several earlier papers on counting other particular lattices; for example, see Erné, Heitzig and Reinhold [27] and [28], and Pawar and Waphare [29].

2. Preliminaries

Here, we overview some concepts and facts we need in the present paper. For a more complex overview, the reader might be interested in G. Grätzer [17] and G. Czédli and G. Grätzer [10]. An element of a lattice is *join-irreducible* if it has exactly one lower cover. A finite lattice L is *slim*, if $\mathrm{Ji}\, L$, the set of join-irreducible elements of L, is included in the union of two chains of L; see G. Czédli and E. T. Schmidt [13]. Note that, in the semimodular case, this concept was first introduced by G. Grätzer and E. Knapp [19] in a different way. We know from G. Czédli and E. T. Schmidt [13] that slim lattices are *planar*, that is, they possess planar diagrams. Remember that all lattices, and thus all diagrams, in this paper are assumed to be finite. If D_1 and D_2 are planar diagrams and $\varphi\colon D_1 \to D_2$ is a bijective map such that φ is a lattice isomorphism and it preserves the left-right order of (upper) covers and that of lower covers of each element of D_1, then φ is called a *similarity map*. Two planar diagrams are *similar* if there exists a similarity map between them. We treat similar diagrams as equal ones. Therefore, when we count planar diagrams, we always do it up to similarity. Adjectives typically used for lattices, like semimodularity, will also be used for their planar diagrams; in this case the diagram is automatically a planar lattice diagram.

A minimal non-chain region of a planar lattice diagram D is called a *cell*. A four-element cell is a 4-*cell*. 4-cells are *covering squares*, that is, cover-preserving four-element Boolean sublattices. A diagram is a *4-cell diagram* if all of its cells are 4-cells. The following statement was proved in G. Grätzer and E. Knapp [19, Lemmas 4 and 5]; see also G. Czédli and E. T. Schmidt [14, Proposition 1] for the present form.

Lemma 2.1. *If D is a slim semimodular diagram, then it is a* 4*-cell diagram, and no two distinct* 4*-cells have the same bottom. Conversely, if D is a* 4*-cell lattice diagram in which no two distinct* 4*-cells have the same bottom, then D is a slim semimodular diagram.*

Following G. Grätzer and E. Knapp [20], a semimodular diagram D is *rectangular* if its *left boundary chain*, denoted by $\mathrm{C}_\ell(D)$, has exactly one doubly irreducible element, $\ell\mathrm{c}(D)$, its *right boundary chain*, $\mathrm{C}_\mathrm{r}(D)$, has exactly one doubly irreducible element, $\mathrm{rc}(D)$, and these two elements, called the *corners* of D, are complementary, that is, $\ell\mathrm{c}(D) \wedge \mathrm{rc}(D) = 0$ and $\ell\mathrm{c}(D) \vee \mathrm{rc}(D) = 1$. It was noticed by E. T. Schmidt, see G. Czédli and G. Grätzer [10, Exercise 1.58], that a slim semimodular lattice L is rectangular iff $\mathrm{Ji}\, L$ is a union of two chains such that no element in the first chain is comparable with some element of the second chain. Associated with a slim rectangular diagram D, the following three numerical parameters will be of particular interest.

Notation 2.2. As usual, the *length* of D is denoted by $\mathrm{length}\, D$. The *left upper length* and the *right upper length* of D, denoted by ${}^{\mathrm{lu}}\mathrm{len}\, D$ and ${}^{\mathrm{ru}}\mathrm{len}\, D$, are the length of the interval $[\ell\mathrm{c}(D), 1]$ and that of $[\mathrm{rc}(D), 1]$, respectively; see Figure 1 for illustration.

A rectangular diagram D is a *patch diagram* if $\ell\mathrm{c}(D)$ and $\mathrm{rc}(D)$ are coatoms. Equivalently, if ${}^{\mathrm{lu}}\mathrm{len}\, D = {}^{\mathrm{ru}}\mathrm{len}\, D = 1$. A *patch lattice* is a lattice that has a patch diagram.

Two prime intervals of a slim semimodular diagram D are *consecutive* if they are opposite sides of a 4-cell. By G. Czédli and E. T. Schmidt [13, Lemma 2.3], covering squares and 4-cells in a slim semimodular diagram are the same, whence the previous sentence can be rephrased as follows: two prime intervals of a slim semimodular diagram D are consecutive if they are opposite sides of a covering square. Therefore, the consecutiveness of two prime intervals in slim semimodular lattice L does not depend on the planar diagram chosen. Maximal sequences of consecutive prime intervals form a *trajectory*, see G. Czédli and E. T. Schmidt [13]. In other words, a trajectory is a class of

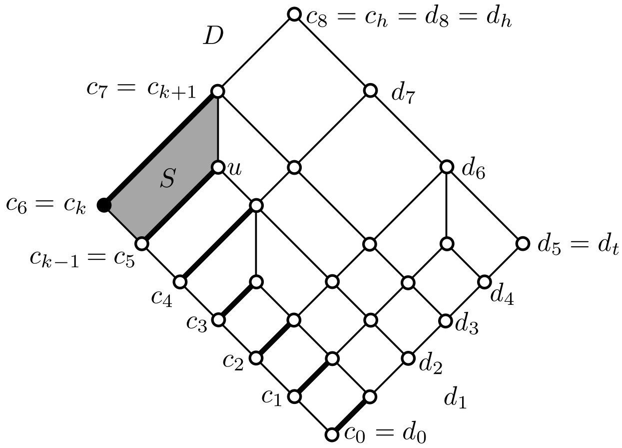

FIGURE 1. A rectangular diagram with $\text{length}\, D = 8$, ${}^{\text{lu}}\text{len}\, D = 2$, and ${}^{\text{ru}}\text{len}\, D = 3$.

the equivalence relation generated by consecutiveness. In [13, Lemma 2.8], the following statement was derived from (the present) Lemma 2.1.

Lemma 2.3. *If T is a trajectory of a slim semimodular diagram D, then T contains exactly one prime interval of $\mathrm{C}_\ell(D)$, and the same holds for $\mathrm{C}_\mathrm{r}(D)$. Going from left to right, T does not branch out. First T goes up* (*possibly in zero steps*)*, then it may turn to the lower right, and finally it goes down* (*possibly, in zero steps*)*. In particular, at most one turn is possible.*

Notation 2.4. We denote the set of *slim rectangular diagrams* of length n and that of *slim semimodular diagrams* of length n by the acronyms $\mathrm{SRectD}(n)$ and $\mathrm{SSmodD}(n)$, respectively. Similarly, the set of *slim rectangular lattices* of length n, that of *slim semimodular lattices* of length n, and that of *slim patch lattices* of length n are denoted by $\mathrm{SRectL}(h)$, $\mathrm{SSmodL}(n)$, and $\mathrm{SPatchL}(n)$.

For a given $n \in \{1, 2, \dots\} = \mathbb{N}$, these five sets above are finite, since we do not make a distinction between similar diagrams or between isomorphic lattices.

Jordan–Hölder permutations associated with semimodular lattices appeared first in H. Abels [1] and R. P. Stanley [26]. Here, following G. Czédli and E. T. Schmidt [16], we define them by means of trajectories. For a slim rectangular diagram D, let $n = \text{length}\, D$, and let

$$\begin{aligned} \mathrm{C}_\ell(D) &= \{0 = c_0 \prec c_1 \prec \dots \prec c_n = 1\}, \\ \mathrm{C}_\mathrm{r}(D) &= \{0 = d_0 \prec d_1 \prec \dots \prec d_n = 1\}. \end{aligned} \tag{2.1}$$

The set of all $\{1, \dots, n\} \to \{1, \dots, n\}$ permutation is denoted by S_n. The (*Jordan–Hölder*) *permutation* $\pi = \pi_D \in S_n$ is defined by the rule $\pi(i) = j$ iff $[c_{i-1}, c_i]$ and $[d_{j-1}, d_j]$ belong to the same trajectory. The following statement was proved in G. Czédli and E. T. Schmidt [16].

Lemma 2.5. *The map* SSmodD$(n) \to S_n$*, defined by* $D \mapsto \pi_D$*, is a bijection.*

In what follows, since this lemma above is obvious for $n \le 1$ and since the length of a slim rectangular lattice is at least 2, we always assume that n denotes an integer greater than 1. Combining Lemma 2.5 with [16, Lemma 4.6] and the definition of π_D, we obtain that

Lemma 2.6. *Let* D_1 *and* D_2 *be slim rectangular diagrams. They determine the same lattice iff* $\pi_{D_1} \in \{\pi_{D_2}, \pi_{D_2}^{-1}\}$.

Planar lattice diagrams have several properties that are easy to believe but not so easy to prove. What we need from them is given by the following lemma, taken from D. Kelly and I. Rival [24, Lemmas 1.2 and 1.5, Propositions 1.6 and 1.7, and Theorem 2.5].

Lemma 2.7. *Let* D *be a planar lattice diagram, and let* $a, b \in D$.

(i) *If* $a \le b$ *and* a *and* b *are on different sides of a maximal chain* C*, then there exists an* $x \in C$ *such that* $a \le x \le b$.
(ii) *A closed interval of* D *is a planar subdiagram.*
(iii) *If* $|D| \ge 3$*, then* D *contains a doubly irreducible element distinct from* 0 *and* 1 *on its left boundary.*
(iv) *If* $a \parallel b$*, then either* a *is on the left of all maximal chains through* b*, or* b *is on the left of all maximal chains through* a*. The same holds with "right" instead of "left".*

Based on Lemma 2.7(iv), if $a \parallel b$ and a is on the left of some (equivalently, all) maximal chains through b, then we say that a is *on the left* of b; analogous terminology is used if "left" is replaced by "right".

3. Description by permutations

For convenience, we introduce the following concept; our terminology will be explained by Proposition 3.3.

Definition 3.1. A permutation $\pi \in S_n$ is called *rectangular* if it satisfies the following three properties.

(i) For all i and j, if $\pi^{-1}(1) < i < j \le n$, then $\pi(i) < \pi(j)$.
(ii) For all i and j, if $\pi(1) < i < j \le n$, then $\pi^{-1}(i) < \pi^{-1}(j)$.
(iii) $\pi(n) < \pi(1)$.

Remark 3.2. If $\pi \in S_n$ is rectangular, then we have

(iv) $\pi^{-1}(n) < \pi^{-1}(1)$.

So, π is rectangular iff π^{-1} is rectangular.

Proof of Remark 3.2. Assume that $\pi \in S_n$ satisfies (i)–(iii). Since π and π^{-1} are injective, (iii) implies that

$$1 < \pi(1), \qquad \pi(n) < n, \qquad 1 < \pi^{-1}(1), \qquad \pi^{-1}(n) < n. \tag{3.1}$$

Suppose, for a contradiction, that (iv) fails. Then $n \geq 2$, and we have that $\pi^{-1}(1) < \pi^{-1}(n)$. By last inequality of (3.1), (i) applies for the pair $\langle i, j \rangle = \langle \pi^{-1}(n), n \rangle$, and we obtain that $n = \pi(\pi^{-1}(n)) < \pi(n)$, a contradiction. □

Now, we are in the position to formulate the main result of this section.

Proposition 3.3. *A slim, semimodular, planar diagram D of length $n \geq 2$ is rectangular if and only if $\pi = \pi_D \in S_n$ is rectangular. Furthermore, if D is rectangular, then*

$$\pi_D(1) = \operatorname{length} D - {}^{\mathrm{ru}}\mathrm{len}\, D + 1, \quad \pi_D^{-1}(1) = \operatorname{length} D - {}^{\mathrm{lu}}\mathrm{len}\, D + 1. \tag{3.2}$$

This proposition trivially implies the following statement.

Corollary 3.4. *A slim, semimodular, planar diagram D of length n is a patch diagram if and only if $\pi_D(1) = n = \pi_D^{-1}(1)$. Therefore, the number of these diagrams is $(n-2)!$.*

Combining Proposition 3.3 and Corollary 3.4 with Lemmas 2.5 and 2.6, we obtain a new description of slim rectangular (or patch) diagrams and lattices by permutations. This description is effective, because G. Czédli and E. T. Schmidt [16, Proposition 2.7 and Theorem 3.3] tell us how to construct D from π_D; however, we do not need these long details here.

The rest of this section is devoted to the proof of Proposition 3.3. The following definition is taken from G. Grätzer and R. W. Quackenbush [23].

Definition 3.5. An element x of a lattice L is called a *narrows* if $L = {\downarrow}x \cup {\uparrow}x$. If, in addition, $x \notin \{0, 1\}$, then x is a *proper narrows.* The set of narrows of L is denoted by $\mathrm{Nar}(L)$. A lattice L is called (*glued sum*) *indecomposable* if $|L| \geq 3$ and $\mathrm{Nar}(L) = \{0, 1\}$.

We know from G. Czédli and E. T. Schmidt [16, after (1.2)] that the set $\mathrm{Nar}(D)$ of narrows of D is $\mathrm{C}_\ell(D) \cap \mathrm{C}_\mathrm{r}(D)$. Note that, by definitions, a glued sum indecomposable diagram is of length at least 2.

Obviously, Lemma 2.1 implies the following statement.

Corollary 3.6. *If D is a* (*glued sum*) *indecomposable, slim, semimodular diagram, then for each $c \in \mathrm{C}_\ell(D) \setminus \{0, 1\}$, there exists a unique c' such that $\{c \wedge c', c, c', c \vee c'\}$ is a 4-cell.*

Lemma 3.7. *If D is an indecomposable, slim, semimodular diagram, $a \prec b$, and $a, b \in \mathrm{C}_\ell(D)$, then exactly one of the following two possibilities holds.*

(i) *a is meet-reducible and b is join-irreducible.* (*In this case, we say that $[a, b]$ is* up-edge.)

(ii) *a is meet-irreducible and b is join-reducible.* (*In this case, we say that $[a, b]$ is* down-edge.)

Proof. Since D is indecomposable, the trajectory starting at $[a, b]$ is not a singleton. In other words, $[a, b]$ is a left edge of a 4-cell S. This implies that a is meet-reducible or b is join-reducible. Hence, G. Czédli and E. T. Schmidt [14, Lemma 4], which says that each of these two cases excludes the other one, completes the proof. □

The name"down-edge" is motivated by the following lemma.

Lemma 3.8. *Let D be a slim semimodular diagram of length n, and assume that $1 \leq i < j \leq n$.*

(i) *If D is glued sum indecomposable and, with the notation given in* (2.1), *$[c_{i-1}, c_i]$ is a down-edge, then $\pi_D(i) < \pi_D(j)$ and $\pi_D(i) < i$.*

(ii) *If c_i is a narrows, then $\pi_D(i) < \pi_D(j)$.*

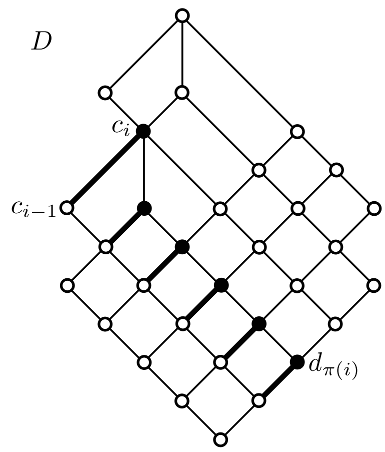

FIGURE 2. Illustrating the proof of Lemma 3.8

Proof. To prove part (i), assume that D is indecomposable. Denote π_D by π. Let T_i be the trajectory that contains $[c_{i-1}, c_i]$; see Figure 2, where T_i consists of the thick edges. Note that T_i consists of at least two edges, because D is indecomposable. Since $[c_{i-1}, c_i]$ is a down-edge, T_i launches to the lower right, and keeps going to this direction without any turn by Lemma 2.3. Hence, the top elements of the edges of T_i, which are the black-filled elements in the figure, form a descending, nontrivial chain. This implies that $d_{\pi(i)} < c_i$, and we conclude that $\pi(i) < i$.

Suppose, for a contradiction, that $\pi(i) > \pi(j)$. This implies that $c_{j-1} \geq c_i > d_{\pi(i)} > d_{\pi(j)}$. Hence, $[c_{j-1}, c_j]$ and $[d_{\pi(j)-1}, d_{\pi(j)}]$ are two comparable prime intervals of the same trajectory. This is a contradiction, since a trajectory cannot have comparable prime intervals by G. Czédli [7, Lemma 3.3]. This proves part (i).

Finally, assume that c_i is a narrows. Clearly, for every 4-cell S, either we have that $S \cap (\downarrow c_i \setminus \{c_i\}) = \varnothing$, or $S \cap (\uparrow c_i \setminus \{c_i\}) = \varnothing$. Hence, no trajectory can cross c_i, and part (ii) follows immediately. □

Next, we generalize some parts of G. Grätzer and E. Knapp [20, Lemmas 3 and 4]. By Lemma 2.7(iii), the element c in the following lemma exists.

Lemma 3.9. *Let D be a glued sum indecomposable, planar lattice diagram. If c is the least doubly irreducible element on the left boundary of D, then the ideal $\downarrow c$ is a chain.*

Proof. Let $\mathrm{C}_\ell(D) \cap \downarrow c = \{0 = c_0 \prec c_1 \prec \cdots \prec c_k = c\}$. It suffices to prove that

$$\{c_1, \ldots, c_k\} \subseteq \mathrm{Ji}\, D. \tag{3.3}$$

Suppose, for a contradiction, that there is an $i \in \{1, \ldots, k\}$ such that c_i is join-reducible. Let i be minimal with respect to this property. The ideal $\downarrow c_i$ is a planar subdiagram by Lemma 2.7(ii). Let $U = \mathrm{C}_\mathrm{r}(\downarrow c_i)$. Take the largest $j \in \{0, \ldots, i-1\}$ such that $c_j \in U$; this j exists, since $c_0 = 0 \in U$. Note that $j \leq i-2$, since c_i is join-reducible. By Lemma 2.7(ii), $D' := [c_j, c_i]$ is a planar subdiagram. Clearly, $|D'| \geq 3$, $\mathrm{C}_\ell(D') = \{c_j, c_{j+1}, \ldots, c_i\}$, and $\mathrm{C}_\mathrm{r}(D') = U \cap [c_j, c_i]$. By Lemma 2.7(iii), there is an $s \in \{j+1, \ldots, i-1\}$ such that c_s is doubly irreducible in D'. By the choice of k, the element c_s is not doubly irreducible in D. The minimality of i yields that c_s is meet-reducible in D. By G. Czédli and E. T. Schmidt [14, Lemma 4], mentioned already in the proof of Lemma 3.7, the join-reducibility of c_i implies that $s \neq i-1$. Hence, $s \leq i-2$. The element c_s has a cover $v \in D$, distinct from c_{s+1}. Since c_s is meet-irreducible in D', we have that $v \notin D'$. We have that $\mathrm{height}\, v = s+1 < i = \mathrm{height}\, c_i$, whence $c_i \nleq v$. We also have that $v \nleq c_i$, since $v \notin D' = [c_j, c_i]$. Thus, $c_i \parallel v$. We conclude from Lemma 2.7(iv) that c_i is on the left of v. That is, v is in the right of all maximal chains through c_i. In particular, if we extend $\mathrm{C}_\mathrm{r}(D')$ to a maximal chain V of D, then v is strictly on the right of V. On the other hand, c_s, which belongs to $\mathrm{C}_\ell(D') \setminus \mathrm{C}_\mathrm{r}(D')$, is strictly on the left of $\mathrm{C}_\mathrm{r}(D')$, whence it is strictly on the left of V. Thus, c_s and v are strictly on different sizes of V while $c_s \prec v$. This contradicts Lemma 2.7(i). □

Lemma 3.10. *Let D be a glued sum indecomposable, slim semimodular diagram of length n. If, with notation (2.1), c_k is the least doubly irreducible element of D on the left boundary chain, then $\pi_D(k+1) = 1$.*

Proof. Clearly, $k \geq 1$. We prove the lemma by induction on k.

First, assume that $k = 1$. Since D is indecomposable, $0 \notin \mathrm{Mi}\, D$. By G. Czédli and E. T. Schmidt [14, Lemma 2],

$$\text{each element of a slim lattice has at most two covers.} \tag{3.4}$$

Hence, there are exactly two atoms, and $c_k = c_1$ is one of them. This clearly implies that $\pi_D(k+1) = \pi_D(2) = 1$.

Next, assume that $k > 1$, and the lemma holds for smaller values. Let $u = c_k'$ by Corollary 3.6. Since c_k has only one cover, and this cover belongs to $\mathrm{C}_\ell(D)$, we have that $c_k \vee u = c_{k+1}$. Similarly, $c_k \wedge u = c_{k-1}$. Hence,

$$S = \{c_{k-1}, c_k, u, c_{k+1}\} \quad \text{is a 4-cell.} \tag{3.5}$$

This 4-cell (or Lemma 3.7) shows that c_{k-1} is meet-reducible; see Figure 1 for an illustration. Let $D' = D \setminus \{c_k\}$; it consists of the empty-filled elements in the figure. Clearly, $c_{k-1} \in \mathrm{C}_\ell(D')$. By (3.4), $c_{k-1} \in \mathrm{Mi}\, D'$. We also have that $c_{k-1} \in \mathrm{Ji}\, D'$, because $c_{k-1} \in \mathrm{Ji}\, D$ by Lemma 3.9. Thus, c_{k-1} is a doubly irreducible element in D'.

Suppose, for a contradiction, that there exists an $i < k-1$ such that c_i is double irreducible in D'. Obviously, it is join-irreducible in D. By the choice of k, c_i is meet-reducible in D. However, its covers are of height $i+1$, which is less than $k = \mathrm{height}\, c_k$. Hence, these covers belong to D', contradicting the assumption that c_i is doubly irreducible in D'. This proves that c_{k-1} is the least doubly irreducible element of D' that belongs to $\mathrm{C}_\ell(D')$.

Let T' be the trajectory of D' such that T' contains $[c_{k-1}, u]$. Obviously, or by G. Czédli [7, Lemma 3.1], the trajectory of D that contains $[c_k, c_{k+1}]$ is $T := T' \cup \{[c_k, c_{k+1}]\}$. Note that the element of height k in $\mathrm{C}_\ell(D')$ is u. By the induction hypothesis, $\pi_{D'}(k) = 1$. This means that $[d_0, d_1] \in T'$. Thus, $[d_0, d_1] \in T$, proving that $\pi_D(k+1) = 1$. □

Proof of Proposition 3.3. By definitions, $\mathrm{SRectD}(n) \subseteq \mathrm{SSmodD}(n)$. Therefore, by Lemma 2.5, it suffices to prove that, for $D \in \mathrm{SSmodD}(n)$, the diagram D is rectangular iff so is the permutation π_D.

To prove the "only if" part of Proposition 3.3, assume that $D \in \mathrm{SRectD}(n)$. Let $k \in \{1, \dots, n-1\}$ denote the height of $\ell\mathrm{c}(D)$, that is, $\ell\mathrm{c}(D) = c_k$. By the rectangularity of D, c_k is the only doubly irreducible element that belongs to the left boundary chain. Thus, Lemma 3.10 yields that

$$\pi(k+1) = 1, \text{ that is, } k+1 = \pi^{-1}(1). \tag{3.6}$$

Next, to verify condition 3.1(i), assume that $\pi^{-1}(1) < i < j \le n$. That is, we assume that $k+1 < i < j \le n$. Since $\ell\mathrm{c}(D) = c_k < c_i$ and c_k is the only doubly irreducible element on the left boundary chain, the element c_i is join-reducible by G. Grätzer and E. Knapp [20, Lemma 3]. Hence, $[c_{i-1}, c_i]$ is a down-edge by Lemma 3.7. Thus, Lemma 3.8(i) yields that $\pi(i) < \pi(j)$, proving that π satisfies 3.1(i).

Next, let t be the height of $\mathrm{rc}(D)$. Again by [20, Lemma 3], d_j is join-reducible for all $t < j \le n$. Hence, for these j, no trajectory can arrive at $[d_{j-1}, d_j]$ from the upper left. On the other hand, c_{n-1} is meet-irreducible and $1 = c_n$ is join-reducible by [20, Lemma 3]. Hence, $[c_{n-1}, c_n]$ is a down-edge, and the trajectory T_n containing this edge goes downwards by Lemma 2.3. Hence, T_n arrives at the right boundary chain from the upper left. Consequently, it cannot arrive at $[d_{j-1}, d_j]$ if $t < j$, and we conclude that $\pi(n) \le t$. If we interchange $\langle \mathrm{left}, \pi, k\rangle$ and $\langle \mathrm{right}, \pi^{-1}, t\rangle$ in the argument proving (3.6), we obtain that $\pi(1) = t+1$. Consequently, 3.1(iii) holds.

Similarly, interchanging $\langle \text{left}, \pi\rangle$ and $\langle \text{right}, \pi^{-1}\rangle$ in the proof of 3.1(i), we obtain that 3.1(ii) holds. Therefore, if D is rectangular, then so is π_D.

Next, to prove the "if" part of Proposition 3.3, assume that $D \in \mathrm{SSmodD}(n)$ but $D \notin \mathrm{SRectD}(n)$. We have to prove that $\pi = \pi_D$ is not rectangular.

First, we assume that D has a nontrivial narrows v. Since $v \in \mathrm{C}_\ell(D) \cap \mathrm{C}_\mathrm{r}(D)$, it is of the form $v = c_s = d_s$ for some $s \in \{1, \dots, n-1\}$. Let T_1' denote the trajectory of the subdiagram $\downarrow v$ that begins with the prime interval $[c_0, c_1]$ of the left boundary chain. It reaches the right boundary of $\downarrow v$ at some $[d_{i-1}, d_i]$, where $i \le s$. Clearly, T_1' is also a trajectory of D, and we obtain that $\pi(1) = i \le s$. The dual argument shows that $\pi(n) \ge s$. (Note, however, the concept of slim rectangular lattices is not selfdual.) Hence, 3.1(iii) fails and π is not rectangular.

Next, we can assume that D is glued sum indecomposable. Since $n \ge 2$, we conclude that 0 is meet-reducible and 1 is join-reducible. By Lemma 2.7(iii), each of $\mathrm{C}_\ell(D)$ and $\mathrm{C}_\mathrm{r}(D)$ has at least one doubly irreducible element. Since D is not rectangular, we obtain from G. Grätzer and E. Knapp [20, Lemma 6] that at least one of $\mathrm{C}_\ell(D)$ and $\mathrm{C}_\mathrm{r}(D)$ has at least two doubly irreducible elements. Note that if we reflect D to a vertical axis, then π turns into π^{-1}. Thus, since the rectangularity of π is equivalent to that of π^{-1} by Remark 3.2, we can assume that, with notation (2.1), there are $1 \le i < j < n$ such that c_i and c_j are the smallest and the largest double irreducible elements that belong to $\mathrm{C}_\ell(D)$, respectively. We have that

$$\pi^{-1}(1) = i + 1 \tag{3.7}$$

by Lemma 3.10. To prove that π is not rectangular, we intend to show that 3.1(i) fails.

First of all, we show that $i + 1 < j$. Suppose, for a contradiction, that $j = i + 1$. Then $[c_i, c_j]$ is a prime interval. Let T denote the trajectory that begins with $[c_i, c_j]$. Since c_i is meet-irreducible, T cannot make its first step to the upper right. Similarly, it cannot make the first step to the lower right, since c_j is join-irreducibly. Thus, T makes no first step, and it consists only of $[c_i, c_j]$. By Lemma 2.3, $\{c_i, c_j\} \in \mathrm{C}_\mathrm{r}(D)$. Hence, c_i and c_j are nontrivial narrows of D, contradicting our assumption. This proves that $i + 1 < j$.

Next, let c_j' be as in Lemma 3.6, that is, $c_j = \ell\mathrm{c}(S)$ and $c_j' = \mathrm{rc}(S)$ for a unique 4-cell S. Since c_j is doubly irreducible, the subdiagram $D' = D \setminus \{c_j\}$ is a slim semimodular lattice diagram by Lemma 2.1. Similarly to (3.5), we have that $\{c_{j-1} = c_j \wedge c_j', c_j, c_j', c_{j+1} = c_j \vee c_j'\}$ is a 4-cell. Let T_{j+1} and T_j denote the trajectories of D beginning with $[c_j, c_{j+1}]$ and with $[c_{j-1}, c_j]$, respectively. Also, let T_{j+1}' and T_j' be the trajectories of D' through $[c_{j-1}, c_j']$ $[c_j', c_{j+1}]$, respectively. Clearly,

$$T_j = T_j' \cup \{[c_{j-1}, c_j]\} \quad \text{and} \quad T_{j+1} = T_{j+1}' \cup \{[c_j, c_{j+1}]\}. \tag{3.8}$$

By Lemma 3.7, the double irreducibility of c_j in D yields that $[c_{j-1}, c_j]$ is an up-edge and $[c_j, c_{j+1}]$ is a down-edge. Hence, by Lemma 2.3, T_{j+1} goes down, without any turn. This, together with (3.8), yields that T'_{j+1} is also a "down-going" trajectory of D'. Thus, either D' is indecomposable and $[c_{j-1}, c'_j]$ is a down-edge, or c'_j is a narrows of D'. In both cases, Lemma 3.8 implies that $\pi_{D'}(j) < \pi_{D'}(j+1)$. This inequality and (3.8) imply that

$$\pi_D(j+1) = \pi_{D'}(j) < \pi_{D'}(j+1) = \pi_D(j).$$

This, together with (3.7) and $i+1 < j$, shows that 3.1(i) fails. □

4. Recursive enumeration

For a rectangular permutation $\pi \in S_n$, we let

$$^{\mathrm{lu}}\mathrm{len}\,\pi = n+1-\pi^{-1}(1) \quad \text{and} \quad ^{\mathrm{ru}}\mathrm{len}\,\pi = n+1-\pi(1).$$

By Proposition 3.3, $^{\mathrm{lu}}\mathrm{len}\,\pi_D = {}^{\mathrm{lu}}\mathrm{len}\,D$ and $^{\mathrm{ru}}\mathrm{len}\,\pi_D = {}^{\mathrm{ru}}\mathrm{len}\,D$ hold for all $D \in \mathrm{SRectD}(n)$. For $2 \le n \in \mathbb{N}$ and $a, b \in \mathbb{N}$, we let

$$\mathrm{RPerm}(n) = \{\pi \in S_n : \pi \text{ is rectangular}\} \text{ and}$$
$$\mathrm{RPerm}(n; a, b) = \{\pi \in \mathrm{RPerm}(n) : {}^{\mathrm{lu}}\mathrm{len}\,\pi = a \text{ and } {}^{\mathrm{ru}}\mathrm{len}\,\pi = b\}.$$

It follows from Definition 3.1 that $\mathrm{RPerm}(n; a, b) \neq \varnothing$ iff $a + b \le n$.

Lemma 4.1. *For $a, b, n \in \mathbb{N}$ with $a + b \le n$,*

$$|\mathrm{RPerm}(n; a, b)| = \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,. \tag{4.1}$$

Proof. For $\pi \in \mathrm{RPerm}(n; a, b)$, we have $\pi^{-1}(1) = n - \big(n+1-\pi^{-1}(1)\big) + 1 = n-a+1$ and, similarly, $\pi(1) = n-b+1$. Since $\pi(n) < \pi(1)$ and $\pi^{-1}(n) < \pi^{-1}(1)$ by 3.1(iii) and 3.2(iv), conditions 3.1(i) and 3.1(ii) can be rephrased as follows:

$$\pi(n-a+1) = 1 < \pi(n-a+2) < \cdots < \pi(n) < n-b+1, \text{ and} \tag{4.2}$$
$$\pi^{-1}(n-b+1) = 1 < \pi^{-1}(n-b+2) < \cdots < \pi^{-1}(n) < n-a+1. \tag{4.3}$$

Conversely, if $\pi \in S_n$ satisfies (4.2) and (4.3), then $\pi \in \mathrm{RPerm}(n; a, b)$. The first and the second binomial coefficients in (4.1) show how many ways conditions (4.3) and (4.2) can be fulfilled, respectively. These conditions take care of the images of $a + b$ elements in $\{1, \ldots, n\}$. Hence, there are $(n-a-b)!$ possibilities for the rest of elements. □

From Lemmas 2.5 and 4.1 and Proposition 3.3, we immediately obtain that

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\le n\\ a,b\in\mathbb{N}}} |\mathrm{RPerm}(n; a, b)|. \tag{4.4}$$

Consequently, the following statement holds.

Proposition 4.2. *For $2 \le n \in \mathbb{N}$, the number of slim rectangular diagrams of length n is*

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\le n\\ a,b\in\mathbb{N}}} \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,.$$

The following lemma belongs to the folklore; see the first sentence in the proof of Proposition 7.13 in M. Bóna [2, page 256], or see G. Czédli, L. Ozsvárt, and B. Udvari [11, Lemma 6.1]. As usual, $(2t-1)!!$ denotes $1\cdot 3\cdot 5\cdot\dots\cdot(2t-1) = (2t)!/(2^t t!)$. Note that $(-1)!! = 1$ by definition. An *involution* is a permutation π such that $\pi^{-1} = \pi$. Let $\mathrm{Invl}(k) = \{\pi \in S_k : \pi = \pi^{-1}\}$ denote the set of involutions acting on the set $\{1, \dots, k\}$.

Lemma 4.3. *For $k \in \mathbb{N}$, the number of involutions in S_k is*

$$|\mathrm{Invl}(k)| = \sum_{j=0}^{\lfloor k/2\rfloor} \binom{k}{k-2j}\cdot(2j-1)!!\ . \tag{4.5}$$

Now, after that $|\mathrm{SRectD}(n)|$ has been determined by Proposition 4.2 and we also have Lemma 4.3, we formulate the following statement.

Proposition 4.4. *For $2 \le n \in \mathbb{N}$, the number of (isomorphism classes) of slim rectangular lattices of length n is*

$$|\mathrm{SRectL}(n)| = \frac{1}{2}\cdot\left(|\mathrm{SRectD}(n)| + \sum_{a=1}^{\lfloor n/2\rfloor}\binom{n-a-1}{a-1}\cdot|\mathrm{Invl}(n-2a)|\right). \tag{4.6}$$

Proof. By Lemmas 2.5 and 2.6, two distinct slim rectangular diagrams, D_1 and D_2, determine the same rectangular lattice iff $\pi_{D_1} = (\pi_{D_2})^{-1}$. Hence, if we count every involution twice and any other permutation once, then we count each lattice in question twice, that is,

$$\begin{aligned} 2\cdot|\mathrm{SRectL}(n)| &= |\mathrm{RPerm}(n)\setminus\mathrm{Invl}(n)| + 2\cdot|\mathrm{RPerm}(n)\cap\mathrm{Invl}(n)|\\ &= |\mathrm{RPerm}(n)| + |\mathrm{RPerm}(n)\cap\mathrm{Invl}(n)|\\ &= |\mathrm{SRectD}(n)| + |\mathrm{RPerm}(n)\cap\mathrm{Invl}(n)|. \end{aligned} \tag{4.7}$$

Therefore, to obtain (4.6), it suffices to prove that

$$|\mathrm{RPerm}(n)\cap\mathrm{Invl}(n)| = \sum_{a=1}^{\lfloor n/2\rfloor}\binom{n-a-1}{a-1}\cdot|\mathrm{Invl}(n-2a)|. \tag{4.8}$$

The argument we need is similar to the one used in the proof of Lemma 4.1. If $\pi = \pi^{-1}$, then $a = b \le n/2$. Hence, an involution π is in $\mathrm{RPerm}(n)$ iff it satisfies (4.2) with $b = a$. There are $\binom{n-a-1}{a-1}$ ways to select the values $\pi(n-a+2) < \dots < \pi(n)$ from $\{2, \dots, n-a\}$. Since π is an involution, each of these selections determines the action of π on the $2a$-element set set

$$\begin{aligned} \{1 = \pi(n-a+1) < \pi(n-a+2) < \dots < \pi(n)\\ < \pi(1) = \pi^{-1}(1) = n-a+1 < n-a+2 < \dots < n\}. \end{aligned}$$

Clearly, π acts as an involution on the $n-2a$ remaining elements. Hence, there are $|\mathrm{Invl}(n-2a)|$ ways to continue the above-mentioned selection to an involution on the whole set $\{1,\dots,n\}$. Finally, $2a = a+b \le n$ gives that $a \le \lfloor n/2 \rfloor$, and we conclude (4.8). □

The situation for slim patch lattices is much easier.

Proposition 4.5. *For $2 \le n \in \mathbb{N}$, the number of (isomorphism classes) of slim patch lattices of length n is* $|\mathrm{SPatchL}(n)| = \big((n-2)! + |\mathrm{Invl}(n-2)|\big)/2$.

Proof. A permutation π from Corollary 3.4 is an involution iff so is its restriction to $\{2,\dots,n-2\}$. Hence, using the idea of (4.7) with "patch" instead of "rectangular", we can obviously conclude our statement from Lemma 2.5 and Corollary 3.4 . □

5. Asymptotic results

For $\mathbb{N} \to \{x \in \mathbb{R} : x > 0\}$ functions f and g, we say that f *is asymptotically* g, if $f(n)/g(n)$ tends to 1 as $n \to \infty$. We denote by $f(n) \sim g(n)$, or sometimes by $\lim_{n\to\infty}\big(f(n)/g(n)\big) = 1$, that f is asymptotically g. In this section, a and b always denote positive integers. Hence, we will not indicate $a, b \in \mathbb{N}$ in range specifications. As usual, e denotes $\sum_{k=0}^{\infty}(k!)^{-1} \approx 2.7182818285$.

Proposition 5.1. *The number of slim rectangular diagrams of length n is asymptotically $(n-2)!\cdot e^2$, that is,* $|\mathrm{SRectD}(n)| \sim (n-2)!\cdot e^2$.

Proof. Based on (4.1), we can compute as follows.

$$\begin{aligned}
|\mathrm{RPerm}(n;a,b)| &= \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)! \\
&= \frac{(n-a-1)\cdots(n-a-b+1)}{(b-1)!}\cdot\frac{(n-b-1)\cdots(n-a-b+1)}{(a-1)!} \\
&\quad\times (n-a-b)! \\
&= \frac{(n-a-1)\cdots(n-a-b+1)}{(b-1)!}\cdot\frac{(n-2)!}{(a-1)!\,(n-2)\cdots(n-b)} \\
&= \frac{(n-2)!}{(a-1)!\,(b-1)!}\cdot\frac{n-a-1}{n-2}\cdot\frac{n-a-2}{n-3}\cdots\frac{n-a-b+1}{n-b}\,. \qquad (5.1)
\end{aligned}$$

Denoting by $q(n,a,b)$ the product of the last $b-1$ factors in (5.1), that is, the product of all but the first factor. In particular, $q(n,a,1) = 1$. Hence,

$$|\mathrm{RPerm}(n;a,b)| = q(n,a,b)\cdot\frac{(n-2)!}{(a-1)!\,(b-1)!}. \qquad (5.2)$$

Since $1 \le a$, $q(n,a,b)$ is the product of factors not greater than 1. Hence, $q(n,a,b) \le 1$ and $|\mathrm{RPerm}(n;a,b)| \le (n-2)!\,((a-1)!\,(b-1)!)^{-1}$. Combining this estimate with (5.2) and using (4.4), we obtain that

$$\begin{aligned} |\mathrm{SRectD}(n)| &\overset{(4.4)}{=} \sum_{a+b\le n} |\mathrm{RPerm}(n;a,b)| \le \sum_{a+b\le n} \frac{(n-2)!}{(a-1)!\,(b-1)!} \\ &\le (n-2)! \cdot \sum_{a=1}^{\infty} \frac{1}{(a-1)!} \cdot \sum_{b=1}^{\infty} \frac{1}{(b-1)!} = (n-2)! \cdot e^2. \end{aligned} \tag{5.3}$$

Next, let ε be an arbitrary (small) positive real number. Since

$$\sum_{a=1}^{\lfloor n/2\rfloor} \frac{1}{(a-1)!} \cdot \sum_{b=1}^{\lfloor n/2\rfloor} \frac{1}{(b-1)!} \le \sum_{a+b\le n} \frac{1}{(a-1)!\,(b-1)!},$$

there exists an $r_1 \in \mathbb{N}$ such that

$$(1-\varepsilon)e^2 \le \sum_{a+b\le n} \frac{1}{(a-1)!\,(b-1)!} \quad \text{for all} \quad n \ge r_1. \tag{5.4}$$

Since each of the $b-1$ factors of $q(n,a,b)$ tends to 1 as $n \to \infty$ while a and b are fixed, and since there are finitely many pairs $(a,b) \in \{1,\dots,r_1\}^2$, there exists an $r_2 \in \mathbb{N}$ such that

$$1-\varepsilon \le q(n,a,b) \quad \text{for all } a \le r_1,\ b \le r_1, \text{ and } n \ge r_2. \tag{5.5}$$

By the previous achievements as indicated below, if n is an arbitrary integer greater than $r = \max\{r_1, r_2\}$, then

$$\begin{aligned} |\mathrm{SRectD}(n)| &\overset{(4.4)}{=} \sum_{a+b\le n} |\mathrm{RPerm}(n;a,b)| \\ &\overset{(5.2)}{=} (n-2)! \sum_{a+b\le n} \frac{q(n,a,b)}{(a-1)!\,(b-1)!} \ge (n-2)! \sum_{a+b\le r_1} \frac{q(n,a,b)}{(a-1)!\,(b-1)!} \\ &\overset{(5.5)}{\ge} (n-2)! \sum_{a+b\le r_1} \frac{1-\varepsilon}{(a-1)!\,(b-1)!} \overset{(5.4)}{\ge} (n-2)! \cdot (1-\varepsilon)^2 e^2. \end{aligned}$$

This and (5.3) imply Proposition 5.1, since $(1-\varepsilon)^2 \to 1$ as $\varepsilon \to 0$. □

Now, we are in the position to formulate and prove our main result.

Theorem 5.2. *The number of (the isomorphism classes of) slim rectangular lattices of length n is asymptotically $(n-2)! \cdot e^2/2$, that is,*

$$\lim_{n\to\infty} \frac{|\mathrm{SRectL}(n)|}{(n-2)! \cdot e^2/2} = 1.$$

Proof. If we divide (4.6) by $(n-2)! \cdot e^2/2$, then the theorem follows from Proposition 5.1, provided we can show that

$$\lim_{n\to\infty} \frac{f(n)}{(n-2)!} = 0, \text{ where } f(n) = \sum_{a=1}^{\lfloor n/2\rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)|. \tag{5.6}$$

Hence, it suffices to deal with (5.6). In order to prove it, recall from S. Chowla, I. N. Herstein, and W. K. Moore [3, Theorem 8] that

$$|\mathrm{Invl}(k)| \sim \frac{1}{\sqrt[4]{4e}} \cdot (k/e)^{k/2} \cdot e^{\sqrt{k}}. \tag{5.7}$$

Since $\sqrt{k} \le k/2$ for $k \ge 4$, (5.7) implies that

$$|\mathrm{Invl}(k)| \le k^{k/2}, \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{5.8}$$

Stirling's formula, $k! \sim \sqrt{2\pi k} \cdot (k/e)^k$, implies that

$$(k/e)^k \le k! \le (k/e)^{k+1}\ , \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{5.9}$$

Denote $n-2$ by m, and assume that m is sufficiently large. Besides (5.8) and (5.9), the following obvious estimates are also needed below. Since the sum of the $\binom{m}{i}$ is 2^m, we have that $\binom{n-a-1}{a-1} \le 2^m$. Since $|\mathrm{Invl}(k)|$ is clearly an increasing function of k, we obtain that $|\mathrm{Invl}(n-2a)| \le |\mathrm{Invl}(m)|$. Clearly, $m \cdot 2^m \le 2^m \cdot 2^m = 4^m$ and $\lfloor n/2 \rfloor \le m$. Let us compute:

$$\frac{f(n)}{(n-2)!} = \sum_{a=1}^{\lfloor n/2\rfloor} \binom{n-a-1}{a-1} \cdot \frac{|\mathrm{Invl}(n-2a)|}{(n-2)!} \le \sum_{a=1}^{m} 2^m \frac{|\mathrm{Invl}(m)|}{m!}$$

$$= m \cdot 2^m \cdot \frac{|\mathrm{Invl}(m)|}{m!} \overset{(5.8, 5.9)}{\le} m \cdot 2^m \cdot \frac{m^{m/2}}{(m/e)^m} \tag{5.10}$$

$$\le 4^m \cdot \frac{(\sqrt{m})^m}{(m/e)^m} = \frac{1}{\left(\frac{\sqrt{m}}{4e}\right)^m} \to 0, \quad \text{as } n \to \infty. \tag{5.11}$$

This completes the proof. □

Remember that SSmodD(n) and SSmodL(n) denote the set of slim semimodular diagrams of length n and that of slim semimodular lattices of length n, respectively. In G. Czédli, L. Ozsvárt, and B. Udvari [11, Proposition 7.1], we proved that $|\mathrm{SSmodL}(n)| \sim n!/2$. This result, $(n-1)/n \sim 1$, Lemma 2.5, and Theorem 5.2 immediately yield the following statement.

Corollary 5.3.

$$\frac{|\mathrm{SRectD}(n)|}{|\mathrm{SSmodD}(n)|} \sim (e/n)^2 \quad \textit{and} \quad \frac{|\mathrm{SRectL}(n)|}{|\mathrm{SSmodL}(n)|} \sim (e/n)^2.$$

Next, we give the asymptotic number of slim patch lattices.

Proposition 5.4. *The number* $|\mathrm{SPatchL}(n)|$ *of* (*the isomorphism classes of*) *slim patch lattices of length* n *is asymptotically* $(n-2)!/2$.

Proof. It follows from (5.10) and (5.11) that

$$|\mathrm{Invl}(n-2)|/((n-2)!) = |\mathrm{Invl}(m)|/(m!) \to 0 \text{ as } n \to \infty.$$

This and Proposition 4.5 imply the statement. □

n	2	3	4	5	6	7	8	9	10	11	12
$\vert\mathrm{SRectD}(n)\vert$	1	3	9	32	139	729	4 515	32 336	263 205	2 401 183	24 275 037
$\vert\mathrm{SRectL}(n)\vert$	1	2	6	19	78	387	2 327	16 384	132 336	1 203 145	12 146 959
$\vert\mathrm{SPatchL}(n)\vert$	1	1	2	5	17	73	398	2 636	20 542	182 750	1 819 148

Computational results for $2 \le n \le 12$

n	200	400	1000
$\vert\mathrm{SRectD}(n)\vert$	$1.4568041 \cdot 10^{371}$	$2.5975960 \cdot 10^{1403}$	$2.9732576 \cdot 10^{2562}$
$\vert\mathrm{SRectL}(n)\vert$	$7.2840205 \cdot 10^{370}$	$1.2987980 \cdot 10^{1403}$	$1.4866288 \cdot 10^{2562}$
$\vert\mathrm{SPatchL}(n)\vert$	$9.9077622 \cdot 10^{369}$	$1.7606738 \cdot 10^{1402}$	$2.0139503 \cdot 10^{2561}$
$\dfrac{\vert\mathrm{SRectL}(n)\vert}{(n-2)! \cdot e^2/2}$	0.99496227	0.99832914	0.99899847

Computational results for $n \in \{200, 600, 1000\}$

6. Results by computer algebra

Based on Propositions 4.2 and 4.4, $|\mathrm{SSmodD}(n)|$ and $|\mathrm{SSmodL}(n)|$ can easily be determined for $n \le 1000$ by computer algebra. Appropriate programs (Maple 5) are available from the authors' web sites. Using a five year old personal computer with Intel Duo CPU 3.00 GHz, 1.98 GHz, and 3.25 GB RAM, these numbers for $n \le 12$, given in the first table, were computed in less than 0.1 seconds. The second table and the exact values for all $n \in \{2, \dots, 100, 200, 600, 1000\}$, available from the authors' web sites, were obtained in 16 minutes. For comparison, note that it took six days on a parallel supercomputer to determine the number of all 18-element lattices in 2001, see Heitzig and Reinhold [28]. Our computer algebraic calculations show that $|1 - |\mathrm{SPatchL}(n)|/((n-2)!/2)|$ and $|1/2 - |\mathrm{SRectL}(n)|/|\mathrm{SRectD}(n)||$ are smaller than 10^{-40} for $n \in \{64, \dots, 100, 200, 600, 1000\}$. This fact and the second table indicate (but do not prove) that the convergence in Proposition 5.4 is much faster than that in Proposition 5.1 and Theorem 5.2.

References

[1] Abels, H.: The gallery distance of flags. Order **8**, 77–92 (1991)

[2] M. Bóna, Combinatorics of permutations. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL (2004)

[3] Chowla, S., Herstein, I.N., Moore, W.K.: On recursions connected with symmetric groups I. Canadian J. Math. **3**, 328-334 (1951)

[4] Czédli, G.: Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices. Algebra Universalis **67**, 313–345 (2012)

[5] Czédli, G.: Patch extensions and trajectory colorings of slim rectangular lattices. Algebra Universalis (in press)

[6] Czédli, G.: The asymptotic number of planar, slim, semimodular lattice diagrams. Order (submitted); `arXiv:1206.3679`

[7] Czédli, G.: Coordinatization of join-distributive lattices. Algebra Universalis (in press)

[8] Czédli, G., Dékány, T., Ozsvárt, L., Szakács, N., Udvari, B.: On the number of slim, semimodular lattices. Mathematica Slovaca (submitted); arXiv:1208.6173
[9] Czédli, G., Grätzer, G.: Notes on planar semimodular lattices. VII. Resections of planar semimodular lattices. Order, DOI 10.1007/s11083-012-9281-1 (published online December 12, 2012)
[10] Czédli, G., Grätzer, G.: Planar Semimodular Lattices and Their Diagrams, in G. Grätzer, F. Wehrung, Lattice Theory: Special Topics and Applications (in preparation)
[11] Czédli, G., Ozsvárt, L., Udvari, B.: How many ways can two composition series intersect?. Discrete Mathematics **312**, 3523–3536 (2012)
[12] Czédli, G., Schmidt, E.T.: How to derive finite semimodular lattices from distributive lattices?. Acta Mathematica Hungarica, **121**, 277–282 (2008)
[13] Czédli, G., Schmidt, E.T.: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices, Algebra Universalis 66, 69–79 (2011)
[14] Czédli, G., Schmidt, E.T.: Slim semimodular lattices. I. A visual approach. Order **29**, 481–497 (2012)
[15] Czédli, G., Schmidt, E.T.: Slim semimodular lattices. II. A description by patchwork systems, Order **30**, 689–721 (2013)
[16] Czédli, G., Schmidt, E.T.: Composition series in groups and the structure of slim semimodular lattices. Acta Sci. Math.(Szeged) (in press)
[17] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser, Basel (2011)
[18] Grätzer, G.: Notes on planar semimodular lattices. VI. On the structure theorem of planar semimodular lattices. Algebra Universalis **69**, 301–304 (2013)
[19] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices. I. Construction. Acta Sci. Math. (Szeged) **73**, 445–462 (2007)
[20] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices. III. Congruences of rectangular lattices. Acta Sci. Math. (Szeged) **75**, 29–48 (2009)
[21] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices. Acta Sci. Math. (Szeged), **76**, 3–26 (2010)
[22] Grätzer, G., Nation, J.B.: A new look at the Jordan-Hölder theorem for semimodular lattices. Algebra Universalis **64**, 309–311 (2010)
[23] Grätzer, G., Quackenbush, R.W.: The variety generated by planar modular lattices. Algebra Universalis **63**, 187–201 (2010)
[24] Kelly, D., Rival, I.: Planar lattices. Canad. J. Math. **27**, 636–665 (1975)
[25] Schmidt, E.T.: Congruence lattices and cover preserving embeddings of finite length semimodular lattices. Acta Sci. Math. Szeged **77**, 47–52 (2011)
[26] Stanley, R.P.: Supersolvable lattices. Algebra Universalis **2**, 197–217 (1972)
[27] Erné, M., Heitzig, J., Reinhold, J.: On the number of distributive lattices. Electron. J. Combin. **9/1** Research Paper 24, 23 pp (2002)
[28] Heitzig, J., Reinhold, J.: Counting finite lattices Algebra Universalis **48** , 43–53 (2002)
[29] Pawar, M.M., Waphare, B.N.: Enumeration of nonisomorphic lattices with equal number of elements and edges. Indian J. Math. **45** , 315–323 (2003)

Gábor Czédli

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~czedli/

Tamás Dékány

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: tdekany@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~tdekany/

Gergő Gyenizse

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: gergogyenizse@gmail.com

Júlia Kulin

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: kulin@math.u-szeged.hu