CONGRUENCE PERMUTABILITY IS PRIME

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ABSTRACT. We give a combinatorial proof that congruence permutability is prime in the lattice of interpretability types of varieties. Thereby, we settle a 1984 conjecture of Garcia and Taylor.

1. INTRODUCTION

A *variety* or *equational class* is a class of all algebras (algebraic structures of a given signature) satisfying a given set of identities. Let Γ be a set of identities over a certain signature of a variety, and let $\mathscr{K}$ be a variety of some (possibly different) signature. We say that Γ *interprets in the variety* $\mathscr{K}$ if by replacing the operation symbols in Γ by term expressions of $\mathscr{K}$—same symbols by same terms with arities kept—the so obtained set of identities holds in $\mathscr{K}$. A *variety* $\mathscr{K}_1$ *interprets in a variety* $\mathscr{K}_2$ if there is a set of identities Γ that defines $\mathscr{K}_1$ and interprets in $\mathscr{K}_2$.

As easily seen, interpretability is a quasiorder on the class of varieties. The blocks of this quasiorder are called the *interpretability types*. In [1] Garcia and Taylor introduced the *lattice of interpretability types of varieties* that is obtained by taking the quotient of the class of varieties quasiordered by interpretability and the corresponding equivalence relation. The join in this lattice is described as follows. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two varieties of disjoint signatures, defined by the sets Σ_1 and Σ_2 of identities, respectively. Their *join* $\mathscr{K}_1 \vee \mathscr{K}_2$ is the variety defined by $\Sigma_1 \cup \Sigma_2$. The so defined join is compatible with the interpretability relation of varieties, and naturally yields the definition of the join operation in the lattice of interpretability types of varieties.

A *digraph* is a pair $\mathbb{G} = (G; E(\mathbb{G}))$ where G is a set and $E(\mathbb{G})$ is a binary relation on G. Here G is called the *vertex set* of $\mathbb{G}$ and $E(\mathbb{G})$ the *edge relation* of $\mathbb{G}$. A *reflexive (symmetric, transitive)* digraph is a digraph whose edge relation is reflexive (symmetric, transitive).

A *compatible digraph of an algebra* $\mathbf{A}$ is a digraph whose vertex set coincides with the base set of $\mathbf{A}$ and whose edge relation is preserved by all of the basic operations of $\mathbf{A}$. A *compatible digraph in a variety* is a compatible digraph of an algebra in the variety.

A *congruence* of an algebra $\mathbf{A}$ is an equivalence ρ on A such that $(A;\rho)$ is a compatible digraph of $\mathbf{A}$. An *algebra* $\mathbf{A}$ *is congruence permutable*, if for any two congruences α and β of $\mathbf{A}$, $\alpha\beta = \beta\alpha$. A *variety is congruence permutable* if all of its members are congruence permutable. For example, the varieties of groups, rings and vector spaces are congruence permutable. The following characterization of congruence permutable varieties is due to Maltsev, cf. [3].

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Theorem 1.1 (Maltsev (1954))**.** *Let $\mathscr{K}$ be a variety. Let $\mathbf{F}_2$ be the algebra freely generated by x and y in $\mathscr{K}$. Then the following are equivalent.*

(1) *$\mathscr{K}$ is congruence permutable.*
(2) *The reflexive digraph whose vertex set is F_2 and whose edge relation is the subalgebra generated by $\{(x,x),\ (x,y),\ (y,y)\}$ in $\mathbf{F}_2^2$ is symmetric.*
(3) *The set $\{m(x,y,y)=x,\ m(y,y,x)=x\}$ of identities where m is a ternary function symbol interprets in $\mathscr{K}$.*

A finite set of identities is called a *strong Maltsev condition*. The set of identities occurring in the third condition of the preceding theorem is a typical example of a strong Maltsev condition. The interpretability types that contain the varieties in which a strong Maltsev condition interprets constitute a principal filter in the lattice of interpretability types of varieties. So the interpretability types containing the congruence permutable varieties form a principal filter in the lattice of interpretability types by Maltsev's theorem. We say that a strong Maltsev condition is *prime* if the related principal filter is a prime filter (equivalently, the smallest element of this principal filter is join-prime). In [1], Garcia and Taylor formulated the conjecture that congruence permutability is prime. Formally, the strong Maltsev condition in item (3) of the above theorem is prime.

In 1996, Tschantz announced a proof of this conjecture. However, his proof has remained unpublished, cf. [5]. In the course of time, two partial results related to Garcia and Taylor's conjecture were published. In his PhD thesis [4], Sequeira proved that congruence permutability is prime with respect to varieties axiomatized by identities with terms of depth at most two. Kearnes and Tschantz gave a proof that congruence permutability is prime in the lattice of interpretability types of idempotent varieties, cf. Lemma 2.8 in [2]. In the present paper, we give a proof of the primeness of congruence permutability in its full generality.

2. A PROOF OF THE PRIMENESS OF CONGRUENCE PERMUTABILITY

A digraph is called a *complete digraph* if its edge relation is the full binary relation. In particular, complete digraphs are reflexive digraphs. The *complement of a digraph* $\mathbb{G}$ is the digraph whose vertex set is G and whose edge set is $G^2 \setminus E(\mathbb{G})$. Let $\mathbb{G}$ be a digraph and u a vertex of $\mathbb{G}$. Let $\mathbb{G}-u$ denote the digraph obtained by removing the vertex u and the edges incident with u from $\mathbb{G}$. We call u a *universal vertex* if for all $g \in G$, $u \leftrightarrow g$ in $\mathbb{G}$.

Let $\mathbb{H}_i$, $i \in I$, be digraphs. Their product $\prod_{i\in I} \mathbb{H}_i$ is the digraph whose vertex set is $\prod_{i\in I} H_i$ and whose edge relation is

$$\{\ (h,h') \in (\textstyle\prod_{i\in I} H_i)^2 :\ h(i) \to h'(i) \text{ in } \mathbb{H}_i \text{ for all } i \in I\ \}.$$

Let $\mathbb{G}$ and $\mathbb{H}$ be two digraphs. We define the digraph $\mathbb{G}^{\mathbb{H}}$ as follows. The vertex set of $\mathbb{G}^{\mathbb{H}}$ is the set of maps from the set H to the set G. The edge relation of $\mathbb{G}^{\mathbb{H}}$ is defined by

$$f \to f' \text{ if and only if } f(x) \to f'(y) \text{ in } \mathbb{G} \text{ for every } x \to y \text{ in } \mathbb{H}.$$

Observe that for any digraphs $\mathbb{G}$, $\mathbb{H}_1$ and $\mathbb{H}_2$, $\mathbb{G}^{\mathbb{H}_1\times\mathbb{H}_2} \cong \left(\mathbb{G}^{\mathbb{H}_1}\right)^{\mathbb{H}_2}$.

The heart of the proof of the main result of this paper is the following theorem on digraph powers.

Theorem 2.1. *For $1 \le i \le 2$, let $\mathbb{G}_i$ be a digraph with a universal vertex u_i and $\mathbb{G}_i^*$ the complement of the digraph $\mathbb{G}_i - u_i$. Let κ be an infinite cardinal where $\kappa \ge \max(|G_1|, |G_2|)$, and $\mathbb{K}$ a complete digraph of κ-many vertices. Then*

$$\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}} \cong \mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}.$$

Proof. When $\mathbb{G}_1$ or $\mathbb{G}_2$ is a one-element complete digraph, then both of the digraphs $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ and $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$ are one-element complete digraphs. So in what follows in this proof, we assume that $|G_1|, |G_2| \ge 2$. Let $G_i^* := G_i \setminus \{u_i\}$ for $1 \le i \le 2$.

For each vertex f of $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$, let

$$C(f) := \{\ (g_1, g_2) \in G_1^* \times G_2^* : \ \text{there exists } k \in K \text{ such that } f(g_2, k) = g_1\ \}.$$

First, we give a characterization of the edge relation of $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ by the use of the sets $C(f)$.

CLAIM 1. *Let $f, f' \in \mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$. The pair (f, f') is not an edge of the digraph $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ if and only if there exist $(g_1, g_2) \in C(f)$ and $(g_1', g_2') \in C(f')$ such that none of (g_1, g_1') and (g_2, g_2') are edges in $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively.*

Clearly, (f, f') is not an edge in $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ if and only if there exist $g_1,\ g_1' \in G_1^*$, $g_2,\ g_2' \in G_2^*$ and $k,\ k' \in K$ such that

$$f(g_2, k) = g_1,\ f(g_2', k') = g_1',$$

$(g_2, k) \to (g_2', k')$ in $\mathbb{G}_2^* \times \mathbb{K}$ and $g_1 \not\to g_1'$ in $\mathbb{G}_1$. By using the definitions of $C(f)$ and $C(f')$ and taking into account that $(g_2, k) \to (g_2', k')$ in $\mathbb{G}_2^* \times \mathbb{K}$ is equivalent to $g_2 \not\to g_2'$ in $\mathbb{G}_2$, we get the claim.

We define the equivalence ρ_1 on $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ by

$$(f, f') \in \rho_1 \text{ if and only if } C(f) = C(f').$$

Clearly, the ρ_1-block that contains the constant u_1 map is a singleton. By the following claim, each of the other blocks has cardinality 2^κ. Also, the subsets of $G_1^* \times G_2^*$ are in a bijective correspondence with the blocks of ρ_1.

CLAIM 2. *For every non-empty subset U of $G_1^* \times G_2^*$*

$$B_U = \{\ f \in G_1^{G_2^* \times K} :\ C(f) = U\ \}$$

is a ρ_1-block of cardinality 2^κ.

For any subset S of K with $|S| = |K| = \kappa$, let g_S be some surjective map in U^S. There is such a map, since $|U| \le |G_1||G_2| \le \kappa$. Now for every $S \subseteq K$ with $|S| = \kappa$, we define $f_S \in \mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ by

$$f_S(g_2, k) := \begin{cases} g_1, \text{ if } k \in S \text{ and } (g_1, g_2) = g_S(k), \\ u_1 \text{ otherwise.} \end{cases}$$

Now clearly, $f_S \in B_U$ for all $S \subseteq K$ with $|S| = \kappa$. Moreover, the number of the f_S coincides with that of the subsets of cardinality κ in K. So this number is 2^κ, hence $2^\kappa \le |B_U|$. On the other hand, $|B_U| \le |G_1|^{|G_2||K|} = 2^\kappa$. Since B_U is non-empty, it is clearly a ρ_1-block. So the claim is proved.

We extend the definition of C onto $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$ by letting

$$C(h) := \{\ (g_1, g_2) \in G_1^* \times G_2^* : \ \text{there exists } k \in K \text{ such that } h(g_1, k) = g_2\ \}$$

for any $h \in G_2^{G_1^* \times K}$. We define the equivalence ρ_2 on $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$ analogously to ρ_1. With the notions defined in this paragraph, the analogues of Claim 1 and Claim 2 obviously hold.

By Claim 2 and its analogue, for every non-empty subset $U \subseteq G_1^* \times G_2^*$, the cardinalities of the ρ_1-block B_U and the ρ_2-block D_U that correspond to U coincide with 2^κ. Hence there exists a bijection from B_U to D_U. If $U = \emptyset$, there is also a bijection from the related ρ_1-block to the related ρ_2-block, since these blocks are one-element, containing the constant u_1 and u_2 maps, respectively. We take such a bijection for every $U \subseteq G_1^* \times G_2^*$. Let η be the union of these bijections. So η is a bijection from $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ to $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$. We finish off the proof by verifying that η is an isomorphism.

Let $f, f' \in \mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$. By Claim 1, the pair (f, f') is not an edge of the digraph $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ if and only if there exist

$$(g_1, g_2) \in C(f) \text{ and } (g_1', g_2') \in C(f')$$

such that none of (g_1, g_1') and (g_2, g_2') are edges in $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. Since

$$C(f) = C(\eta(f)) \text{ and } C(f') = C(\eta(f')),$$

the latter one is equivalent to the condition that there exist

$$(g_1, g_2) \in C(\eta(f)) \text{ and } (g_1', g_2') \in C(\eta(f'))$$

such that none of (g_1, g_1') and (g_2, g_2') are edges in $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. Now by the analogue of Claim 1, this is equivalent that the pair $(\eta(f), \eta(f'))$ is not an edge of the digraph $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$. Thus (f, f') is not an edge of the digraph $\mathbb{G}_1^{\mathbb{G}_2^* \times \mathbb{K}}$ if and only if $(\eta(f), \eta(f'))$ is not an edge of the digraph $\mathbb{G}_2^{\mathbb{G}_1^* \times \mathbb{K}}$. So η is an isomorphism. □

The following corollary of the preceding theorem is an essential tool in the proof of our main result.

Corollary 2.2. *Let I be an arbitrary set. For every $i \in I$, let $\mathbb{G}_i$ be a non-complete digraph with a universal vertex u_i. Then there exist a digraph $\mathbb{X}$ and a non-complete digraph $\mathbb{T}$ with a universal vertex such that $\mathbb{G}_i^{\mathbb{X}} \cong \mathbb{T}$ for all $i \in I$.*

Proof. Let $\mathbb{N}$ be the countably infinite digraph whose edge relation is the equality. Let $\mathbb{X} := \prod_{i \in I} (\mathbb{G}_i^* \times \mathbb{K})^{\mathbb{N}}$ where for each $i \in I$, $\mathbb{G}_i^*$ is the complement of the digraph $\mathbb{G}_i - u_i$, and $\mathbb{K}$ is an infinite complete digraph whose cardinality is greater than or equal to that of any of the $\mathbb{G}_i$. Since for any digraph $\mathbb{G}$

$$\mathbb{G} \times \mathbb{G}^{\mathbb{N}} \cong \mathbb{G}^{\mathbb{N}},$$

for any $i \in I$

$$\mathbb{X} \cong (\mathbb{G}_i^* \times \mathbb{K}) \times \mathbb{X}.$$

Then by Theorem 2.1, for any i and j

$$\mathbb{G}_i^{\mathbb{X}} \cong \mathbb{G}_i^{(\mathbb{G}_j^* \times \mathbb{K}) \times \mathbb{X}} \cong \left(\mathbb{G}_i^{\mathbb{G}_j^* \times \mathbb{K}}\right)^{\mathbb{X}} \cong \left(\mathbb{G}_j^{\mathbb{G}_i^* \times \mathbb{K}}\right)^{\mathbb{X}} \cong \mathbb{G}_j^{(\mathbb{G}_i^* \times \mathbb{K}) \times \mathbb{X}} \cong \mathbb{G}_j^{\mathbb{X}}.$$

Let i be some element of I. Since for each $j \in I$, $\mathbb{G}_j$ is a non-complete digraph, $\mathbb{G}_j^*$ has an edge. Hence $\mathbb{X}$ has an edge, too. Therefore, the constant maps from X to G_i induce a subdigraph of $\mathbb{G}_i^{\mathbb{X}}$ isomorphic to $\mathbb{G}_i$. Hence $\mathbb{G}_i^{\mathbb{X}}$ is a non-complete digraph. Moreover, the constant u_i map is a universal vertex in $\mathbb{G}_i^{\mathbb{X}}$. Then we let $\mathbb{T} := \mathbb{G}_i^{\mathbb{X}}$. Clearly, the so defined $\mathbb{X}$ and $\mathbb{T}$ satisfy the claim. □

Now we have all the tools at our disposal to prove our main result.

Theorem 2.3. *Congruence permutability is prime.*

Proof. We are going to prove that the join of two non-permutable varieties $\mathscr{K}$ and $\mathscr{L}$ is non-permutable. As $\mathscr{K}$ and $\mathscr{L}$ are not congruence permutable, by Theorem 1.1, there exist non-symmetric reflexive digraphs $\mathbb{G}_0$ and $\mathbb{H}_0$ that are compatible digraphs in the varieties $\mathscr{K}$ and $\mathscr{L}$, respectively. Through a series of definitions, starting from $\mathbb{G}_0$ and $\mathbb{H}_0$, we define some compatible digraphs $\mathbb{G}_i$ in $\mathscr{K}$ and some compatible digraphs $\mathbb{H}_i$ in $\mathscr{L}$. The compatibility of all of these digraphs will be automatic, since their vertex sets and edge relations will be defined from compatible digraphs by existentially quantified (maybe infinite) conjunctions of atomic formulas in the language of digraphs extended by the equality.

We define the digraph $\mathbb{G}_1$ on the vertex set

$$G_1 := \{\ (a,b,c,d) \in G_0^4 : a \to b,c,d \text{ and } b \to c,d \text{ and } c \to d \text{ in } \mathbb{G}_0\ \}$$

with the edge relation

$$E(\mathbb{G}_1) := \{\ ((a,b,c,d),(a,b',c',d)) \in G_1^2 : b \to c' \text{ and } b' \to c \text{ in } \mathbb{G}_0\ \}.$$

Now we prove that each of the components of $\mathbb{G}_1$ has a universal vertex. Let C be a component of $\mathbb{G}_1$. Then C is contained in the set

$$W = \{\ (a,b,c,d) \in G_0^4 : a \to b,c \text{ and } b \to c,d \text{ and } c \to d \text{ in } \mathbb{G}_0\ \}$$

for some $a \to d$ in $\mathbb{G}_0$. Then (a,a,d,d) is a vertex of $\mathbb{G}_1$ as $\mathbb{G}_0$ is reflexive. Moreover, $(a,a,d,d) \to (a,b,c,d) \to (a,a,d,d)$ in $\mathbb{G}_1$ for every $(a,b,c,d) \in W$. Hence $C = W$ and (a,a,d,d) is a universal vertex in C. If (a,d) is an edge but (d,a) is not an edge of $\mathbb{G}_0$, then $((a,d,d,d),(a,a,a,d))$ is not an edge in the component of (a,a,d,d) in $\mathbb{G}_1$. So at least one of the components of $\mathbb{G}_1$ is a non-complete digraph.

Let $\mathbb{R}$ be a non-complete component of $\mathbb{G}_1$. We define the digraph $\mathbb{G}_2$ on the vertex set

$$G_2 := \{\ f \in G_1^{\{0\}\cup R} : f(0) \leftrightarrow f(x) \text{ in } \mathbb{G}_1 \text{ for all } x \in R\ \}$$

with the edge relation

$$E(\mathbb{G}_2) := \{\ (f,f') \in G_2^2 : f(0) \to f'(0) \text{ in } \mathbb{G}_1 \text{ and } f(x) = f'(x) \text{ for all } x \in R\ \}.$$

Let f be any vertex of $\mathbb{G}_2$ and f_u the vertex obtained from f by changing the value of f at 0 to the universal vertex u of the component of $f(0)$ in $\mathbb{G}_1$. Then f_u is a universal vertex of the component of f in $\mathbb{G}_2$.

Moreover, the vertex set

$$\{\ f \in G_2 :\ f(x) = u_{\mathbb{R}} \text{ for all } x \in R\ \}$$

where $u_{\mathbb{R}}$ is a universal vertex of $\mathbb{R}$ induces a component isomorphic to $\mathbb{R}$ in $\mathbb{G}_2$. Thus, $\mathbb{G}_2$ contains a non-complete component. The digraph $\mathbb{G}_2$ also contains a complete component. Indeed, the vertex set

$$\{\ f \in G_2 :\ f(x) = x \text{ for all } x \in R\ \}$$

gives a component isomorphic to the clique of the universal vertices of $\mathbb{R}$ in $\mathbb{G}_2$.

We define $\mathbb{H}_1$ and $\mathbb{H}_2$ from $\mathbb{H}_0$ by the pattern as $\mathbb{G}_1$ and $\mathbb{G}_2$ are defined from $\mathbb{G}_0$. Thus all components of $\mathbb{H}_2$ and $\mathbb{G}_2$ contain universal vertices, and there are both complete and non-complete components in each of these digraphs.

By Corollary 2.2, there exist a digraph $\mathbb{X}$ and a non-complete digraph $\mathbb{T}$ with a universal vertex such that for every non-complete component $\mathbb{C}$ of the digraphs $\mathbb{G}_2$ and $\mathbb{H}_2$, $\mathbb{C}^{\mathbb{X}}$ is isomorphic to $\mathbb{T}$. We define the digraph $\mathbb{G}_3$ on the vertex set

$$G_3 := \{\, f \in G_2^X : \text{ there exists } u \in G_2 \text{ such that for all } x \in X,\ f(x) \to u \text{ in } \mathbb{G}_2 \,\}$$

with the edge relation

$$E(\mathbb{G}_3) := \{\, (f, f') \in G_3^2 : f(x) \to f'(y) \text{ in } \mathbb{G}_2 \text{ for all } x \to y \text{ in } \mathbb{X} \,\}.$$

The digraph $\mathbb{G}_3$ is the subdigraph of $\mathbb{G}_2^{\mathbb{X}}$ induced by the vertices whose ranges lie in a single component of $\mathbb{G}_2$. Thus the components of $\mathbb{G}_3$ are the $\mathbb{X}$-th powers of the components of $\mathbb{G}_2$. Hence, $\mathbb{G}_3$ has only two kinds of components, the ones isomorphic to $\mathbb{T}$ and the complete ones. We define $\mathbb{H}_3$ from $\mathbb{H}_2$ similarly. The digraph $\mathbb{H}_3$ also has only two kinds of components, some isomorphic to $\mathbb{T}$ and some complete components.

Now there are some powers of $\mathbb{G}_3$ and $\mathbb{H}_3$ of the same infinite cardinality, say λ. Since $\mathbb{G}_3$ and $\mathbb{H}_3$ are compatible in $\mathscr{K}$ and $\mathscr{L}$, respectively, so are these powers. Therefore, the complete digraph $\mathbb{K}_\lambda$ of cardinality λ and the λ-many element digraph $\mathbb{Q}_\lambda$ whose edge relation is the equality are compatible digraphs in both varieties $\mathscr{K}$ and $\mathscr{L}$. Hence the digraph $\mathbb{G}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ is compatible in $\mathscr{K}$, and the digraph $\mathbb{H}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ is compatible in $\mathscr{L}$. Clearly, $\mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ is a digraph with λ-many components where each component is isomorphic to K_λ. Hence, both of the digraphs $\mathbb{G}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ and $\mathbb{H}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ consist of λ-many complete components of size λ and λ-many non-complete components isomorphic to $\mathbb{T} \times \mathbb{K}_\lambda$. Therefore, the digraphs $\mathbb{G}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ and $\mathbb{H}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ are isomorphic and compatible in each of the varieties $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{K} \vee \mathscr{L}$. Since $\mathbb{G}_3 \times \mathbb{K}_\lambda \times \mathbb{Q}_\lambda$ has a non-complete component that contains a universal vertex, there exist vertices u, v, w in it such that $v \to u \to u \to w$ and $v \not\to w$. Then, the third condition in Theorem 1.1 does not hold for $\mathscr{K} \vee \mathscr{L}$, for otherwise there would be a ternary term m in the language of $\mathscr{K} \vee \mathscr{L}$ such that $v = m(v, u, u) \to m(u, u, w) = w$, a contradiction. Hence, $\mathscr{K} \vee \mathscr{L}$ is not congruence permutable. □

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