THE STRUCTURE OF POLYNOMIAL OPERATIONS ASSOCIATED WITH SMOOTH DIGRAPHS

GERGŐ GYENIZSE, MIKLÓS MARÓTI AND LÁSZLÓ ZÁDORI

Abstract. With every digraph we associate an algebra whose fundamental operations are the polymorphisms of the digraph. In a 2012 paper the second and third authors proved that the digraph of endomorphisms of any finite connected reflexive digraph is connected, provided that the algebra associated with the digraph lies in a congruence join-semidistributive over modular variety. In the same paper, this connectivity result led to a proof of the statement that, if the algebra associated with a finite reflexive digraph generates a congruence modular variety, then the digraph has a near-unanimity polymorphism.

A digraph is smooth, if it has no sinks and no sources. Smooth digraphs of algebraic length 1 are a broad generalization of reflexive digraphs. In a 2009 paper, Barto et al. proved that every finite smooth digraph of algebraic length 1 whose associated algebra lies in a congruence meet-semidistributive over modular variety has a loop edge. This is a powerful theorem that has nice applications in algebra and computer science.

In the present paper we prove that the digraph of unary polynomial operations of the algebra associated with a finite smooth connected digraph of algebraic length 1 is connected, provided that the algebra lies in a congruence join-semidistributive over modular variety. This generalizes our connectivity result mentioned above and implies a restricted version of the result of Barto et al. in the congruence join-semidistributive over modular case. We also give a characterization of locally finite idempotent congruence join-semidistributive over modular varieties via smooth compatible digraphs of algebraic length 1.

It remains as an open question whether the congruence modularity of the variety generated by the algebra associated with a finite smooth digraph of algebraic length 1 implies the existence of a near-unanimity polymorphism of the digraph.

1. Introduction

First, we require the definition of exponentiation for relational structures. Let $\mathcal{R}$ be a fixed signature of relational symbols. Let $\mathbb{A} = (A; \mathcal{R})$ and

Date: July 25, 2014.

Key words and phrases. Smooth digraphs of algebraic length 1, connectivity, digraph of unary polynomial operations, twin relation, twin congruence, congruence join-semidistributivity (over modular), congruence meet-semidistributivity (over modular), congruence modularity, near unanimity polymorphism.

The authors' research was supported by the European Union and co-funded by the European Social Fund under the project Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences of project number TAMOP-4.2.2.A-11/1/KONV-2012-0073, and by OTKA grants K83219 and K104251.

$\mathbb{B} = (B; \mathcal{R})$ be similar relational structures. For a set $F \subseteq B^A$ of maps we define the relational structure $\mathbb{F} = (F; \mathcal{R})$ as follows. For any k-ary relational symbol $\varrho \in \mathcal{R}$ and maps $f_1, \dots, f_k : A \to B$

$$(f_1, \dots, f_k) \in \varrho_{\mathbb{F}} \text{ iff } (a_1, \dots, a_k) \in \varrho_{\mathbb{A}} \implies (f_1(a_1), \dots, f_k(a_k)) \in \varrho_{\mathbb{B}}.$$

In particular, by $\mathbb{B}^{\mathbb{A}}$ we mean the relational structure of all maps from A to B.

At this point we have to warn the reader that in the literature $\mathbb{B}^{\mathbb{A}}$ is sometimes used to denote the structure of the homomorphisms from $\mathbb{A}$ to $\mathbb{B}$, e.g., this is the case in our paper [9]. We opted for the definition of exponentiation given here, since it is general enough to unify the other notions, and the usual properties of exponentiation remain valid for it, as seen below.

In the present paper, we let $\mathrm{Hom}(\mathbb{A}, \mathbb{B})$ denote the structure of all homomorphisms from $\mathbb{A}$ to $\mathbb{B}$. Note that $\mathrm{Hom}(\mathbb{A}, \mathbb{B})$ contains precisely the maps $f \in \mathbb{B}^{\mathbb{A}}$ for which

$$(f, \dots, f) \in \varrho_{\mathbb{B}^{\mathbb{A}}}$$

for all relational symbols $\varrho \in \mathcal{R}$. For a set A let $\mathbb{I}_A = (A; \mathbb{R})$ be the relational structure with the (diagonal) relations

$$\varrho_{\mathbb{I}_A} = \{\, (a, \dots, a) \in A^k \mid a \in A \,\}$$

for all relational symbols $\varrho \in \mathcal{R}$. It is also easy to see that the n-fold Cartesian power $\mathbb{A}^n$ of $\mathbb{A}$ is precisely $\mathbb{A}^{\mathbb{I}_{\{1,\dots,n\}}}$. For similar structures $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ we have

$$(\mathbb{C}^{\mathbb{B}})^{\mathbb{A}} = \mathbb{C}^{\mathbb{B} \times \mathbb{A}}, \quad \mathbb{C}^{\mathbb{A}} \times \mathbb{B}^{\mathbb{A}} = (\mathbb{C} \times \mathbb{B})^{\mathbb{A}}, \quad \text{and}$$

the composition map $\circ : \mathbb{C}^{\mathbb{B}} \times \mathbb{B}^{\mathbb{A}} \to \mathbb{C}^{\mathbb{A}}$ defined as $(f \circ g)(a) = f(g(a))$ is a homomorphism.

The elements of $\mathrm{Hom}(\mathbb{A}^n, \mathbb{A})$ are called *n-ary polymorphisms*. Unary polymorphisms are called *endomorphisms*. Let $\mathrm{End}(\mathbb{A}) = \mathrm{Hom}(\mathbb{A}, \mathbb{A})$. With every relational structure $\mathbb{A}$ we associate an algebra denoted by $\mathrm{Alg}(\mathbb{A})$ whose underlying set is A and fundamental operations are the polymorphisms of $\mathbb{A}$. For an algebra $\mathbf{A}$, let $\mathrm{Pol}_n(\mathbf{A})$ denote the set of n-ary polynomial operations. Notice that $\mathrm{Pol}_n(\mathbf{A})$ has an algebraic structure, more precisely, $\mathrm{Pol}_n(\mathbf{A})$ is the subalgebra of $\mathbf{A}^{A^n}$ generated by the n-ary constant operations and the n-ary projection operations of A. Clearly, $\mathrm{End}(\mathbb{A}) \subseteq \mathrm{Pol}_1(\mathrm{Alg}(\mathbb{A})) \subseteq \mathbb{A}^{\mathbb{A}}$. Hence both $\mathrm{End}(\mathbb{A})$ and $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{A}))$ are relational structures with the relations inherited from $\mathbb{A}^{\mathbb{A}}$. At the same time, they both are subalgebras of $\mathrm{Alg}(\mathbb{A})^A$. In the present paper, this two-sided feature of $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{A}))$ stands in the center of our investigations on the structure of algebras associated with finite digraphs.

A *digraph* is a relational structure $\mathbb{G} = (G; \to)$ where $\to \subseteq G^2$. The *induced subdigraph* of $\mathbb{G}$ on the subset $A \subseteq G$ is the digraph $(A; \to \cap A^2)$. A digraph $\mathbb{G}$ is called *connected* if for any two elements $a, b \in G$ there exists an oriented path $a = a_0 \to a_1 \leftarrow \cdots \to a_n = b$ in $\mathbb{G}$ of length $n \geq 0$ where the arrows can point in either way. The *components* of a digraph $\mathbb{G}$ are the maximal connected induced subdigraphs of $\mathbb{G}$. The digraph $\mathbb{G}$ is called *smooth*, if the binary relation $\to \subseteq G^2$ is subdirect, i.e., each vertex has at least one incoming and one outgoing edge. All of the one-element digraphs are connected, but only the ones that have a loop are smooth. The *algebraic length* of an oriented path is the number of forward edges minus the number of backward edges. The *algebraic length* of a connected digraph is the smallest of the positive algebraic lengths of closed paths. It is easy to see that the algebraic length of a connected digraph equals the greatest common divisor of all positive algebraic lengths of oriented closed paths.

A *variety* is a class of all algebras of the same signature that satisfy a set of identities. In the following definitions, let P stand for the lattice property join-semidistributive, meet-semidistributive or modular. We say that *a variety is congruence* P, if the congruence lattice of any algebra in the variety is P. We say that *a variety is congruence* P *over modular*, if the congruence lattice $\mathbf{L}$ of any algebra in the variety has a lattice congruence α such that the quotient lattice $\mathbf{L}/\alpha$ is P and all α-blocks are modular lattices. In [4] Hobby and McKenzie elaborated the foundations of the tame congruence theory and used their theory to give various characterizations of certain classes of locally finite varieties based on the shape of congruence lattices of the algebras in the varieties. The classes we deal with in this paper form a poset with respect to containment as displayed in Figure 1.

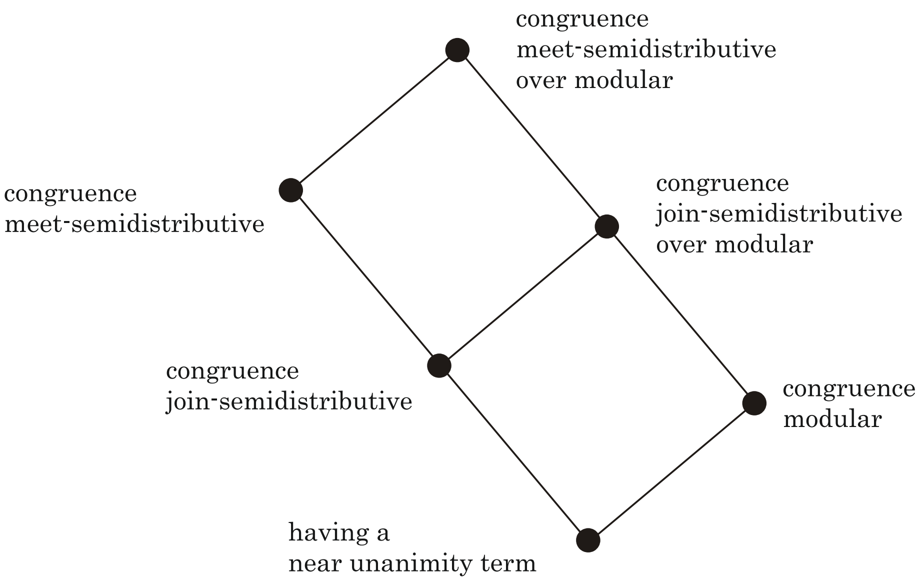

FIGURE 1. A poset of some classes of locally finite varieties.

A n-ary term f is called an *idempotent term* with respect to an algebra or a variety if it satisfies the identity:

$$f(x, x, \ldots, x) = x.$$

We remark that, by one of the characterizations in [4], the top element of the poset in Figure 1 consists of the locally finite varieties that obey non-trivial sets of idempotent identities (identities involving only idempotent terms).

An algebra or a variety is *idempotent* if all terms are idempotent with respect to it. An n-ary term f is a *near unanimity term* with respect to an algebra or a variety if $n \geq 3$ and f satisfies the identities

$$f(y,x,\dots,x) = f(x,y,x,\dots) = \cdots = f(x,\dots,x,y) = x$$

in two variables x and y. A *majority term* is a ternary near unanimity term.

Theorem 2.6 in [9] obtained by the second and third authors states that the digraph of endomorphisms of any finite connected reflexive digraph is connected, provided that the algebra associated with the digraph generates a congruence join-semidistributive over modular variety. In [9], this connectivity result led to a proof of the statement that, if the algebra associated with a finite reflexive digraph generates a congruence modular variety, then the digraph has a near-unanimity polymorphism.

The goal of the present paper is to extend the above connectivity result from reflexive digraphs to smooth digraphs of algebraic length 1. Even to state the generalization is not straightforward, in the sense that there are finite smooth connected digraphs of algebraic length 1 where the digraph of endomorphisms is disconnected, even when the digraph has a majority polymorphism. Such an example is depicted in Figure 2 in the next section. Nevertheless, we shall prove that connectivity is inherited for the digraph of polynomial operations, see Corollary 9 in the next section. The proof will be much more involved than in the reflexive digraph case.

In [3] (for another proof see [2]), Barto et al. proved that every finite smooth digraph of algebraic length 1 whose associated algebra lies in a congruence meet-semidistributive over modular variety has a loop edge. This powerful theorem, often called the Loop Lemma, has nice applications in algebra and computer science, see e.g. [10], [6] and [2]. We shall see that the Loop Lemma restricted to the join-semidistributive over modular case is an easy consequence of our new connectivity result. We also use our main result to obtain a characterization of locally finite idempotent congruence join-semidistributive over modular varieties via smooth compatible digraphs of algebraic length 1.

It remains as an open question whether the congruence modularity of the variety generated by the algebra associated with a finite smooth digraph of algebraic length 1 implies the existence of a near-unanimity polymorphism of the digraph. We remark that the question has positive answers in two important special cases, apart from our reflexive digraph result in [9]: Barto settled the congruence distributive case in [1] and Kazda did the congruence permutable case in [5]. Note that the results of Kazda and Barto do not require smoothness and algebraic length 1 of the digraph. On the other hand, as we noted in the Concluding Remarks of [9], there are examples which yield a negative answer if we drop smoothness and algebraic length 1 in the above question.

2. Results

First, we point out that our connectivity result stated for reflexive digraphs in [9] does not hold for smooth digraphs of algebraic length 1. Indeed, the smooth digraph

$$\mathbb{G} = (\{0,1\}^2; \{((a,b),(b,c)) : \ a,b,c \in \{0,1\}\})$$

of algebraic length 1 in Figure 2 has a majority polymorphism that acts componentwise, and the digraph $\mathrm{End}(\mathbb{G})$ of endomorphisms is easily seen to be disconnected as $\{\mathrm{id}\}$ is a component of it.

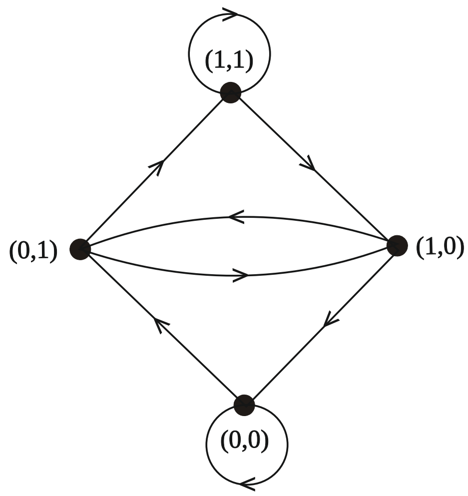

FIGURE 2. A smooth connected digraph $\mathbb{G}$ of algebraic length 1 such that $\mathbb{G}$ has a majority polymorphism and $\mathrm{End}(\mathbb{G})$ induces a disconnected subdigraph of $\mathbb{G}^{\mathbb{G}}$.

The example shows that, in general, the digraph $\mathrm{End}(\mathbb{G})$ of endomorphisms may not be large enough to induce a connected subdigraph of $\mathbb{G}^{\mathbb{G}}$ for a smooth digraph $\mathbb{G}$ of algebraic length 1. We shall replace $\mathrm{End}(\mathbb{G})$ by the larger digraph $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ of polynomial operations that, at least if $\mathrm{Alg}(\mathbb{G})$ lies in a congruence join-semidistributive over modular variety, will work. Our goal in this section is to give a proof of this fact.

The notion of twin relation of polynomial operations plays a crucial role in our proof. For an algebra $\mathbf{A}$ we say that $p, q \in \mathrm{Pol}_n(\mathbf{A})$ are *twins* if there exists a term t of $n+m$ variables and constants $\bar{a}, \bar{b} \in A^m$ such that $p(\bar{x}) = t(\bar{x}, \bar{a})$ and $q(\bar{x}) = t(\bar{x}, \bar{b})$ for all $\bar{x} \in A^n$. The transitive closure τ of the twin relation is easily seen to be a congruence on the algebra $\mathrm{Pol}_n(\mathbf{A})$, which we call the *twin congruence*. Next, we prove two lemmas of algebraic nature on the twin congruence on $\mathrm{Pol}_1(\mathbf{A})$.

Lemma 1. *For every finite algebra* $\mathbf{A}$ *that generates a congruence join-semidistributive variety the twin congruence of* $\mathrm{Pol}_1(\mathbf{A})$ *coincides with the largest congruence.*

Proof. Let τ be the twin congruence on $\mathrm{Pol}_1(\mathbf{A})$. For an element $a \in A$ let $\pi_a : \mathrm{Pol}_1(\mathbf{A}) \to \mathbf{A}$ be the projection defined by $\pi_a(p) = p(a)$, and let η_a be the kernel of π_a. Fix two polynomial operations $p, q \in \mathrm{Pol}_1(\mathbf{A})$ and an element $a \in A$. Let $b = p(a)$, $c = q(a)$ and denote by $\hat{b}$ and $\hat{c}$ the constant maps. Then

$$p \; \eta_a \; \hat{b} \; \tau \; \hat{c} \; \eta_a \; q.$$

This proves that $\tau \vee \eta_a = 1$ for all $a \in A$. By repeatedly applying the join-semidistributive congruence identity

$$\alpha \vee \beta_1 = \alpha \vee \beta_2 \implies \alpha \vee \beta_1 = \alpha \vee (\beta_1 \wedge \beta_2)$$

in the congruence lattice of $\mathrm{Pol}_1(\mathbf{A})$, we get that

$$\tau = \tau \vee 0 = \tau \vee (\bigwedge_{a \in A} \eta_a) = 1.$$

□

In Section 7 of [4], Hobby and McKenzie define the notions of solvable algebras and the solvability congruence of the congruence lattice of an algebra. They prove also that if $\mathbf{A}$ is a finite algebra in a congruence join-semidistributive over modular variety, then modding out the congruence lattice of $\mathbf{A}$ by the solvability congruence, the resulting lattice is join-semidistributive, cf. item (3) of Theorem 7.7 and Theorem 9.8 in [4].

Lemma 2. *Let $\mathbf{A}$ be a finite algebra in a congruence join-semidistributive over modular variety, and let τ be the twin congruence on $\mathrm{Pol}_1(\mathbf{A})$. Then $\mathrm{Pol}_1(\mathbf{A})/\tau$ is a solvable algebra.*

Proof. We prove that τ and 1 are related by the solvability congruence. This yields immediately, by the definitions of solvable algebras and the solvability congruence that $\mathrm{Pol}_1(\mathbf{A})/\tau$ is a solvable algebra. The proof goes along the lines of the preceding proof. In this case, modding out the congruence lattice of $\mathrm{Pol}_1(\mathbf{A})$ with the solvability congruence yields a join-semidistributive lattice. By applying join semi-distributivity for the solvability congruence blocks of the congruences that occur in the preceding proof instead of doing it for the congruences themselves, we get that the solvability congruence blocks of τ and 1 are the same. □

We require the following well known and easy to prove lemma on digraphs.

Lemma 3. *Let $\mathbb{G}$ be a smooth digraph of algebraic length 1. If $\mathbb{G}$ is connected, then $\mathbb{G}^n$ is connected for all natural numbers n. Conversely, if $\mathbb{G}^n$ is connected for some n, then $\mathbb{G}$ is connected.* □

Next we prove some combinatorial lemmas on the twin congruence blocks of $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$, where $\mathbb{G}$ is a finite smooth connected digraph of algebraic length 1.

Lemma 4. *Let $\mathbb{G}$ be a finite smooth connected digraph of algebraic length 1. Then each block of the twin congruence of $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ induces a connected subdigraph in $\mathbb{G}^{\mathbb{G}}$.*

Proof. Let us consider a pair p, q of twin unary polynomial operations of the algebra $\mathbf{G} = \mathrm{Alg}(\mathbb{G})$. By definition there exist a homomorphism $t \in \mathrm{Hom}(\mathbb{G}^{n+1}, \mathbb{G})$ and constants $\bar{a}, \bar{b} \in \mathbb{G}^n$ such that $p = t(x, \bar{a})$ and $q = t(x, \bar{b})$.

By Lemma 3 the digraph $\mathbb{G}^n$ is connected. Since $\mathbb{G}^{\mathbb{G}^{n+1}} = \mathbb{G}^{\mathbb{G}\times\mathbb{G}^n} = (\mathbb{G}^{\mathbb{G}})^{\mathbb{G}^n}$, we may regard t as a homomorphism from $\mathbb{G}^n$ to $\mathbb{G}^{\mathbb{G}}$. Thus t maps the path connecting $\bar{a}$ with $\bar{b}$ to a path connecting p and q. In fact, the elements of this path are all unary polynomial operations. $\square$

The *smooth part* of $\mathbb{G}$ is the unique maximal smooth induced subdigraph of $\mathbb{G}$. The *smooth components* of $\mathbb{G}$ are the components of its smooth part. We say that a unary map $r \in \mathbb{G}^{\mathbb{G}}$ is *idempotent* if $r^2 = r$, it is a *retraction* if it is idempotent and $r \in \mathrm{End}(\mathbb{G})$, and it is *proper* if $r \neq \mathrm{id}$. It is well known that for any unary map $f \in A^A$ on a finite set A there exists an integer m (we can uniformly choose $m = |A|!$) such that f^m is idempotent, i.e. $f^{2m} = f^m$. We will denote the idempotent iterate f^m of f by f^*. In the proof of the following lemma and in later proofs throughout the paper, we frequently use the fact that for any maps $f_1, f_2, g_2, g_2 \in \mathbb{G}^{\mathbb{G}}$ if $f_1 \to f_2$ and $g_1 \to g_2$ then $f_1 \circ g_1 \to f_2 \circ g_2$.

Lemma 5. *Let $\mathbb{G}$ be a finite digraph, and let $\mathbb{C}$ be the smooth component of* id *in the digraph* $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$*. If $\mathbb{C}$ contains a non-permutation, then it contains a proper retraction.*

Proof. Choose a path from id to a non-permutation in $\mathbb{C}$. In this path there exist a permutation g and a non-permutation f such that either $g \to f$ or $f \to g$. Without loss of generality we may assume that $g \to f$. By iterating, we obtain that $\mathrm{id} \to f^k$ for some k, where f^k is a non-permutation in C.

Choose a non-permutation map $f_n \in C$ such that there exists a path $\mathrm{id} \to f_1 \to \cdots \to f_n \to f_{n+1}$ in C and $f_n(G)$ is of minimal size. For $i \leq n+1$ put $g_i = f_1 \circ f_2 \circ \cdots \circ f_i$. Clearly, $\mathrm{id} \to g_1 \to \cdots \to g_n \to g_{n+1}$, and $G \supseteq g_1(G) \supseteq g_2(G) \supseteq \cdots \supseteq g_n(G) \supseteq g_{n+1}(G)$. Since $g_n = g_{n-1} \circ f_n$ and $f_n(G)$ is of minimal size, we have $|g_n(G)| = |f_n(G)|$ and $G \neq g_n(G) = g_{n+1}(G)$.

Let $h_i = g_i^*$ be the idempotent iterate of g_i. Thus we have $\mathrm{id} \to h_1 \to \cdots \to h_n \to h_{n+1}$ in C, $h_i^2 = h_i$ for all i, and $G \neq h_n(G) = h_{n+1}(G)$. In particular, both h_n and h_{n+1} are the identity on the set $h_n(G)$, so $h_{n+1} \circ h_n = h_n$. For $i \leq n+1$ put $t_i = h_i \circ h_{i-1} \circ \cdots \circ h_1$. Clearly $\mathrm{id} \to t_1 \to \cdots \to t_n \to t_{n+1}$, and $t_{n+1} = h_{n+1} \circ h_n \circ t_{n-1} = h_n \circ t_{n-1} = t_n$. Therefore, t_n is a non-permutation homomorphism, and the idempotent iterate of t_n is a proper retraction in $\mathbb{C}$. $\square$

Lemma 6. *Let $\mathbb{G}$ be a finite smooth connected digraph of algebraic length 1. If the twin congruence block of* id *in the algebra* $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ *contains a non-permutation, then it contains a proper retraction.*

Proof. Observe that the twin congruence block of id induces a smooth subdigraph of $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$, and by Lemma 4, this subdigraph is also connected. Moreover, composition of functions preserves the twin relation, hence it preserves the twin congruence block of id. Therefore, the proof of the preceding lemma translates into the proof of the present one by replacing $\mathbb{C}$ with the twin congruence block of id. $\square$

We now have all the tools at our disposal to prove the main theorem of the paper.

Theorem 7. *Let $\mathbb{G}$ be a finite smooth connected digraph of algebraic length 1 such that the variety generated by* $\mathrm{Alg}(\mathbb{G})$ *is congruence join-semidistributive over modular. Then the twin congruence coincides with the largest congruence on* $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$.

Proof. Let τ denote the twin congruence on $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$. If $\mathbb{G}$ has one element, then the claim is obvious. Let us assume that the claim is not true, and let $\mathbb{G}$ be of minimal size such that $\tau \neq 1$. Then, by Lemma 2, $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))/\tau$ is a solvable algebra. Let C be the τ-block of the identity map in $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$. Next, we prove that C has a non-permutation.

Let us suppose to the contrary that C contains only permutations. Since $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))/\tau$ is a finite solvable algebra in a congruence join-semidistributive over modular variety, Theorem 7.2 and item (3) of Theorem 7.11 in [4] yield that the variety generated by $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))/\tau$ is congruence permutable. So, there is a ternary term m in the language of $\mathrm{Alg}(\mathbb{G})$ such that m obeys the identities

$$m(\bar{f}, \bar{g}, \bar{g}) = m(\bar{g}, \bar{g}, \bar{f}) = \bar{f}$$

on $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))/\tau$. Hence for all g in $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ we have

$$m(\mathrm{id}_G, g, g)\ \tau\ m(g, g, \mathrm{id}_G)\ \tau\ \mathrm{id}_G.$$

Since $m(\mathrm{id}_G, g, g)\ \tau\ m(\mathrm{id}_G, g, h)$ for all constant polynomial operations g and h, we get that $\mathrm{id}_G\ \tau\ m(\mathrm{id}_G, g, h)$ for all constant polynomial operations g and h. So, $m(\mathrm{id}_G, g, h)$ is in C for all constant polynomial operations g and h. Similarly, $m(g, h, \mathrm{id}_G)$ is in C for all constant polynomial operations g and h. As C contains only permutations, $m(\mathrm{id}_G, g, h)$ and $m(g, h, \mathrm{id}_G)$ are permutations for all constant polynomial operations g and h, hence, by Lemma 2.10 of Kiss in [7], there is a Maltsev term for $\mathrm{Alg}(\mathbb{G})$. Now, by Kazda's result in [5] there is a majority term for $\mathrm{Alg}(\mathbb{G})$. This implies that the variety generated by $\mathrm{Alg}(\mathbb{G})$ is congruence join-semidistributive. Then, by Lemma 1, $\tau = 1$, a contradiction.

So C must have a non-permutation. By Lemma 6, C contains a proper retraction, say r. Since r is an endomorphism of $\mathbb{G}$, $r(\mathbb{G})$ is a smooth digraph of algebraic length 1. Moreover, the set of n-ary operations of $\mathrm{Alg}(r(\mathbb{G}))$ is of the form

$$\mathrm{Hom}(r(\mathbb{G})^n, r(\mathbb{G})) = \{rf|_{r(G)} :\ f \in \mathrm{Hom}(\mathbb{G}^n, \mathbb{G})\}.$$

By Theorem 9.8 in [4], the class of locally finite varieties that are congruence join-semidistributive over modular is characterized by the existence of certain linear identities. Linear identities are preserved under retraction, and so $\mathrm{Alg}(r(\mathbb{G}))$ generates a variety that is congruence join-semidistributive over modular. Then, by the minimality of $\mathbb{G}$, the twin congruence coincides with the largest congruence on $\mathrm{Pol}_1(\mathrm{Alg}(r(\mathbb{G})))$. Thus, $\mathrm{id}_{r(G)}$ is twin congruence related to a constant operation g of $r(\mathbb{G})$, that is, there is a sequence of polynomial operations $f_0, \dots, f_m$ in $\mathrm{Pol}_1(\mathrm{Alg}(r(\mathbb{G})))$ such that $f_0 = \mathrm{id}_{r(G)}$, $f_m = g$ and f_i is twin related to f_{i+1} for all i. So the sequence

$r = f_0 \circ r, \ldots, f_m \circ r = g \circ r$ witnesses the fact that r is τ-related to a constant operation in $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$. On the other hand id_G is τ-related to r and hence, by transitivity of τ, id_G is τ-related to a constant operation, and so $\tau = 1$. This contradiction concludes the proof. □

A repeated application of the previous theorem and lemma yields the following corollary.

Corollary 8. *Let* $\mathbb{G}$ *be a finite smooth connected digraph of algebraic length 1 such that the variety generated by* $\mathrm{Alg}(\mathbb{G})$ *is congruence join-semidistributive over modular. Then* $\mathbb{G}$ *has a loop.*

Proof. We prove the claim by induction on the size of G. If G has one element, then the claim obviously holds. Suppose that G has more than one element. By the previous theorem, the twin congruence block of the identity is the entire $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$. The constant operations are in $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$, so, by the previous lemma, $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ contains a proper retraction r. Notice that $r(\mathbb{G})$ is a finite smooth connected digraph of algebraic length 1 and the variety generated by $\mathrm{Alg}(r(\mathbb{G}))$ is congruence join-semidistributive over modular. Now, by the induction hypothesis, $r(\mathbb{G})$ has a loop, and so $\mathbb{G}$ also has a loop. □

Lemma 4 and Theorem 7 give the following corollary.

Corollary 9. *Let* $\mathbb{G}$ *be a finite smooth connected digraph of algebraic length 1 such that the variety generated by* $\mathrm{Alg}(\mathbb{G})$ *is congruence join-semidistributive over modular. Then* $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ *induces a connected subdigraph of* $\mathbb{G}^{\mathbb{G}}$*.*

Next, we provide an example of a finite smooth digraph $\mathbb{G}$ of algebraic length 1 such that $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ is not connected. We call a digraph $\mathbb{G}$ *dismantlable* if $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ is connected. The notion of dismantlability is well known for posets, the definition we gave here generalizes that notion. Note that posets, being reflexive, are smooth digraphs of algebraic length 1. It was checked in [8] that poset $\mathbb{P}$ depicted in Figure 3 is non-dismantlable. We also remark that in [8] it was proved that $\mathbb{P}$ has a semilattice polymorphism. Hence $\mathrm{Alg}(\mathbb{P})$ generates a variety that is not congruence join-semidistributive, but is congruence meet-semidistributive.

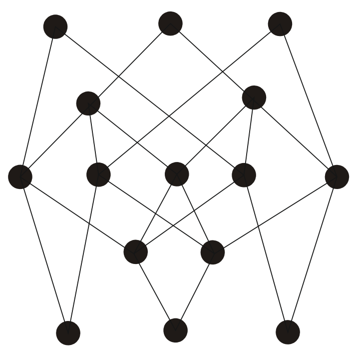

FIGURE 3. A non-dismantlable poset $\mathbb{P}$ with a semilattice polymorphism.

A *compatible structure* in a variety is relational structure $\mathbb{B}$ such that there is an algebra in the variety whose operations are polymorphisms of $\mathbb{B}$. Theorem 4.4 of [8] states that a locally finite idempotent variety is congruence join-semidistributive over modular if and only if every finite connected compatible poset in the variety is dismantlable. This theorem and the preceding corollary yield the following.

Corollary 10. *A locally finite idempotent variety is congruence join-semidistributive over modular if and only if every finite smooth connected compatible digraph of algebraic length 1 in the variety is dismantlable.*

Finally, we prove a connectivity result for finite smooth digraphs of algebraic length 1 whose associated algebras generate congruence modular varieties. Congruence modularity of varieties are characterized by an infinite sequence of finite sets of idempotent identities. The terms occurring in this characterization are called Gumm terms.

The ternary terms $d_0, \dots, d_n$, and p are called *Gumm terms* if they obey the identities

$$\begin{aligned} x &= d_0(x, y, z), \\ d_i(x, y, x) &= x \text{ for all } i, \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \text{ for even } i, \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \text{ for odd } i, \\ d_n(x, y, y) &= p(x, y, y), \text{ and} \\ p(x, x, y) &= y. \end{aligned}$$

Let $\mathrm{IPol}_k(\mathbf{A})$ denote the set of the k-ary idempotent polynomial operations of an idempotent algebra $\mathbf{A}$. Similarly to $\mathrm{Pol}_k(\mathbf{A})$, $\mathrm{IPol}_k(\mathbf{A})$ has a structure of both an algebra and a digraph. Let $\mathrm{IAlg}(\mathbb{G})$ be the full idempotent reduct of $\mathrm{Alg}(\mathbb{G})$, where $\mathbb{G}$ is a digraph.

Corollary 11. *Let $\mathbb{G}$ be a finite smooth connected digraph of algebraic length 1 such that* $\mathrm{Alg}(\mathbb{G})$ *generates a congruence modular variety. Then for every k the twin congruence on* $\mathrm{IPol}_k(\mathrm{IAlg}(\mathbb{G}))$ *equals with the largest congruence, and x and y are in the same connected component of* $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$.

Proof. By Theorem 7, there is a sequence $f_0, \dots, f_m$ of unary polynomial operations in $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ such that $f_0 = \mathrm{id}$, $f_m = \hat{c}$, where $\hat{c}$ is a constant operation and f_{i-1} and f_i are twins for all $i \geq 1$. Then the sequence

$$d_i(x, f_0(x), y), \dots, d_i(x, f_m(x), y) = d_i(x, f_m(y), y), \dots, d_i(x, f_0(y), y)$$

witnesses the fact that $d_i(x, x, y)$ and $d_i(x, y, y)$ are twin-connected in the algebra $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$ for all i. Moreover, $p(x, y, y)$ and $p(\hat{c}, \hat{c}, y)$ are twin-connected by the sequence

$$p(f_0(x), f_0(y), y), \dots, p(f_m(x), f_m(y), y).$$

Now, by applying the Gumm identities we obtain a path that twin-connects x and y in $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$. Since $\mathrm{Pol}_1(\mathrm{Alg}(\mathbb{G}))$ is connected, the consecutive elements in this path are connected by a path in the digraph $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$.

Hence, x and y are in the same connected component of $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$. To see that $\mathrm{IPol}_k(\mathrm{IAlg}(\mathbb{G}))$ is twin-connected just plug in all pairs $f, g \in \mathrm{IPol}_k(\mathrm{IAlg}(\mathbb{G}))$ for every occurrence of x and y in a path twin-connecting x and y in $\mathrm{IPol}_2(\mathrm{IAlg}(\mathbb{G}))$. $\square$

References

[1] L. Barto, *Finitely related algebras in congruence distributive varieties have near unanimity terms*, Canadian Journal of Mathematics **65**:1 (2013), 3–21.

[2] L. Barto, M. Kozik, *Absorbing subalgebras, cyclic terms and the constraint satisfaction problem*, Logical Methods in Computer Science **8**:1:07 (2012), 1–26.

[3] L. Barto, M. Kozik, T. Niven, *The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell)*, SIAM Journal on Computing **38**:5 (2009), 1782–1802.

[4] D. Hobby and R. McKenzie, *The structure of finite algebras*, Contemporary Mathematics,**76**, American Mathematical Society, Providence, RI, 1988.

[5] A. Kazda, *Maltsev digraphs have a majority polymorphism*, European Journal of Combinatorics **32** (2011), 390–397.

[6] K. A. Kearnes, P. Marković, and R. N. McKenzie, *Optimal strong Mal'cev conditions for omitting type 1 in locally finite varieties*, preprint.

[7] E. W. Kiss, *An easy way to minimal algebras*, Internat. J. Algebra Comput. **7** (1997), 55–75.

[8] B. Larose and L. Zádori, *Finite posets and topological spaces in locally finite varieties*, Algebra Universalis **52**:2-3 (2004), 119–136.

[9] M. Maróti and L. Zádori, *Reflexive digraphs with near-unanimity polymorphisms*, Discrete Mathematics **12**:15 (2012), 2316–2328.

[10] M. Siggers, *A Strong Mal'cev Condition for Varieties Omitting the Unary Type*, Algebra Universalis **64**:1 (2010), 15–20.

Bolyai Intézet, Aradi vértanúk tere 1, H-6720, Szeged, Hungary
E-mail address: gergogyenizse@gmail.com
E-mail address: mmaroti@math.u-szeged.hu
E-mail address: zadori@math.u-szeged.hu