ON LATTICE REPRESENTATIONS WITH DCC POSETS

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ABSTRACT. In this paper, we investigate the class of lattices representable with posets satisfying the DCC condition. We describe a way to decide whether a finite lattice is in this class. We also give a necessary condition for an arbitrary lattice to be in this class. This hints at a notion that would be a weaker version of lower boundedness. lattice representation and lower bounded lattices and DCC posets and D relation

1. INTRODUCTION

Probably the most basic representation theorem of lattice theory is that every lattice is embeddable into the lattice of equivalences of a large enough set ([8]). A complementary result is proved by Schein in [1]: any lattice is representable as a lattice of posets on a set. (The partial orders of a set do not form a lattice with respect to inclusion. The reflexive and transitive binary relations do, however, and when we talk about a lattice of posets, we mean a sublattice of this containing only antisymmetric relations.)

For finite lattices, the equivalance representation can be given on an underlying finite set ([5]). This is not true for the poset representation. Siványk in [7] gave a characterization for lattices representable as lattices of posets on a finite set using the notion of *small congruences* of a lattice (a kind of congruences where every congruence class has at most two elements). In [2], the authors note that this characterization precisely describe the class of finite *lower bounded* lattices.

This can be seen as a generalization of a theorem of Caspard [3], namely that the lattice of permutations of a finite set is (both lower and upper) bounded. Here the lattice of permutations means the lattice induced by the following order: we fix a linear order on the underlying set, and a permutation π_1 will be smaller than the (distinct) permutation π_2 iff for every pair $i < j$ of the underlying set, $\pi_1(i) > \pi_1(j)$ implies $\pi_2(i) > \pi_2(j)$. The result about lattices of posets is a generalization of this because the lattice of permutations can be seen as dually isomorphic to a lattice of posets: for every permutation π, take the poset $\leq_\pi$ defined by $i \leq_\pi j$ iff either $i = j$, or $i < j$ and $\pi(i) < \pi(j)$. We leave it to the reader to check that $\pi \to \leq_\pi$ indeed defines an injective dual lattice homomorphism.

Semenova in [6] proves something more general (see Proposition 1.6 and Theorem 4.6): any finite lattice embeddable into the suborder lattice of a poset containing no infinite chain must be lower bounded, and all lower

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bounded lattices are embeddable into such a lattice. Being embeddable into a suborder lattice of a poset containing no infinite chain means (at least for lattices with a largest element) being representable with posets satisfying both the ascending and descending chain conditions. (A poset satisfies both the DCC and ACC if and only if it has no infinite chain.)

So any lattice can be represented as a lattice of posets, but if we require a finiteness condition for the appearing posets, then we get a much smaller class, which is closely related to the well-known class of lower bounded lattices. This paper is concerned with the following question: what if we require a weaker finiteness condition–only the DCC, but not the ACC–of the posets? Obviously, all lower bounded lattices will be representable in such a way. But it turns out that even among finite lattices, there are representable ones that are not lower bounded.

We give an algorithmic characterization of representable finite lattices, and a general necessary condition. The latter is a weaker version of lower boundedness: instead of D-cycles, it forbids cycles of D-cycles.

The reader is invited to find a class that is closely related to the class of lattices representable with DCC posets (for example, having the same finite members), thus discovering a connection alike to the one between lower bounded lattices and lattices representable with posets containing no infinite chains.

2. Preliminaries

A *preorder* on a set X is a reflexive and transitive binary relation on X. The set of all preorders on X with respect to inclusion forms a lattice, denoted by $\operatorname{Pre} X$. Naturally, the meet operation is the set-theoretic intersection, and the join operation is equivalent to

$$(a,b) \in \alpha \vee \beta \Leftrightarrow \exists k \in \mathbb{N} : \exists c_1, \dots, c_k \in X : \; a = c_1, \; b = c_k, \; \forall i : (c_i, c_{i+1}) \in \alpha \cup \beta.$$

A sublattice of $\operatorname{Pre} X$ that only contains antisymmetric preorders is called a *lattice of posets* on X. A lattice of posets can be considered as a set $\mathfrak{C}$ of posets on the same underlying set such that $\mathfrak{C}$ is closed to set-theoretic intersection and for any $\alpha, \beta \in \mathfrak{C}$ the transitive closure of $\alpha \cup \beta$ is also in $\mathfrak{C}$.

One says that a poset satisfies the descending chain condition (DCC) if there are no elements $a_1, a_2, \dots$ such that (with respect to this poset) $a_1 > a_2 > \dots$. The dual of the DCC is the ascending chain condition (ACC). We call a lattice of posets a *lattice of DCC-posets* if all of its elements satisfy the DCC.

We use the following definitions and statements from [4] to introduce the notions of (and related to) lower bounded lattices.

An element l of a lattice $\mathbf{L}$ is *join irreducible* if there are no elements $l_1, l_2 < l$ such that $l_1 \vee l_2 = l$. It is *completely join irreducible* if there is a largest element l_* among all the elements of the lattice smaller than l. If an element is completely join irreducible, then it is also join irreducible, and for finite lattices the converse is also true. The set of the join irreducible elements of $\mathbf{L}$ is denoted by $J(L)$. The element l is *join prime* if for all l_1, l_2 satisfying $l_1 \vee l_2 \geq l$ either $l_1 \geq l$ or $l_2 \geq l$.

The join dependency relation, D, is a binary relation on the set of join irreducible elements of $\mathbf{L}$. It is defined by

$$a\ D\ b \Leftrightarrow a \neq b,\ (\exists c:\ a \leq b \vee c,\ \forall d < b : a \nleq d \vee c).$$

If b is completely join irreducible, this is simplified into

$$a\ D\ b \Leftrightarrow a \neq b,\ (\exists c:\ a \leq b \vee c,\ a \nleq b_* \vee c).$$

A lattice $\mathbf{L}$ is *lower bounded* if there is a finitely generated free lattice $\mathbf{F}_n$ and a congruence $\theta \in \operatorname{Con} \mathbf{F}_n$ such that all θ-classes have a smallest element, and $\mathbf{L} \cong \mathbf{F}_n/\theta$. A lower bounded lattice cannot contain an infinte *D-path* (that is, the graph induced by D does not contain an infinite path).

A *D-cycle* is a cycle in the graph of D (vertices appearing multiple times is permitted). As a D-cycle naturally gives rise to an infinite D-path, it also cannot appear in a lower bounded lattice. For finite lattices, the opposite is also true: if the lattice contains no D-cycle, it is lower bounded.

Finally, any lower bounded lattice is join semidistributive.

3. Representation of finite lattices

The problem of whether a finite lattice is representable with DCC posets (i.e. is isomorphic to a lattice of DCC posets) is decidable, as Theorem 3.1 will show. We will start with the idea for the characterization, then state the theorem and give the (quite technical) proof.

Suppose that we have a finite lattice $\mathbf{L}$, and DCC posets δ_l for each $l \in L$ on the same underlying set, $l \to \delta_l$ is order-preserving, but not quite a lattice homomorphism. Either it does not commute with meet or it does not commute with join, so there are elements l_1 and l_2 such that either $\delta_{l_1 \wedge l_2} < \delta_{l_1} \wedge \delta_{l_2}$ or $\delta_{l_1 \vee l_2} > \delta_{l_1} \vee \delta_{l_2}$. In the first case the natural idea is to add edges to $\delta_{l_1 \wedge l_2}$ so that it coincides with $\delta_{l_1} \wedge \delta_{l_2}$. After this $l \to \delta_l$ may not be order-preserving anymore, hence we need to add any $\delta_{l''}$-edge to $\delta_{l'}$ for all $l' > l''$. The second case can be addressed with adding an extra element x to our underlying set for each $(y_1, y_2) \in \delta_{l_1 \vee l_2} \backslash (\delta_{l_1} \vee \delta_{l_2})$ edge, and adding either (y_1, x) or (x, y_2) to δ_{l_1}, and the other to δ_{l_2}.

This process may be infinite, but it has a limit, at which $l \to \delta_l$ is a homomorphism. The problem is that while each step maintains the DCC for the appearing posets, the whole process does not necessarily does so. Note that it is not always a problem if the process if infinite: if we need an extra element x_1 for the edge (y_1, y_2), then an extra element x_2 for (x_1, y_2), then an extra element x_3 for (x_2, y_2), and so on–always a new element for the *top* segment–then we will get an infinite ascending chain, but no infinite descending one. On the other hand, if we always need a new element for the *bottom* segment, we will get an infinite descending chain. This shows that it matters that which of (y_1, x) and (x, y_2) goes into δ_{l_1} and which into δ_{l_2}.

Unfortunately, the process described above is too complicated, as it needs to correct problems both with meets and with joins. Therefore, we will use a different process: we will concentrate only on the images of the join irreducible elements of $\mathbf{L}$. Instead of the mapping preserving meets and join, we will concentrate on the mapping being order-preserving, and having the following property: if $l \leq l_1 \vee \cdots \vee l_k$ for some $l, l_1, \ldots, l_k \in J(L)$, then δ_l must be included in $\delta_{l_1} \vee \cdots \vee \delta_{l_k}$. We will try to ensure this the same way

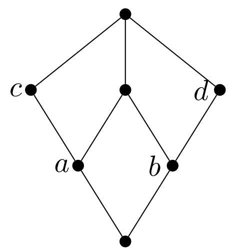

FIGURE 1. The lattice $\mathbf{D}_2$

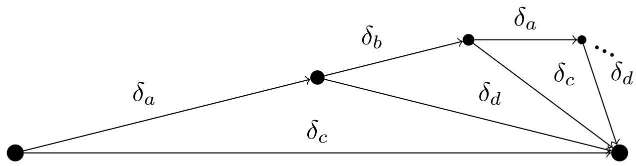

FIGURE 2. Edges added by our process beginning with a δ_c-edge

we wanted to ensure that δ preserved joins, only in this case we need to add $k-1$ extra elements to the underlying set.

To illustrate this process, let us take for example the lattice $\mathbf{D}_2$ (Figure 1). We will only consider the nonzero join irreducible elements of this lattice, which are marked on the figure. Suppose that we have or DCC posets $\delta_a < \delta_c$ and $\delta_b < \delta_d$. If, for example, $\delta_c \leq \delta_a \vee \delta_d$ does not hold because of an edge (y_1, y_2), our process will continously add elements to develop a picture like the one seen on Figure 2. Note that in Figure 2, the bottom segments are in γ_a or γ_b, while the top segments are in γ_c or γ_d. This results in the DCC poset seen on the figure.

Definition 3.1. For a finite lattice $\mathbf{L}$, let $\mathcal{C}_\mathbf{L}$ denote the set of nontrivial join covers of join irreducibles by join irreducibles, i.e. the set

$$\{(l, l_1, \ldots, l_k) : k \geq 2, (l, l_1, \ldots, l_k) \in J(L)^{k+1} : l \leq l_1 \vee l_2 \vee \cdots \vee l_k,$$
$$l \not\leq (l_1)_* \vee l_2 \vee \cdots \vee l_k, l \not\leq l_1 \vee (l_2)_* \vee \cdots \vee l_k, \ldots, l \not\leq l_1 \vee l_2 \vee \cdots \vee (l_k)_*\}$$

Theorem 3.1. *A finite lattice $\mathbf{L}$ is isomorphic to a lattice of DCC posets if and only if there is a mapping $s \colon \mathcal{C}_\mathbf{L} \to J(L)$ satisfying the following:*

- *for any $(l, l_1, \ldots, l_k) \in \mathcal{C}_\mathbf{L}$, $s(l, l_1, \ldots, l_k) \in \{l_1, \ldots, l_k\}$,*
- *s is symmetrical in all but the first variable, i.e. for any permutation $\pi \in S_k$,*
$$s(l, l_1, \ldots, l_k) = s(l, l_{\pi(1)}, \ldots, l_{\pi(k)}),$$
- *for the binary relations*
$$T := \{(l, l_i) : (l, l_1, \ldots, l_k) \in \mathcal{C}_\mathbf{L}, s(l, l_1, \ldots, l_k) \neq l_i\}$$
and
$$U := \mathrm{Tr}(\{(l, l) : l \in L\} \cup \{(l, l_i) : (l, l_1, \ldots, l_k) \in \mathcal{C}_\mathbf{L}, s(l, l_1, \ldots, l_k) = l_i\}),$$
*the relation $U \circ T$ does not contain a cycle (*Tr *denotes the transitive closure).*

Proof. $\Rightarrow$ Suppose that there is such a mapping s. We will use the process described above, with s telling us that when adding the extra elements to our underlying set, into which poset the top segment will go.

The process will yield a set X, partial orders $\gamma_l \subseteq X^2$ for all $l \in L$ such that the mapping $l \to \gamma_l$ is an injective lattice homomorphism from $\mathbf{L}$ into $\operatorname{Pre} X$, and all γ_l satisfy DCC.

We begin with a set $X^{(0)} = \{x_{l,1} : l \in L\} \cup \{x_{l,2} : l \in L\}$, and for any $l \in J(L)$ we define the binary relation $\gamma_l^{(0)} := \{(x_{l,1}, x_{l,2})\}$ on $X^{(0)}$.

Now we recursively define for all natural j and $l \in J(L)$ the sets $X^{(j)}$ and binary relations $\gamma_l^{(j)}$ on $X^{(j)}$. Firstly, $X^{(j)}$ will be a set containing $X^{(j-1)}$, and $\gamma_l^{(j)}$ a relation containing $\gamma_l^{(j-1)}$. Secondly, for any $(l, l_1, \dots, l_k) \in \mathcal{C}_{\mathbf{L}}$ and any $(y_1, y_2) \in \gamma_l^{(j-1)} \setminus (\gamma_{l_1}^{(j-1)} \vee \cdots \vee \gamma_{l_k}^{(j-1)})$, add the elements $x_{y_1,y_2,l,l_1,\dots,l_k,1}, \dots, x_{y_1,y_2,l,l_1,\dots,l_k,k-1}$ to $X^{(j)}$. Use the notion $x_{y_1,y_2,l,l_1,\dots,l_k,0} = y_1$, $x_{y_1,y_2,l,l_1,\dots,l_k,k} = y_2$. For $0 \leq r < k$, we add the edge $(x_{y_1,y_2,l,l_1,\dots,l_k,r}, x_{y_1,y_2,l,l_1,\dots,l_k,r+1})$ into one of the $\gamma_{l_i}^{(j)}$ in a way that exactly one edge goes into each of the $\gamma_{l_i}^{(j)}$, and the last edge (that is, $(x_{y_1,y_2,l,l_1,\dots,l_k,k-1}, y_2)$) goes into $\gamma_{s(l,l_1,\dots,l_k)}^{(j)}$.

Finally, set $X = \bigcup X^{(j)}$ and for any $l \in L$ set

$$\gamma_l = \operatorname{Tr}(\{(x,x) : x \in X\} \cup \bigcup \{\gamma_{l'}^{(j)} : j \in \mathbb{N},\, l' \in J(L),\, l' \leq l\}).$$

It is easy to see that all γ_l are partial orders on X, and that $\gamma_{l_1} < \gamma_{l_2}$ iff $l_1 < l_2$. (The inequality is strict because there is an $l' \in J(L)$ such that $l' \leq l_2$ and $l' \not\leq l_1$, and then $(x_{l',1}, x_{l',2})$ is in $\gamma_{l_2} \setminus \gamma_{l_1}$.)

We will prove that $l \to \gamma_l$ preserves meets and joins. For the "join" part, as $l \to \gamma_l$ is order-preserving, it is enough to show that $\gamma_{l_1 \vee l_2} \leq \gamma_{l_1} \vee \gamma_{l_2}$ and to do that, it is enough to show that for all $l' \in J(L)$ with $l' \leq l_1 \vee l_2$, and all j, $\gamma_{l'}^{(j)} \subseteq \gamma_{l_1} \vee \gamma_{l_2}$.

$\mathbf{L}$ is a finite lattice, so each of its elements is a join of join irreducible elements: $l_1 = p_1 \vee \cdots \vee p_m$, and $l_2 = q_1 \vee \cdots \vee q_n$. Now as $l' \leq l_1 \vee l_2 = p_1 \vee \dots p_m \vee q_1 \cdots \vee q_n$,

$$\gamma_{l'}^{(j)} \subseteq \operatorname{Tr}(\gamma_{p_1}^{(j+1)} \cup \cdots \cup \gamma_{p_m}^{(j+1)} \cup \gamma_{q_1}^{(j+1)} \cup \cdots \cup \gamma_{q_n}^{(j+1)}) \subseteq \gamma_{l_1} \vee \gamma_{l_2}.$$

For the "meet" part, take a (non-loop) edge $(x_1, x_2) \in \gamma_{l_1} \wedge \gamma_{l_2}$. If both x_1 and x_2 are in $X^{(0)}$, there is an $l' \in J(L)$ such that $(x_1, x_2) = (x_{l',1}, x_{l',2})$. Therefore, $l' \leq l_1$ and $l' \leq l_2$, so $l' \leq l_1 \wedge l_2$, and $(x_1, x_2) \in \gamma_{l_1 \wedge l_2}$.

Otherwise, we can assume that both x_1 and x_2 are in $X^{(j)}$, but x_1 is not in $X^{(j-1)}$, so $x_1 = x_{y_1,y_2,l',l'_1,\dots,l'_k,r}$, where $y_1, y_2 \in X^{(j-1)}$, $l' \leq l'_1 \vee \cdots \vee l'_k$, and $(y_1, y_2) \in \gamma_{l'}^{(j-1)} \setminus (\gamma_{l'_1}^{(j-1)} \vee \cdots \vee \gamma_{l'_k}^{(j-1)})$. Suppose that $x_2 = x_{y_1,y_2,l',l'_1,\dots,l'_k,r'}$ for some $r < r'$, in which case $(x_1, x_2) \in \gamma_{l_1} \wedge \gamma_{l_2}$ is only possible if

$$(x_{y_1,y_2,l',l'_1,\dots,l'_k,r}, x_{y_1,y_2,l',l'_1,\dots,l'_k,r+1}), \dots, (x_{y_1,y_2,l',l'_1,\dots,l'_k,r'-1}, x_{y_1,y_2,l',l'_1,\dots,l'_k,r'}) \in \gamma_{l_1} \wedge \gamma_{l_2},$$

because any γ_l on $X^{(j)}$ coincides with the reflexive and transitive closure of $\gamma_l^{(j)}$. Hence, $l_1, l_2 \leq l'_r, \dots, l'_{r'-1}$, and thus $l_1 \wedge l_2 \leq l'_r, \dots, l'_{r'-1}$, and $(x_1, x_2) \in \gamma_{l_1 \wedge l_2}$.

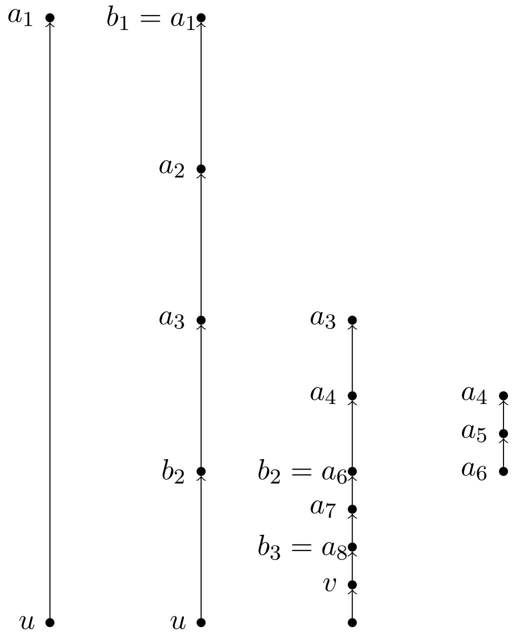

FIGURE 3. How to get the sequence $(b_i)_{i\in\mathbb{N}}$ from the sequence $(a_i)_{i\in\mathbb{N}}$. Nodes to the left/right of each other are meant to represent the same element.

If x_2 is not of the form $x_{y_1,y_2,l',l'_1,\dots,l'_k,r'}$, then (again because γ_l on $X^{(j)}$ coincides with the reflexive and transitive closure of $\gamma_l^{(j)}$) $(x_1,x_2)\in\gamma_{l_1}\wedge\gamma_{l_2}$ is only possible if (x_1,y_2) and (y_2,x_2) are both in $\gamma_{l_1}\wedge\gamma_{l_2}$. As $y_2 = x_{y_1,y_2,l',l'_1,\dots,l'_k,k}$, by the previous argument, $(x_1,y_2)\in\gamma_{l_1\wedge l_2}$.

We still need that $(y_2,x_2)\in\gamma_{l_1\wedge l_2}$. If x_2 is in $X^{(j-1)}$, then we are done by induction. Otherwise, by repeating to (y_2,x_2) what we did to (x_1,x_2), we will get an $y'_1\in X^{(j-1)}$ such that it is enough to prove $(y_2,y'_1)\in\gamma_{l_1\wedge l_2}$ to prove $(y_2,x_2)\in\gamma_{l_1\wedge l_2}$. That, again, can be done by induction.

We have shown that $l\to\gamma_l$ is strictly order-preserving and is a lattice homomorphism. Therefore, it is a lattice embedding. The only thing left to show is that γ_1 satisfies DCC (1 denoting the largest element of $\mathbf{L}$).

For any $l\in L$, the *difficulty* of l will denote the length of the longest $U\circ T$-path starting from l. This is finite because $\mathbf{L}$ is a finite lattice, and $U\circ T$ contains no cycle.

An edge of the type $(x_{l,1},x_{l,2})$ will be called an *original edge*, an edge of the type $(x_{y_1,y_2,l,l_1,\dots,l_k,r},x_{y_1,y_2,l,l_1,\dots,l_k,r+1})$ with $r<k-1$ a *lower part*, an edge $(x_{y_1,y_2,l,l_1,\dots,l_k,k-1},y_2)$ an *upper part* of the edge (y_1,y_2). An edge is a *core edge* if it is either original, or a lower or upper part of a (core) edge. Thus, each γ_l is the reflexive and transitive closure of the set of the core γ_l edges. An edge is an *upper edge* if it is of the form $(x_{y_1,y_2,l,l_1,\dots,l_k,h},y_2)$ for some $0<h<k$.

Suppose $(a_i)_{i\in\mathbb{N}}$ is an infinite (strictly) decreasing sequence in γ_1. We can assume that each of the (a_{i+1},a_i) is a core edge (otherwise, it is the concatenation of core edges, and we split it to core edges). Now after any upper core edge $(x_{y_1,y_2,l,l_1,\dots,l_k,k-1}, y_2)$ in the sequence, the following edge is either another upper core one, or it is the lower core edge $(x_{y_1,y_2,l,l_1,\dots,l_k,k-2}, x_{y_1,y_2,l,l_1,\dots,l_k,k-1})$, after which the only possible lower core one is $(x_{y_1,y_2,l,l_1,\dots,l_k,k-3}, x_{y_1,y_2,l,l_1,\dots,l_k,k-2})$, and so on. We can assume that there is an upper core edge before reaching $(y_1, x_{y_1,y_2,l,l_1,\dots,l_k,1})$, because otherwise we can exchange all these edges to their concatenation (y_1, y_2), which is still a core edge. Thus, by concatenating any upper part edge with all the lower part ones following it before the next upper part, we get a sequence of γ_1-edges $(b_{i+1}, b_i)_{i\in\mathbb{N}}$ such that all of those edges are upper edges (though not necessarily cores).

An illustration of this process can be seen on Figure 3: the edge (a_2, a_1) is an upper part (of four parts) of the edge (u, a_1), so $a_2 = x_{u,a_1,l,l_1,\dots,l_4,3}$ for some $(l, l_1, \dots, l_4) \in \mathcal{C}_{\mathbf{L}}$. We choose the smallest j such that there is an element of the sequence $(a_i)_{i\in\mathbb{N}}$ smaller than $x_{u,a_1,l,l_1,\dots,l_4,j}$. We can assume that $(a_i)_{i\in\mathbb{N}}$ does not have an element below u, because otherwise u would equal a_h to some h, and we could throw out the elements $a_2, \dots, a_{h-1}$ from the sequence. So $j = 1$, and we set $b_2 = x_{u,a_1,l,l_1,\dots,l_4,j} = a_6$. Now, as (a_7, a_6) is an upper part of (u, b_2), we choose the smallest j' such that there is an element of $(a_i)_{i\in\mathbb{N}}$ smaller than $x_{u,b_2,l',l'_1,\dots,l'_4,j'}$ (here $l', l'_1, \dots, l'_4$ are such that $a_7 = x_{u,b_2,l',l'_1,\dots,l'_4,3}$). Assuming that no a_i is smaller than v, j' must be 2, and we choose b_3 accordingly.

The following claim presents an immediate contradiction to the existence of such $(b_i)_{i\in\mathbb{N}}$.

Claim 1. *For any $b = x_{y_1,y_2,l,l_1,\dots,l_k,r}$, any path of upper edges ending in b has length not greater than the difficulty of the l_j satisfying $(x_{y_1,y_2,l,l_1,\dots,l_k,r-1}, b) \in \gamma_{l_j}$.*

Let $d = x_{y_1,y_2,l,l_1,\dots,l_k,r-1}$.

The claim is proved by induction on the difficulty of l_j. Suppose first that it is 0. This means that there is no T-edge with source l_j, consequently, l_j is a join prime, and there is no element of the form $x_{d,b,h,h_1,\dots,h_t,q}$, and no upper edge ending in b.

Now suppose that the difficulty of l_j is positive. Let (c, b) be an upper edge, thus c is of the form $x_{z,b,m,m_1,\dots,m_t,q}$. The edge (z, b) can be obtained by repeatedly taking upper parts of the edge (y_1, b), therefore $(l_j, m) \in U$. Suppose that $(x_{z,b,m,m_1,\dots,m_t,q-1}, c)$, which is a lower part of the edge (z, b), is in $\gamma_{m_{j'}}$, then $(m, m_{j'}) \in T$. Thus $(l_j, m_{j'}) \in U \circ T$, so the difficulty of $m_{j'}$ is smaller than the difficulty of l_j. By the inductive hypothesis, any path of upper part core edges ending in c has length smaller than the difficulty of l_j. The claim is proved.

$\Leftarrow$ Assume that $l \to \gamma_l$ is a lattice embedding from $\mathbf{L}$ into $\operatorname{Pre} X$. For any edge $(z_1, z_2) \in \gamma_1$ there is a smallest $k \in L$ such that $(z_1, z_2) \in \gamma_k$, the *weight* of this edge.

An edge (c, d) is called *contained* in the edge (a, b) if both (a, c) and (d, b) are in γ_1, but $(c, d) \neq (a, b)$. It is *properly contained* if moreover $d < b$.

Let $(l, l_1, \dots, l_k) \in \mathcal{C}_{\mathbf{L}}$, and take an edge $(a_1, a_2) \in \gamma_l \backslash \gamma_{l_*}$, this edge has weight l. There are elements $a_1 = b_0, b_1, \dots, b_r = a_2$ in X such that for all $0 \leq i < r$, $(b_i, b_{i+1}) \in \gamma_{l_1} \cup \dots \cup \gamma_{l_k}$. Among these edges there must be at least one with weight l_j for all $1 \le j \le l$, otherwise

$$(a_1, a_2) \in \gamma_l \cap (\gamma_{l_1} \vee \dots \vee \gamma_{(l_j)_*} \vee \dots \vee \gamma_{l_k}) \subseteq \gamma_{l_*}.$$

So each edge with weight l contains at least one edge of weight l_1, at least one of weight l_2, e.c., and it contains these edges properly with at most one exception.

Now define the mapping s on $(l, l_1, \dots, l_k)$ so that if there are infinitely many edges with weight l, but only finitely many contains an edge of weight l_j properly, then $s(l, l_1, \dots, l_k) = l_j$. If there is no such j, then set $s(l, l_1, \dots, l_k) = l_1$.

If $(l, l') \in U \circ T$, then all edges of weight l must contain an edge of weight l'. Suppose there is a cycle of $U \circ T$ containing the edge (l, l'). There is an $m \in L$ so that $(l, m) \in U$ and $(m, l') \in T$. All edges of weight l must contain an edge of weight l. Starting from an edge (f_1, e_1) of weight l we can get the edges (f_2, e_2), (f_3, e_3) etc., each contained in the previous, and each having weight l. Furthermore, for each $j > 0$ either (f_j, e_j) can be chosen so that $e_j \neq e_{j-1}$, or it can only be chosen so that there is an edge of weight m containing (f_j, e_j) and contained in (f_{j-1}, e_{j-1}) that does not properly contain any edge of weight l'. There are only finitely many such edges of weight m. Therefore, the (f_j, e_j) edges can be chosen so that the sequence $(e_j)_{j \in \mathbb{N}}$ contains an infinite strictly decreasing subsequence in γ_1, a contradiction. □ □

This theorem gives an algorithm deciding whether $\mathbf{L}$ is representable with DCC posets. The algorithm is in $\mathcal{EXPTIME}$.

Problem 3.1. Are there real numbers k and α such that for any finite lattice $\mathbf{L}$, $|\mathcal{C}_\mathbf{L}| < k|L|^\alpha$?

If the answer to this is "yes", then the algorithm is actually in $\mathcal{NP}$.

Conjecture 3.1. Deciding whether a finite lattice is representable with DCC posets is an $\mathcal{NP}$-hard problem.

For comparison, deciding whether a finite lattice is lower bounded is in $\mathcal{P}$, because D is computable in polynomial time.

4. Representation of arbitrary lattices

For a lattice $\mathbf{L}$, denote with $\mathcal{CY}_\mathbf{L}$ the set of D-cycles of $\mathbf{L}$ consisting of completely join irreducible elements. Introduce a binary relation on $\mathcal{CY}_\mathbf{L}$:

$$E_\mathbf{L} := \{((\beta_1, \dots, \beta_l), (\alpha_1, \dots, \alpha_k)) : \\ \exists i : \exists j : \alpha_{j+1} \le \beta_i \vee \alpha_j,\ \alpha_{j+1} \not\le \beta_i \vee (\alpha_j)_*,\ \alpha_{j+1} \not\le (\beta_i)_* \vee \alpha_j\},$$

with the index j meant as modulo k and the index i as modulo l.

The following is a necessary condition for a lattice to be representable with DCC posets.

Theorem 4.1. *If $\mathbf{L}$ is representable with DCC posets, then $E_\mathbf{L}$ does not contain a cycle.*

Proof. Suppose the contrary, that

$$\underline{\alpha}^{(1)}\, E_{\mathbf{L}}\, \underline{\alpha}^{(2)}\, E_{\mathbf{L}} \dots E_{\mathbf{L}}\, \underline{\alpha}^{(t)}\, E_{\mathbf{L}}\, \underline{\alpha}^{(1)},$$

with $\underline{\alpha}^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_{k_i}^{(i)})$ for all $1 \le i \le t$. For all $1 \le i \le t$ and $1 \le j \le k_i$, there is a $\gamma_j^{(i)}$ that $\alpha_j^{(i)} \le \gamma_j^{(i)} \vee \alpha_{j+1}^{(i)}$ and $\alpha_j^{(i)} \not\le \gamma_j^{(i)} \vee (\alpha_{j+1}^{(i)})_*$, with the index j taken modulo k_i. By the definition of $E_{\mathbf{L}}$, we may assume that for all i there is an index m_i such that $\gamma_{m_i}^{(i)} = \alpha_1^{(i+1)}$ for all $1 \le i \le t$ (taken modulo t), and furthermore, $\alpha_{m_i}^{(i)} \not\le (\alpha_1^{(i+1)})_* \vee \alpha_{m_i+1}^{(i)}$ is satisfied. Set μ as the join of all the $\alpha_j^{(i)}$.

Let the underlying set of the posets be X. Start with the inequality

$$\alpha_{m_1}^{(1)} \le \gamma_{m_1}^{(1)} \vee (\alpha_{m_1+1}^{(1)} \wedge (\gamma_{m_1+1}^{(1)} \vee (\alpha_{m_1+2}^{(1)} \wedge \dots \vee (\alpha_{m_1-1}^{(1)} \wedge (\gamma_{m_1-1}^{(1)} \vee \alpha_{m_1}^{(1)}))))) =$$
$$\alpha_1^{(2)} \vee (\alpha_{m_1+1}^{(1)} \wedge (\gamma_{m_1+1}^{(1)} \vee (\alpha_{m_1+2}^{(1)} \wedge \dots \vee (\alpha_{m_1-1}^{(1)} \wedge (\gamma_{m_1-1}^{(1)} \vee \alpha_{m_1}^{(1)}))))) =$$
$$(\alpha_1^{(2)} \wedge (\gamma_1^{(2)} \vee (\alpha_2^{(2)} \wedge \dots \vee (\alpha_{m_2-1}^{(2)} \wedge (\gamma_{m_2-1}^{(2)} \vee \alpha_{m_2}^{(2)})))))\vee$$
$$(\alpha_{m_1+1}^{(1)} \wedge (\gamma_{m_1+1}^{(1)} \vee (\alpha_{m_1+2}^{(1)} \wedge \dots \vee (\alpha_{m_1-1}^{(1)} \wedge (\gamma_{m_1-1}^{(1)} \vee \alpha_{m_1}^{(1)}))))).$$

We will call a sequence of L-elements $a = c_0, \dots, c_r = b$ a *realization of the $\alpha_{m_1}^{(1)}$ edge* (a, b) if the following are satisfied:

- For all $1 \le s \le r$, (c_s, c_{s+1}) is an edge of either $\alpha_{m_1}^{(1)}$, or $\alpha_{m_2}^{(2)}$, or $\gamma_j^{(2)}$ for an $1 \le j < m_2$, or $\gamma_j^{(1)}$ for an $1 \le j < k_1$, $j \ne m_1$.
- There are indices $0 = r_0 < r_1 < \dots < r_h = r$ such that
  - For any even $0 \le h' < h$, $(c_{r_{h'}}, c_{r_{h'+1}}) \in \alpha_1^{(2)}$, and for any odd $0 \le h' < h$, $(c_{r_{h'}}, c_{r_{h'+1}}) \in \alpha_{m_1+1}^{(1)}$,
  - For even h', if $r_{h'} \le s_1 < s_2 \le r_{h'+1}$ and i_0 are such that the set
  $$\{s : ((c_s, c_{s+1}) \in \alpha_{m_2}^{(2)} \vee \exists i : i_0 \le i < m_2, (c_s, c_{s+1}) \in \gamma_i^{(2)})\}$$
  contains all s' satisfying $s_1 \le s' < s_2$ but does not contain $s_1 - 1$ and s_2, then $(c_{s_1}, c_{s_2}) \in \alpha_{i_0}^{(2)}$,
  - For odd h', if $r_{h'} \le s_1 < s_2 \le r_{h'+1}$ and i_0 are such that the set
  $$\{s : ((c_s, c_{s+1}) \in \alpha_{m_1}^{(1)} \vee \exists i : i \in \{i_0, i_0+1, \dots, m_1 - 1\}, (c_s, c_{s+1}) \in \gamma_i^{(2)})\}$$
  contains all s' satisfying $s_1 \le j' < s_2$ but does not contain $s_1 - 1$ and s_2, then $(c_{s_1}, c_{s_2}) \in \alpha_{i_0}$.

Note the connection between the definition of a realization and the inequality preceding it: the inequality tells us that each $\alpha_{m_1}^{(1)}$ edge will have a realization.

For each $\alpha_{m_1}^{(1)}$ edge we choose a single realization by the axiom of choice to get a *canonical realization.* We likewise obtain canonical realizations for $\alpha_{m_i}^{(i)}$ edges for all $1 \le i \le t$ (just switch the lower indices everywhere in the definition cyclically by $i - 1$). If an edge is at the same time an $\alpha_{m_{i_1}}^{(i_1)}$ and

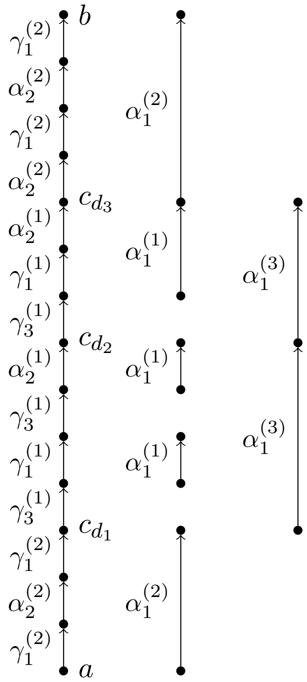

FIGURE 4. A realization of the $\alpha_{m_1}^{(1)}$ edge (a, b) with $t = 3$, $k_1 = 3$, $k_2 = 2$, $k_3 = 3$, $m_1 = 2$, $m_2 = 2$, $m_3 = 1$. Nodes to the left/right of each other are meant to represent the same element.

an $\alpha_{m_{i_2}}^{(i_2)}$ edge, it will get a canonical realization both as an $\alpha_{m_{i_1}}^{(i_1)}$ and as an $\alpha_{m_{i_2}}^{(i_2)}$ edge.

For an edge $(x_1, x_2) \in \alpha_{m_i}^{(i)}$, denote with $\mathcal{R}_{(x_1,x_2)}^{(i)}$ the set of the edges of the canonical realization of (x_1, x_2) as an $\alpha_{m_i}^{(i)}$ edge, and set

$$\mathcal{K}_{(x_1,x_2)}^{(i)} = \{(y_1, y_2) \in \mathcal{R}_{(x_1,x_2)}^{(i)} : y_2 \neq x_2, (y_1, y_2) \in \alpha_{m_i}^{(i)} \cup \alpha_{m_{i+1}}^{(i+1)}\}.$$

Define a *rank* of the edge (x_1, x_2): it will be zero if $\mathcal{K}_{(x_1,x_2)}^{(i)} = \emptyset$, otherwise, the rank is recursively defined as the maximal rank of the elements of $\mathcal{K}_{(x_1,x_2)}^{(i)}$ plus one. As μ satisfies DCC, all $\alpha_{m_i}^{(i)}$ edge has a (finite) rank.

Take an $\alpha_{m_1}^{(1)}$ edge (x_1, x_2) of rank 0. In its canonical realization there is at most one edge that is also an $\alpha_{m_1}^{(1)}$ or an $\alpha_{m_2}^{(2)}$ edge (the last edge of the realization). This means that

$$\begin{aligned}(x_1,x_2)\in\eta_0^{(1)} :=&\\ &(\alpha_{m_1}^{(1)}\wedge((\alpha_1^{(2)}\wedge(\gamma_1^{(2)}\vee(\alpha_2^{(2)}\wedge\cdots\vee(\alpha_{m_2-1}^{(2)}\wedge\gamma_{m_2-1}^{(2)}))))\vee\\ &(\alpha_{m_1+1}^{(1)}\wedge(\gamma_{m_1+1}^{(1)}\vee(\alpha_{m_1+2}^{(1)}\wedge\cdots\vee(\alpha_{m_1-1}^{(1)}\wedge(\gamma_{m_1-1}^{(1)}\vee\alpha_{m_1}^{(1)})))))))\vee\\ &(\alpha_{m_1}^{(1)}\wedge((\alpha_1^{(2)}\wedge(\gamma_1^{(2)}\vee(\alpha_2^{(2)}\wedge\cdots\vee(\alpha_{m_2-1}^{(2)}\wedge(\gamma_{m_2-1}^{(2)}\vee\alpha_{m_2}^{(2)})))))\vee\\ &(\alpha_{m_1+1}^{(1)}\wedge(\gamma_{m_1+1}^{(1)}\vee(\alpha_{m_1+2}^{(1)}\wedge\cdots\vee(\alpha_{m_1-1}^{(1)}\wedge\gamma_{m_1-1}^{(1)})))))).\end{aligned}$$

We likewise define the L-elements $\eta_0^{(i)}$ for all $1\le i\le t$, with each containing all the $\alpha_{m_i}^{(i)}$ edges of rank 0. Note that the long expression above is not to be read as one would initially think: it is *not* a join of four subexpression, but two, and each of those two is a meet of $\alpha_{m_1}^{(1)}$ and an other (lengthy) expression.

We recursively define L-elements $\eta_n^{(i)}\le\alpha_{m_i}^{(i)}$ for all $1\le i\le t$ and all nonnegative integer n: for $n=0$ they are already defined, and for $n>0$ we set

$$\begin{aligned}\eta_n^{(i)} :=&\\ &(\alpha_{m_i}^{(i)}\wedge((\alpha_1^{(i+1)}\wedge(\gamma_1^{(i+1)}\vee(\alpha_2^{(i+1)}\wedge\cdots\vee(\alpha_{m_{i+1}-1}^{(i+1)}\wedge(\gamma_{m_{i+1}-1}^{(i+1)}\vee\eta_{n-1}^{(i+1)})))))\vee\\ &(\alpha_{m_i+1}^{(i)}\wedge(\gamma_{m_i+1}^{(i)}\vee(\alpha_{m_i+2}^{(i)}\wedge\cdots\vee(\alpha_{m_i-1}^{(i)}\wedge(\gamma_{m_i-1}^{(i)}\vee\alpha_{m_i}^{(i)})))))))\vee\\ &(\alpha_{m_i}^{(i)}\wedge((\alpha_1^{(i+1)}\wedge(\gamma_1^{(i+1)}\vee(\alpha_2^{(i+1)}\wedge\cdots\vee(\alpha_{m_{i+1}-1}^{(i+1)}\wedge(\gamma_{m_{i+1}-1}^{(i+1)}\vee\alpha_{m_{i+1}}^{(i+1)})))))\vee\\ &(\alpha_{m_i+1}^{(i)}\wedge(\gamma_{m_i+1}^{(i)}\vee(\alpha_{m_i+2}^{(i)}\wedge\cdots\vee(\alpha_{m_i-1}^{(i)}\wedge(\gamma_{m_i-1}^{(i)}\vee\eta_{n-1}^{(i)}))))))).\end{aligned}$$

If (x_1,x_2) is an $\alpha_{m_i}^{(i)}$ edge (x_1,x_2) of rank n, then in its canonical realization all the $\alpha_{m_i}^{(i)}$ and $\alpha_{m_{i+1}}^{(i+1)}$ edges except perhaps one have rank at most $n-1$. From this fact we can easily prove by induction that $(x_1,x_2)\in\eta_n^{(i)}$.

As $\alpha_{m_i}^{(i)}$ is a completely join irreducible element of $\mathbf{L}$, and it is the union of all $\eta_n^{(i)}$ (as they form an increasing chain, and all $\alpha_{m_i}^{(i)}$ edge has finite rank), there is an n_i so that $\alpha_{m_i}^{(i)}=\eta_{n_i}^{(i)}$. We can suppose that among all the n_i, n_1 is (one of) the smallest.

As $\eta_{n_1}^{(1)}$ is defined as the join of two elements of L that are smaller or equal than $\alpha_{m_1}^{(1)}$, and $\alpha_{m_1}^{(1)}$ is join irreducible in $\mathbf{L}$, $\alpha_{m_1}^{(1)}$ is equal to either

$$\begin{aligned}&\alpha_{m_1}^{(1)}\wedge((\alpha_1^{(2)}\wedge(\gamma_1^{(2)}\vee(\alpha_2^{(2)}\wedge\cdots\vee(\alpha_{m_2-1}^{(2)}\wedge(\gamma_{m_2-1}^{(2)}\vee\eta_{n_1-1}^{(2)})))))\vee\\ &\quad(\alpha_{m_1+1}^{(1)}\wedge(\gamma_{m_1+1}^{(1)}\vee(\alpha_{m_1+2}^{(1)}\wedge\cdots\vee(\alpha_{m_1-1}^{(1)}\wedge(\gamma_{m_1-1}^{(1)}\vee\alpha_{m_1}^{(1)}))))))\end{aligned}$$

or

$$\begin{aligned}&\alpha_{m_1}^{(1)}\wedge((\alpha_1^{(2)}\wedge(\gamma_1^{(2)}\vee(\alpha_2^{(2)}\wedge\cdots\vee(\alpha_{m_2-1}^{(2)}\wedge(\gamma_{m_2-1}^{(2)}\vee\alpha_{m_2}^{(2)})))))\vee\\ &\quad(\alpha_{m_1+1}^{(1)}\wedge(\gamma_{m_1+1}^{(1)}\vee(\alpha_{m_1+2}^{(1)}\wedge\cdots\vee(\alpha_{m_1-1}^{(1)}\wedge(\gamma_{m_1-1}^{(1)}\vee\eta_{n_1-1}^{(1)})))))).\end{aligned}$$

From both equalities follows an inequality of the type

$$\alpha_{m_1}^{(1)}\le(\alpha_1^{(2)}\wedge\delta_1)\vee(\alpha_{m_1+1}^{(1)}\wedge\delta_2).$$

Recall the definition of the $\alpha_j^{(i)}$ to see that a consequence of this is that $\alpha_1^{(2)} \leq \delta_1$ and $\alpha_{m_1+1}^{(1)} \leq \delta_2$.

From the $\alpha_1^{(2)} \leq \delta_1$ type inequality, we can deduce in the first case that

$$\alpha_{m_1-1}^{(1)} \leq \gamma_{m_1-1}^{(1)} \vee \eta_{n_1-1}^{(1)},$$

and in the second that

$$\alpha_{m_1-1}^{(1)} \leq \gamma_{m_1-1}^{(1)} \vee \eta_{n_1-1}^{(1)}.$$

Both are impossible: by the choice of n_1, $\eta_{n_1-1}^{(2)} \leq (\alpha_{m_2}^{(2)})_*$ and $\eta_{n_1-1}^{(1)} < (\alpha_{m_1}^{(1)})_*$, so either of these inequalities contradicts the fact that $\alpha_j^{(i)} \not\leq \gamma_j^{(i)} \vee (\alpha_{j+1}^{(i)})_*$ is satisfied by all possible i and j. □ □

Problem 4.1. Is it true that a lattice $\mathbf{L}$ is representable with DCC posets if and only if $E_{\mathbf{K}}$ contains no cycle for any $\mathbf{K} \leq \mathbf{L}$? Is it true if, moreover, $\mathbf{L}$ is finite?

Here is an overview of the known properties of the class of representable lattices.

Theorem 4.2. *For the class $\mathcal{R}$ of lattices representable with DCC posets:*

- *$\mathcal{R}$ is closed under taking sublattices and products,*
- *$\mathcal{R}$ does not contain the lattice $\mathbf{M}_3$,*
- *$\mathcal{R}$ contains all lower bounded lattices,*
- *even the finite part of $\mathcal{R}$ is not contained in $SD_\vee$,*
- *it is algorithmically decidable if a finite lattice $\mathbf{L}$ is in $\mathcal{R}$,*
- *if a lattice $\mathbf{L}$ is in $\mathcal{R}$, then $E_{\mathbf{L}}$ contains no cycle.*

Proof. The last two statements have been proved. $\mathbf{M}_3$ fails the condition of the last statement, because the three middle elements form a D-cycle, and there is an E-loop on that cycle. As it was mentioned in the introduction, all lower bounded lattices are representable with posets satisfying both DCC and ACC by Theorem 4.6 of [6].

For the statement about join semidistributivity, consider the lattice $\mathbf{D}_2$ (Figure 1). It is not join semidistributive, but we can show that it is in $\mathcal{R}$. This can be done either by following through on the construction described in the beginning of Section 3, or by directly applying Theorem 3.1: it is easy to check that $\mathcal{C}_{\mathbf{D}_2} = \{(c, a, d), (c, d, a), (d, b, c), (d, c, b)\}$, and we can define s so it maps the first two elements of $\mathcal{C}_{\mathbf{D}_2}$ to c and the second two to d. Now

$$T = \{(c, a), (d, b)\}$$

and

$$U = \{(a, a), (b, b), (c, c), (c, d), (d, c), (d, d)\},$$

so indeed $U \circ T$ does not contain a cycle.

The only item left is that $\mathcal{R}$ is closed to direct products. If $\mathbf{L}_i \in \mathcal{R}$ for all $i \in I$ such that $\mathbf{L}_i$ is represented by DCC posets on X_i (which can be assumed to be disjoint), $\prod_{i \in I} \mathbf{L}_i$ can be represented on $\bigcup_{i \in I} X_i$: the element $(l_i)_{i \in I}$ will be represented by the (disjoint) union of the posets representing the individual l_i. □ □

Problem 4.2. Is it true that $\mathcal{R}$ contains $SD_\vee$? Is there a nontrivial lattice quasi-identity satisfied by all members of $\mathcal{R}$?

References

[1] B.M. Schein: *A representation theorem for lattices*, Algebra universalis 2 (1972), pages 177–178.

[2] K.V. Adaricheva, V.A. Gorbunov, V.I. Tumanov: *Join-semidistributive lattices and convex geometries*, Advances in Mathematics, 173 (1) (2003), pages 1–49.

[3] N. Caspard: *The Lattice of Permutations Is Bounded*, International Journal of Algebra and Computation, 10 (4) (2000), pages 481–490.

[4] Ralph Freese, Jaroslav Ježek, and J. B. Nation: *Free Latices*, Mathematical Surveys and Monographs, vol. 42, Amer. Math. Soc., Providence, RI, 1995.

[5] P. Pudlák and J. Tůma: *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis 10 (1980), pages 74–95.

[6] M. V. Semenova: *Lattices that Are Embeddable in Suborder Lattices*, Algebra and Logic, 44 (4) (2005), pages 270–285.

[7] B. Sivák: *Representations of finite lattices by orders on finite sets*, Mathematica Slovaca, 28 (2) (1978), pages 203–215.

[8] P. M. Whitman: *Lattices, equivalence relations, and subgroups*, Bull. Amer. Math. Soc. 52 (1946), pages 507–522.