ON THE SYMMETRIC PARTS OF FINITELY GENERATED FREE LATTICES

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Abstract. An element of a finitely generated free lattices is called symmetric if it is fixed by all automorphisms of the lattice. We examine the lattice formed by these elements. It is known to be nonfinitely generated, and containing a sublattice isomorphic to any finitely generated free lattice. We show that it is also weakly atomic, and prove two further statements regarding the doubly prime elements of this lattice.

1. Introduction

Free structures are essential in universal algebra. For some algebraic classes (for example, semigroups and abelian groups), free algebras are quite trivial. They can also be almost hopelessly complicated. For the class of lattices, neither is the case: the study of free lattices yielded a large amount of deep yet digestible literature.

One of the earliest results in this subfield is due to Whitman [7]: for $m \geq n \geq 3$ free lattices generated by m elements are embeddable into the free lattice generated by n elements. In 2016, Czédli proved a sharper version of this: this embedding can be done in a *self-dual* way [1]. In 2019, this was strengtened further by him, Kunos, and the author: it can be done in a self-dual and *symmetric* way, that is, so the image of the embedding is closed to all the automorphisms and anti-automorphisms of the n-generates free lattice [2]. On the way to that result, we have made some observations about the sublattices of the finitely generated free lattices induced by the symmetric terms. In particular, these sublattices were shown to be nonfinitely generated for $n \geq 3$.

In the present paper, we continue the study of these lattices. We prove that they are weakly atomic. We extend the description of their structure by proving that for odd $n \geq 3$, they contain an element t so that each element of the lattice is either smaller or equal than t or larger or equal than its dual. Finally, we prove that it is decidable whether a given element is join(meet) prime/irreducible in the symmetric part of a free lattice.

As free lattices are crucial in the study of lattice varieties, we hope that these results will be useful in the study of lattice varieties characterized by symmetric identities.

Key words and phrases. lattices, free lattices, symmetrical terms, doubly prime, weakly atomic.

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2. Preliminaries

The lattice freely generated by the generating set $\{x_1, \ldots, x_n\}$ will be denoted by $\mathrm{FL}(n)$. The elements of $\mathrm{FL}(n)$ are called (n-ary) *lattice terms*. Free lattices (and their sublattices) satisfy *Whitman's Condition*: the inequality $u_1 \wedge \cdots \wedge u_k \leq v1 \vee \cdots \vee v_l$ holds in $\mathrm{FL}(n)$ if and only if there is either a $1 \leq i \leq k$ so that $u_i \leq v_1 \vee \cdots \vee v_l$ or a $1 \leq j \leq l$ so that $u_1 \vee \cdots \vee u_k \leq v_j$ (Theorem 1.8 of [4]).

The *dual* of a lattice term t, denoted by $\bar{t}$, is the term obtained from t by substituting meets with joins, and joins with meets. (In other words, $\bar{t}$ is the image of t by the unique anti-automorphism of $\mathrm{FL}(n)$ that fixes all the x_i.)

We denote by $\mathbf{S}_n$ the symmetric group on the set $\{1, 2, \ldots, n\}$. Let $\sigma \in S_n$. Then there is a unique automorphism of $\mathrm{FL}(n)$ mapping x_i to $x_{\sigma(i)}$. The image of t by this automorphism will be denoted by t_σ. The term t is *symmetric* if $t = t_\sigma$ for all σ. The symmetric terms form a sublattice of $\mathrm{FL}(n)$ denoted by $\mathrm{SFL}(n)$.

There are two *symmetrizations* of the term t:

$$\bigvee_\pi t := \bigvee_{\sigma \in S_n} t_\sigma,$$

and

$$\bigwedge_\pi t := \bigwedge_{\sigma \in S_n} t_\sigma.$$

(In this paper, we reserve the latter π for symmetrizations, and do not use it to refer to any particular permutation.) Clearly, the symmetrizations of a term are symmetric, and both symmetrizations of a symmetric term are the term itself.

The smallest element of $\mathrm{SFL}(n)$ is $0_{\mathrm{FL}(n)} = x_1 \wedge \cdots \wedge x_n$, and its largest element is $1_{\mathrm{FL}(n)} = x_1 \vee \cdots \vee x_n$. By [2], $\mathrm{SFL}(n) \backslash \{0_{\mathrm{FL}(n)}, 1_{\mathrm{FL}(n)}\}$ is a sublattice of $\mathrm{SFL}(n)$, we will denote this lattice by $\mathrm{SFL}^*(n)$. The smallest element of $\mathrm{SFL}^*(n)$ is

$$m := \bigvee_\pi (x_1 \wedge \cdots \wedge x_{n-1}),$$

and the largest element of $\mathrm{SFL}(n)$ is $\overline{m}$. All the elements of $\mathrm{SFL}^*(n)$ are *near-unanimity operations*, i.e. they satisfy the idenetities

$$f(x, x, \ldots, x, y) \approx f(x, x, \ldots, y, x) \approx \cdots \approx f(y, x, \ldots, x) \approx x.$$

A key concept in the study of free lattices and lattice varieties is boundedness, introduced by McKenzie in [6]. A lattice homomorphism $f : \mathbf{L}_1 \to \mathbf{L}_2$ is *lower bounded* if for all $l_2 \in L_2$, the set $\{l_1 \in L_1 : f(l_1) \leq l_2\}$ has a largest element. It is *upper bounded* if for all $l_2 \in L_2$, the set $\{l_1 \in L_1 : f(l_1) \geq l_2\}$ has a smallest element. A homomorphism is *bounded* if it is both lower and upper bounded. A lattice $\mathbf{L}$ is bounded if there is a bounded surjective homomorphism $\mathrm{FL}(n) \to \mathbf{L}$ for some n. By Theorem 2.13 of [4], if $\mathbf{L}$ is bounded, then *any* lattice homomorphism mapping to $\mathbf{L}$ is bounded. By Corollary 2.17 of [4], the class of finite bounded lattices is closed to taking homomorphic images, sublattices, and finite direct products.

We define the *depth* of a term $t \in \mathrm{SL}(n)$ recursively as follows: the depth of the x_i are 0. Let $k \geq 0$, and assume that we have defined terms of depth s for $s \leq 2k$. Then the depth of a term t will be $2k+1$ if it does not already have a (smaller) depth, and it is the meet of some terms of depth at most $2k$. The depth of t will be $2k+2$ if it does not already have depth at most $2k+1$, but it is a join of terms with depths at most $2k+1$.

An element l of a lattice $\mathbf{L}$ is *join prime*, if for any $k_1, k_2 \in L$, $k_1 \vee k_2 \geq l$ implies that $k_1 \geq l$ or $k_2 \geq l$. It is *join irreducible* if $k_1 \vee k_2 = l$ implies that $k_1 = l$ or $k_2 = l$. Meet primeness and meet irreducibility are defined dually. An element is *doubly prime* if it is both join and meet prime, and *doubly irreducible* if it is both join and meet irreducible. Join/meet/doubly prime elements are clearly join/meet/doubly irreducible. Also, doubly irreducible elements must be contained in any generating set of a lattice. Accordingly, in $\mathrm{FL}(n)$ the only doubly irreducible elements are the free generators, which are also doubly prime.

The lattice $\mathbf{L}$ is *weakly atomic* if for any $l_1, l_2 \in L$ satisfying $l_1 < l_2$, there are elements $u, v \in L$ so that $u_1 \leq u \prec v \leq u_2$. (In other words, all intervals contain a covering pair.)

In this paper, we will use the lattices $\mathbf{2}$ (the two element chain), $\mathbf{M}_n$ (for $n \geq 3$), as well as $\mathbf{N}_5$ to $\mathbf{L}_4$. We note that all of these are subdirectly irreducible lattice that generate small lattice varieties (see [5]).

3. Weak atomicity

In [4], it is proved–following Alan Day–that $\mathrm{FL}(n)$ is weakly atomic for all n (Corollary 2.85). Actually, weak atomicity is proved there for a substantially larger class, including, for example, all finitely generated projective lattices, however, all elements of this class are finitely generated. As we have seen in [2], $\mathrm{SFL}(n)$ is not finitely generated for $n \geq 3$, because it contains infinitely many doubly prime elements. Still, by adapting Day's argument we can prove that weak atomicity holds for these lattices.

Theorem 3.1. *For any natural $n > 2$, the lattice $\mathrm{SFL}(n)$ is weakly atomic.*

Proof. Suppose that $p < q$ holds in $\mathrm{SFL}(n)$. By Theorem 2.84 of [4], there is a finite bounded lattice $\mathbf{K}$ and a surjective homomorphism $f : \mathrm{FL}(n) \to \mathbf{K}$ so that $f(p) < f(q)$. Let $q' \in \mathrm{SFL}(n)$ be such that $p < q' \leq q$, $f(p) < f(q')$, but there is no $q_1 \in \mathrm{SFL}(n)$ so that $p \leq q_1 < q'$ and $f(p) < f(q_1) < f(q')$. Such q' exists because of the finiteness of $\mathbf{K}$.

We choose an enumeration of the elements of S_n: $S_n := \{\sigma_1, \ldots, \sigma_{n!}\}$, and then we define the mapping

$$\widehat{f} : \mathrm{FL}(n) \to \mathbf{K}^{n!}, \quad \widehat{f}(r) := (f(r_{\sigma_i}))_{i=1}^{n!}.$$

The mapping $\widehat{f}$ is bounded, because $\mathbf{K}^{n!}$ is bounded. Let $\theta := \ker \widehat{f}$.

There is a largest element s in the θ-class of p. As p is a symmetrical term, $\widehat{f}(p)$ is a constant tuple. As s is in the same θ-class as p, $\widehat{f}(s)$ is also a constant tuple. This means that for all $1 \leq i \leq n!$, $\widehat{f}(s_{\sigma_i}) = \widehat{f}(s)$, and so $s_{\sigma_i} \leq s$ by the definition of s. Hence, $s = \bigvee_\pi s$ (notice that the right side contains s), which means that s is symmetric. Likewise, the smallest element t in the θ-class of q' is also symmetric.

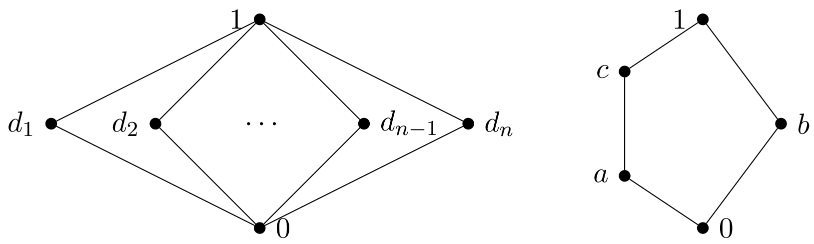

FIGURE 1. The lattices $\mathbf{M}_n$ and $\mathbf{N}_5$

We claim that $p \leq (p \vee t) \wedge s \prec p \vee t \leq q' \leq q$ holds in SFL(n). The inequalities are obvious, and $(p \vee t) \wedge s$ and $p \vee t$ cannot coincide, because the former is in the θ-class of p, and the latter is in the θ-class of q'.

Suppose that $(p \vee t) \wedge s < u < p \vee t$ for some $u \in$ SFL(n). As $u \geq p$, $u \geq t$ cannot hold, so u is not in the θ-class of q'. Also, as $u \leq s$ is impossible, u cannot be in the θ-class of p. Thus, $\widehat{f}(p) < \widehat{f}(u) < \widehat{f}(q')$, but as both p, u, and q' are symmetrical, this means that $f(p) < f(u) < f(q')$, contradicting the choice of q'. Therefore, $p \vee t$ indeed covers $(p \vee t) \wedge s$ in SFL(n), and we are ready.

□

4. THE CUT INDUCED BY $\mathbf{N}_5$

In [2] we introduced a *natural cut* on SFL(n), a homomorphism SFL(n) $\to$ $\mathbf{2}$ that mapped a term t to 0 iff it mapped $\bar{t}$ to 1. This cut was induced by the lattice $\mathbf{M}_n$ in the sense that t was mapped into 0 if and only if $t(d_1, \ldots, d_n)$ held in $\mathbf{M}_n$. In this section, we describe a somewhat similar cut that is induced by the lattice $\mathbf{N}_5$.

Definition 4.1. Fix an $n \geq 3$, $1 \leq k \leq n-1$, a finite lattice $\mathbf{L}$ and a tuple $(a_1, \ldots, a_n)$ so that $\{a_1, \ldots, a_n\}$ is a (not necessarily n-element) generating set of $\mathbf{L}$. We define the n-ary lattice terms $p_{a_i}^{(0)} := x_i$ and then recursively

$$p_l^{(k+1)} := \bigvee_{\substack{d \in \mathbf{N} \\ l_1 \wedge \cdots \wedge l_d \leq l}} (p_{l_1}^{(k)} \wedge \cdots \wedge p_{l_d}^{(k)}).$$

(If $a_i = a_j$, then $p_{a_i}^{(0)}$ and $p_{a_j}^{(0)}$ will be considered different, but for $k > 0$, $p_{a_i}^{(k)}$ and $p_{a_j}^{(k)}$ will be the same.) These terms are called the *upper limit terms* corresponding to the tuple $(a_1, \ldots, a_n)$.

It is easy to see the following:

Proposition 4.2. *For each k and $l \in L$, $p_l^{(k)}$ is the largest n-ary lattice term of depth at most $2k$ that induces a term function of $\mathbf{L}$ that maps $(a_1, \ldots, a_n)$ to an element smaller or equal than l.* □

As $a_1 \wedge \cdots \wedge a_n$ is the smallest element of $\mathbf{L}$, $p_l^{(k)}$ is defined for every l for $k \geq 1$. It is obvious from the definition that for such k, the mapping

$$\mathbf{L} \to \mathrm{FL}(n),\ l \mapsto p_l^{(k)}$$

is monotone. It is also injective for large enough k: as $\{a_1, \dots, a_n\}$ is a generating set, there is a term s so that $s(a_1, \dots, a_n) = l$. Obviously, s cannot be smaller than any term that maps $(a_1, \dots, a_n)$ to anything strictly smaller than l, so if $2k$ is not smaller than the depth of s, then $p_l^{(k)}(a_1, \dots, a_n)$ must be l.

Definition 4.3. A finite lattice $\mathbf{L}$ together with an n-tuple of elements $(a_1, \dots, a_n)$ is called *cut-inducing* if

$$|\{f(a_1, \dots, a_n) : f \in \mathrm{SFL}^*(n)\}| = 2,$$

and in this case the the kernel of the homomorphism

$$\mathrm{SFL}^*(n) \to \mathbf{L},\ f \mapsto f(a_1, \dots, a_n)$$

is called the *cut induced* by this tuple. This cut is *self-dual* if for all $f \in \mathrm{SFL}^*(n)$, f and $\overline{f}$ are in different classes of the cut.

So the aforementioned natural cut of $\mathbf{M}_n$ is the cut induced by $(a_1, \dots, a_n)$ in that lattice.

Now we consider the lattice $\mathbf{N}_5$. Observe that if $f \in \mathrm{SFL}(n)$ is a near unanimity operation, and $1 \leq k \leq n-2$, then $f(\underbrace{a \dots a}_{k} \underbrace{c \dots c}_{n-k-1} b)$ can be only a or c. As $m(\underbrace{a \dots a}_{k} \underbrace{c \dots c}_{n-k-1} b) = a$ and $\overline{m}(\underbrace{a \dots a}_{k} \underbrace{c \dots c}_{n-k-1} b) = c$, the tuple $(\underbrace{a, \dots, a,}_{k} \underbrace{c, \dots c,}_{n-k-1} b)$ is cut-inducing. Obviously, the induced cut will be self-dual precisely if $k = \frac{n-1}{2}$.

A straightforward calculation shows the following:

Proposition 4.4. *The upper limit terms corresponding to* $(\underbrace{a, \dots, a,}_{k} \underbrace{c, \dots c,}_{n-k-1} b)$ *in* $\mathbf{N}_5$ *are*

$$p_0^{(1)} = (x_1 \wedge x_n) \vee \dots \vee (x_{n-1} \wedge x_n)$$
$$p_a^{(1)} = x_1 \vee \dots \vee x_k \vee (x_{k+1} \wedge x_n) \vee \dots \vee (x_{n-1} \wedge x_n)$$
$$p_b^{(1)} = x_n$$
$$p_c^{(1)} = x_1 \vee \dots \vee x_{n-1}$$
$$p_1^{(1)} = x_1 \vee \dots \vee x_n$$
$$p_0^{(2)} = p_b^{(1)} \wedge p_c^{(1)} = x_n \wedge (x_1 \vee \dots \vee x_{n-1})$$
$$p_a^{(2)} = p_a^{(1)} \vee (p_b^{(1)} \wedge p_c^{(1)}) = x_1 \vee \dots \vee x_k \vee (x_n \wedge (x_1 \vee \dots \vee x_{n-1}))$$
$$p_b^{(2)} = p_b^{(1)} = x_n$$
$$p_c^{(2)} = p_c^{(1)} = x_1 \vee \dots \vee x_{n-1}$$
$$p_1^{(2)} = x_1 \vee \dots \vee x_n$$
$$p_l^{(k)} = p_l^{(2)} \text{ for any } k > 2 \text{ and } l \in N_5$$

□

As for any symmetric n-ary term f, $f(\underbrace{a\dots a}_{k}\underbrace{c\dots c}_{n-k-1}b) = c$ is equivalent to $\overline{f}(\underbrace{c\dots c}_{k}\underbrace{a\dots a}_{n-k-1}b) = c$:

Corollary 4.5. *Let $n \geq 3$, $1 \leq k \leq n-2$, and $f \in \mathrm{SFL}^*(n)$. Then $f(\underbrace{a\dots a}_{k}\underbrace{c\dots c}_{n-k-1}b) = a$ if and only if*

$$f \leq \bigwedge_{\pi}(x_1 \vee \dots \vee x_k \vee (x_n \wedge (x_1 \vee \dots \vee x_{n-1}))),$$

and $f(\underbrace{a\dots a}_{k}\underbrace{c\dots c}_{n-k-1}b) = c$ if and only if

$$f \geq \bigvee_{\pi}(x_1 \wedge \dots \wedge x_k \wedge (x_n \vee (x_1 \wedge \dots \wedge x_{n-1}))).$$

□

This means that in these cuts, both congruence classes have a smallest and a largest element. This was not the case for the cut induced by $\mathbf{M}_n$: the upper limit terms there did not converge, and neither did the $\bigwedge_\pi p_0^{(k)}$. On, the other hand, there is a similarity: in the case of $\mathbf{M}_3$, all the $\bigwedge_\pi p_0^{(k)}$ and $\bigvee_\pi p_1^{(k)}$ were doubly prime elements. This also holds for $\mathbf{N}_5$:

Proposition 4.6. *Let $n \geq 3$, $1 \leq k \leq n-2$, and*

$$t := \bigwedge_{\pi}(x_1 \vee \dots \vee x_k \vee (x_n \wedge (x_1 \vee \dots \vee x_{n-1}))) \in \mathrm{SFL}(n).$$

Then t is a doubly prime element of $\mathrm{SFL}(n)$.

Proof. Suppose first that $t \leq h_1 \vee h_2$ for some $h_1, h_2 \in \mathrm{SFL}^*(n)$. By Whitman's Condition, either $t \leq h_1$, $t \leq h_2$, or there is a permutation σ such that

$$(x_1 \vee \dots \vee x_k \vee (x_n \wedge (x_1 \vee \dots \vee x_{n-1})))_\sigma \leq h_1 \vee h_2.$$

As $h_1 \vee h_2$ is symmetric, in the latter case

$$x_1 \vee \dots \vee x_k \vee (x_n \wedge (x_1 \vee \dots \vee x_{n-1})) \leq h_1 \vee h_2,$$

and so $x_1 \leq h_1 \vee h_2$. As x_1 is a double prime element of $\mathrm{FL}(n)$, either $x_1 \leq h_1$ or $x_1 \leq h_2$. But as h_1 and h_2 are symmetric, this can only happen if one of them equals $x_1 \vee \dots \vee x_n$, the largest element of $\mathrm{SFL}(n)$. Therefore, t is join prime.

It is a straightforward consequence of Corollary 4.5 that t is meet prime: if $h_1, h_2 \not\leq t$, then $h_1, h_2 \geq \overline{t}$, and $h_1 \wedge h_2 \geq \overline{t}$, so $h_1 \wedge h_2 \not\leq t$. □

It is not hard to find other cut-inducing tuples: for example, the triple (a, b, c) is cut-inducing in $\mathbf{L}_4$. (This is because $m(a, b, c) = 0$, $M(a, b, c) = a \vee b$, and there cannot be an $f \in \mathrm{SFL}(3)$ mapping (a, b, c) to a or b, for $\mathbf{L}_4$ has an automorphism swapping a and b and fixing c.) We could not find any that induced a symmetric cut different form the one induced by $\mathbf{N}_5$. (Different triples *can* induce this cut: take, for example, the triple (aa, bb, cc) of $\mathbf{N}_5^2$.) Thus we pose the following questions:

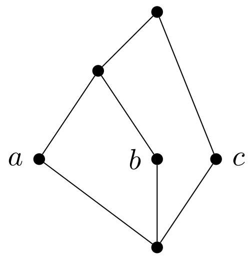

FIGURE 2. The lattice $\mathbf{L}_4$

Problem 4.7. Is it true that if n is even, then no n-element tuple of any finite lattice induces a symmetrical cut? Is it true that for odd n, the only tuple-induced symmetrical cut is the one induced by the tuple $(\underbrace{a,\dots,a,}_{\frac{n-1}{2}}\underbrace{c,\dots c,}_{\frac{n-1}{2}}b)$ of $\mathbf{N}_5$? If so, is it true that for odd n, there are precisely two homomorphisms $\iota : \mathrm{SFL}(n) \to \mathbf{2}$ satisfying $\iota(f) \neq \iota(\overline{f})$ for all f?

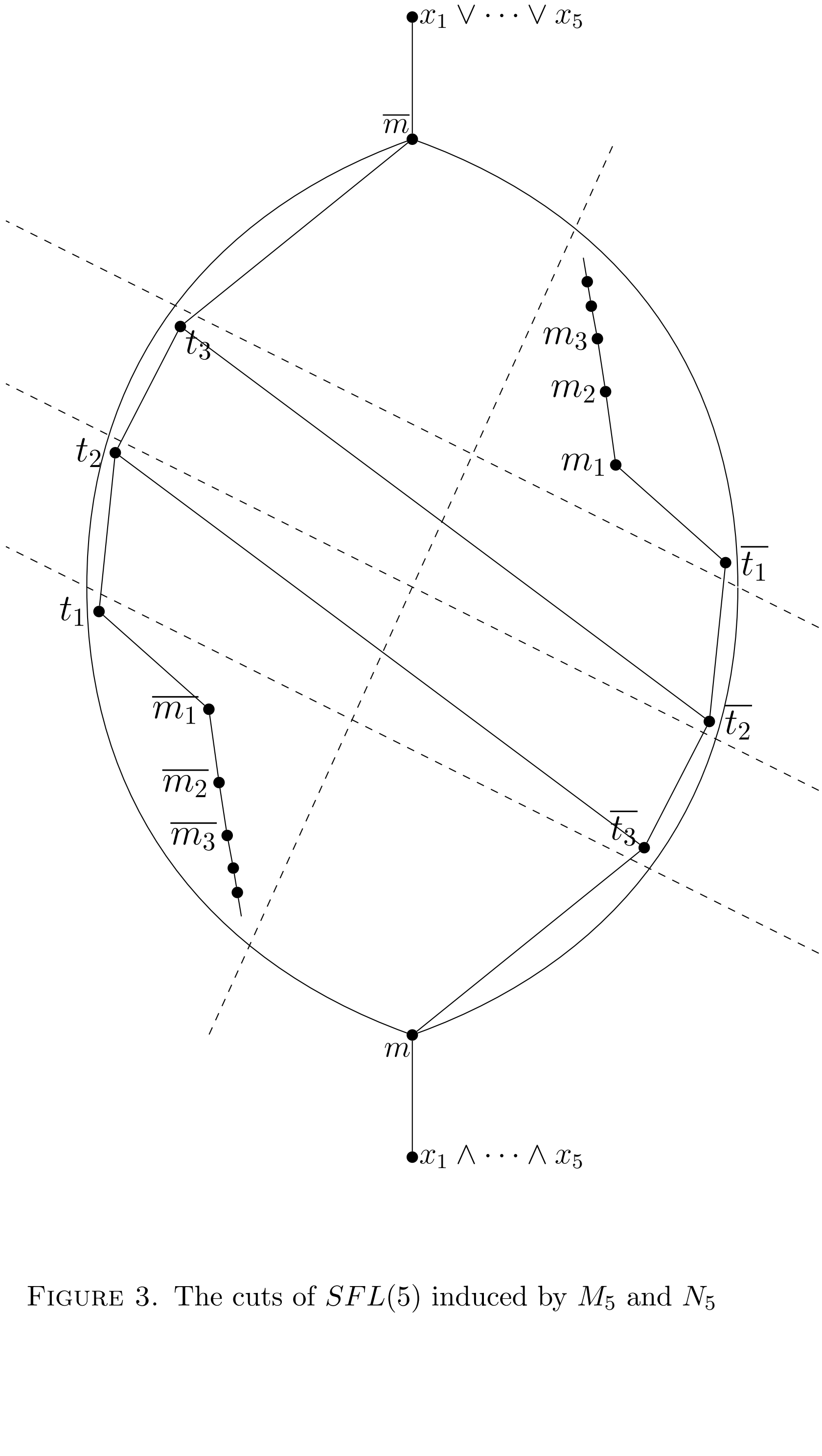

FIGURE 3. The cuts of $SFL(5)$ induced by M_5 and N_5

5. PRIMENESS AND REDUCIBILITY

Proposition 4.6 gives us new doubly prime elements of the lattice SFL(n). Recall that in [2], we proved that there is a chain of doubly irreducible elements $m_1 < m_2 < \dots$ in $\mathrm{SFL}(n)$ defined by

$$q_i^{(0)} := x_i$$
$$q_i^{(k)} := x_i \vee \bigvee_{\substack{j_1 \neq j_2 \\ j_1, j_2 \neq i}} (q_{j_1}^{(k-1)} \wedge q_{j_2}^{(k-1)})$$
$$m_k := \bigvee_{i_1 \neq i_2} (q_{i_1}^{(k-1)} \wedge q_{i_2}^{(k-1)}).$$

(Here i, j_1 and j_2 run from 1 to n.) These are not exactly the upper limit terms corresponding to $(d_1, \dots, d_n)$ of $\mathbf{M}_n$, but very similar to them. Indeed, defining m_k instead as the upper limit term $p_0^{(k)}$ corresponding to this tuple would likewise yield an increasing chain of double prime elements of $\mathrm{SFL}(n)$. (This fact can be proved the same way as the statement was proved for the original m_i in [2].)

In Figure 4, we show (in the case $n = 5$) how the new doubly prime elements come into this picture. Note that this picture shows the *poset* induced by these elements, so for example, $t_1 \vee \overline{t_3} = t_2$ is not true.

We pose the following questions about the double prime elements of $\mathrm{SFL}(n)$:

Problem 5.1. What is the largest antichain in $\mathrm{SFL}(n)$ containing only doubly prime elements?

Problem 5.2. Is $\mathrm{SFL}(n)$ generated by its doubly prime elements? In other worlds: Is $\mathrm{SFL}(n)$ a lattice that is freely generated by a poset?

The first of these problems is theoretically researchable by computer:

Theorem 5.3. *It is algorithmically decidable whether a given element of* $\mathrm{SFL}(n)$ *is join (meet) prime, and that whether it is join (meet) reducible.*

Proof. Fix a $t \in \mathrm{FL}(n)$. Denote by $\mathrm{Sub}(t)$ the subposet of $\mathrm{FL}(n)$ induced by the subterms of t. For any $u \in \mathrm{FL}(n)$, use the notation

$$D(u) := \{t' \in \mathrm{Sub}(t) : t' \leq u\}.$$

A consequence of Whitman's Condition is that for $s_1, s_2 \in \mathrm{FL}(n)$, $s_1 \vee s_2 \geq t$ depends only on $D(s_1)$ and $D(s_2)$. (This fact is immediate by induction on the depth of t.)

From now on, we assume t to by symmetric. The join primeness of t can be decided in the following way: first, compute $\mathrm{Sub}(t)$, and then all the order ideals of it. For each order ideal I, compute

$$I^* := D(\bigvee_{t' \in I} \bigvee_{\pi} t'),$$

this is an order ideal containing I. The order ideal I will be called *closed* if $I^* = I$. Note that if $h \in \mathrm{SFL}(n)$, then $D(h)$ is a closed order ideal. Take all pairs (I_1, I_2) of closed order ideals such that neither I_1 nor I_2 contains t. For each pair, it can be determined whether $D(u_1) = I_1$ and $D(u_2) = I_2$ implies $u_1 \vee u_2 \geq t$ or $u_1 \vee u_2 \not\geq t$.

If there is a pair so that $D(u_1) = I_1$ and $D(u_2) = I_2$ implies $u_1 \vee u_2 \geq t$, then t is not join prime in $\mathrm{SFL}(n)$: let

$$h_j = \bigvee_{t' \in I_j} \bigvee_{\pi} t'$$

for $j = 1, 2$, then $D(h_j) = I_j^* = I_j$, so $h_1 \vee h_2 \geq t$, and $h_1, h_2 \ngeq t$. Also, h_1 and h_2 are symmetrical, as they are the join of some symmetrical terms $\bigvee_{\pi} t'$.

On the other hand, if there is no such pair, then t is join prime in $\mathrm{SFL}(n)$: if $h_1 \vee h_2 \geq t$, and $h_1, h_2 \ngeq t$ hold for some $h_1, h_2 \in \mathrm{SFL}(n)$, then $D(h_1)$ and $D(h_2)$ are closed ideals not containing t, and $D(u_1) = I_1$ and $D(u_2) = I_2$ clearly cannot imply $u_1 \vee u_2 \ngeq t$.

Now we consider the join reducibility of t. Again, take all pairs (I_1, I_2) of closed order ideals such that neither I_1 nor I_2 contains t. For each pair, calculate

$$(\bigvee_{t' \in I_1} \bigvee_{\pi} t') \vee (\bigvee_{t' \in I_2} \bigvee_{\pi} t').$$

If any of these equals t, then t is reducible in $\mathrm{SFL}(n)$, for both $\bigvee_{t' \in I_1} \bigvee_{\pi} t'$ and $\bigvee_{t' \in I_2} \bigvee_{\pi} t'$ are symmetrical, and neither can equal t. (As $D(t)$ obviously contains t, while $D(\bigvee_{t' \in I_1} \bigvee_{\pi} t') = I_1^* = I_1$ and $D(\bigvee_{t' \in I_2} \bigvee_{\pi} t') = I_2^* = I_2$ do not.)

However, if neither of the aforementioned joins equal t, then t is irreducible in $\mathrm{SFL}(n)$. To see this, assume that $t = h_1 \vee h_2$ for some $h_1, h_2 < t$ in $\mathrm{SFL}(n)$. Let $I_1 = D(h_1)$ and $I_2 = D(h_2)$, then I_1 and I_2 are closed order ideals not containing t. Note that for $j = 1, 2$, $\bigvee_{t' \in I_j} \bigvee_{\pi} t' \leq h_j$ and $D(\bigvee_{t' \in I_j} \bigvee_{\pi} t') = I_j^* = I_j = D(I_j)$. From the former fact follows

$$(\bigvee_{t' \in I_1} \bigvee_{\pi} t') \vee (\bigvee_{t' \in I_2} \bigvee_{\pi} t') \leq h_1 \vee h_2 = t,$$

and from the latter

$$(\bigvee_{t' \in I_1} \bigvee_{\pi} t') \vee (\bigvee_{t' \in I_2} \bigvee_{\pi} t') \geq t,$$

as for any $s_1, s_2 \in \mathrm{FL}(n)$, $s_1 \geq s_2 \geq t$ depends only on $D(s_1)$ and $D(s_2)$. Hence, if t is a nontrivial join, then it is a nontrivial join of the form

$$(\bigvee_{t' \in I_1} \bigvee_{\pi} t') \vee (\bigvee_{t' \in I_2} \bigvee_{\pi} t')$$

for some pair (I_1, I_2) of closed order ideals not containing t.

Obviously, meet primeness and meet irreducibility can be decided in a dual way.

□

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