OUTER AND INNER MEDIANS IN SOME SMALL LATTICES

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Abstract. By median we mean a scheme that inputs three element of a lattice, and inputs an element that is an average of the three inputs in a certain sense. The medians of a given finite lattice form a new lattice that is usually larger than the original, but generates a (not necessarily strictly) smaller variety. A median is called inner if it is a term function. The inner median lattice is closely related to the symmetric part of the equational basis of the lattice. We examine the outer and inner median lattices of all lattices of six elements, and all subdirectly irreducible lattices of seven elements.

1. Preliminaries and general observations

Definition 1. Let $\mathbf{L}$ be a lattice. A mapping $f : L^3 \to L$ that is a symmetric monotone majority operation is called a *median* of $\mathbf{L}$. A median is *inner* if it is also a term function, and *outer* otherwise. The medians of $\mathbf{L}$ form a sublattice of $\mathbf{L}^{L^3}$, this will be called the *outer median lattice* of $\mathbf{L}$, and denoted by $\operatorname{OutMed}\mathbf{L}$. In this, the inner medians form a sublattice, this *inner median lattice* will be denoted by $\operatorname{InnMed}\mathbf{L}$.

Definition 2. Let $\mathcal{V}_1, \mathcal{V}_2$ be lattice varieties. Then the class of lattices $\mathcal{V}_1 \circ \mathcal{V}_2$ is defined by

$$\mathbf{L} \in \mathcal{V}_1 \circ \mathcal{V}_2 \Leftrightarrow \exists \theta \in \operatorname{Con}\mathbf{L} : \mathbf{L}/\theta \in \mathcal{V}_2, \text{ all the } \theta\text{-classes are in } \mathcal{V}_1$$

In the case when $\mathcal{V}_1$ is the variety of distributive lattices, we use the notation $\mathcal{V}_2^d$ instead of $\mathcal{V}_1 \circ \mathcal{V}_2$.

The following if an easy and well-known property of varietial products.

Proposition 3. *For all lattice varieties $\mathcal{V}_1, \mathcal{V}_2$, $\mathcal{V}_1 \circ \mathcal{V}_2$ is also a lattice variety.* □

Definition 4. Define the lattice terms

$$m(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z),$$

and

$$M(x, y, z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z),$$

the *lower* and *upper lattice medians*, respectively.

By Lemmas 4.4 and 4.5 of [1], m and M are the smallest and largest elements of $\operatorname{OutMed}\mathbf{L}$ (and therefore also $\operatorname{InnMed}\mathbf{L}$), respectively. It is well known that a lattice is distributive iff the lower and upper medians induce the same term functions on it.

Key words and phrases. lattices, free lattices, symmetrical terms, lattice median, inner median, variety product.

Supported by the ÚNKP-21-4-SZTE-512 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund.

FIGURE 1. The lattices $\mathbf{M}_3$ and $\mathbf{N}_5$

Definition 5. For any lattice $\mathbf{L}$, define the congruence $\theta^d \in \operatorname{Con}\mathbf{L}$ as the congruence generated by all the pairs $(m(a,b,c), M(a,b,c))$, where $a,b,c \in L$.

Proposition 6. *For any lattice variety $\mathcal{V}$ and lattice $\mathbf{L}$,*

$$\mathbf{L} \in \mathcal{V}^d \Leftrightarrow \textit{all the } \theta^d\textit{-clases of } \mathbf{L} \textit{ are in } \mathcal{V}.$$

Proof. Suppose that $\mathbf{L} \in \mathcal{V}^d$, then there is a congruence θ of $\mathbf{L}$ so that $\mathbf{L}/\theta$ is distributive and the θ-classes are in $\mathcal{V}$. As the lower and upper medians of $\mathbf{L}/\theta$ coincide:

$$m(a,b,c)/\theta = m(a/\theta, b/\theta, c/\theta) = M(a/\theta, b/\theta, c/\theta) = M(a,b,c)/\theta$$

for all $a,b,c \in L$, and hence $\theta^d \leq \theta$. Therefore the θ^d-classes are sublattices of θ-classes, and thus are in $\mathcal{V}$.

The other direction is trivial. □

Theorem 7. *Suppose that $\mathcal{V}$ is a lattice variety, and $\mathbf{L} \in \mathcal{V}^d$. Then* $\operatorname{OutMed}\mathbf{L} \in \mathcal{V}$.

Proof. Suppose that $f \in \operatorname{OutMed}\mathbf{L}$, and $a,b,c \in L$. As f is monotone and a majority operation, $f(a,b,c) \geq f(a\wedge b, a\wedge b, c) = a\wedge b$, $f(a,b,c) \geq f(a\wedge c, b, a\wedge c) = a \wedge c$, and $f(a,b,c) \geq f(a, b\wedge c, b\wedge c) = b\wedge c$. Thus $f(a,b,c) \geq m(a,b,c)$, and similarly, $f(a,b,c) \leq M(a,b,c)$. So any median of $\mathbf{L}$ maps the triple (a,b,c) into the interval $[m(a,b,c), M(a,b,c)]$. So

$$\operatorname{OutMed}\mathbf{L} \leq \Pi_{(a,b,c)\in L^3}[m(a,b,c), M(a,b,c)].$$

By Proposition 6, the lattice $[m(a,b,c), M(a,b,c)]$ is in $\mathcal{V}$, and therefore $\operatorname{OutMed}\mathbf{L}$ is also in $\mathcal{V}$. □

2. Calculating outer and inner median lattices

Definition 8. Let $\mathbf{L}$ be a lattice. For each triple $(l_1, l_2, l_3) \in L^3$, we call $[m(l_1,l_2,l_3), M(l_1,l_2,l_3)]$ the *permitted interval* of this triple.

The poset $\mathcal{T}_{\mathbf{L}}$ is defined the following way: its underlying set is the set of three-element subsets of $\mathbf{L}$ having nontrivial permitted interval. The order on $\mathcal{T}_{\mathbf{L}}$ is defined by

$$(l_1,l_2,l_3) \leq (k_1,k_2,k_3) \Leftrightarrow \exists\,\sigma \in S_{\{1,2,3\}} : \forall\, 1 \leq i \leq 3 : \, l_i \leq k_{\sigma(i)}.$$

An order homomorphism $\mathcal{T}_{\mathbf{L}} \to \mathbf{L}$ is called *permitted* if it maps each triple to an element of its permitted interval.

There is a natural bijection the medians of $\mathbf{L}$ and the permitted homomorphisms. (If $\mathbf{L}$ is distributive, then $\mathcal{T}_{\mathbf{L}}$ is empty. In this case, the empty set is such a homomorphism, and indeed, $\operatorname{OutMed}\mathbf{L}$ is trivial.)

When calculating $\mathcal{T}_{\mathbf{L}} \to \mathbf{L}$, it is helpful to remember that all triples in it contain either three pairwise incomparable elements, or two comparable elements and a third that is incomparable to the other two.

Proposition 9. *If $\mathbf{L}$ is either the linear sum of the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$, or is obtained by gluing $\mathbf{L}_2$ to $\mathbf{L}_1$, then*

$$\operatorname{OutMed}\mathbf{L} \cong \operatorname{OutMed}\mathbf{L}_1 \times \operatorname{OutMed}\mathbf{L}_2$$

Proof. If $(k_1, k_2, k_3) \in (L_1 \cup L_2)^3 \backslash (L_1^3 \cup L_2^3)$, then there is either a smallest or a largest element in the set $\{k_1, k_2, k_3\}$, which means $\{k_1, k_2, k_3\} \notin \mathcal{T}_{\mathbf{L}}$. Therefore,

$$\mathcal{T}_{\mathbf{L}} = \mathcal{T}_{\mathbf{L}_1} \cup \mathcal{T}_{\mathbf{L}_2}.$$

Any element of the permitted interval of some triple in $\mathcal{T}_{\mathbf{L}_1}$ is smaller or equal than any element of the permitted interval of some triple in $\mathcal{T}_{\mathbf{L}_2}$. Thus the set-theoretic join of a permitted homomorphism of $\mathbf{L}_1$ and a permitted homomorphism of $\mathbf{L}_2$ is a permitted homomorphism of $\mathbf{L}$, and we are ready. □

Proposition 10. *If the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ generate the same lattice variety, then $\operatorname{InnMed}\mathbf{L}_1 \cong \operatorname{InnMed}\mathbf{L}_2$.*

Proof. We define two congruences of SFL(3): for $i = 1, 2$ and $h_1, h_2 \in \mathrm{SFL}(3)$

$$(h_1, h_2) \in \theta_i \Leftrightarrow \forall, (a, b, c) \in L_i^3 : h_1(a, b, c) = h_2(a, b, c).$$

Obiously, $\operatorname{InnMed}\mathbf{L}_i \cong \mathrm{SFL}(3)/\theta_i$. As $(h_1, h_2) \in \theta_i$ if and only if the identity $h_1 \approx h_2$ holds in $\mathbf{L}_i$, and $\mathbf{L}_1$ and $\mathbf{L}_2$ satisfy the same identities, $\theta_1 = \theta_2$, and the result follows. □

Theorem 11. *For a finite lattice $\mathbf{L}$, the following are equivalent.*

(1) $|\operatorname{OutMed}\mathbf{L}| \le 2$,
(2) $\operatorname{OutMed}\mathbf{L} = \operatorname{InnMed}\mathbf{L}$,
(3) *of the 3-generated sublattices of $\mathbf{L}$, at most one is nondistributive, and if there is a nondistributive 3-generated sublattice, then it is isomorphic to $\mathbf{N}_5$.*

Proof.

- (1) ⇒ (2): As the smallest and largest elements of $\operatorname{OutMed}\mathbf{L}$ are inner medians, this is obvious.
- (2) ⇒ (3): Suppose first $\mathbf{M}_3$ is in the variety generated by $\mathbf{L}$. As $\mathbf{L}$ is finite, this implies by Jónsson's Lemma that $\mathbf{L}$ has a sublattice $\mathbf{L}'$ that has a congruence μ so that $\mathbf{L}'/\mu \cong \mathbf{M}_3$. Choose $l_1, l_2, l_3 \in L'$ so that

  $$[l_1/\mu, l_2/\mu, l_3/\mu] = \mathbf{L}'/\mu,$$

  and define

  $$l_1^* := (m(l_1, l_2, l_3) \vee l_1) \wedge M(l_1, l_2, l_3).$$

Now l_1^* is in the μ-class of l_1, and it is in the permitted interval of $\{l_1, l_2, l_3\}$. We define the mapping

$$\mathcal{T}_{\mathbf{L}} \to \mathbf{L} : \{k_1, k_2, k_3\} \mapsto \begin{cases} M(k_1, k_2, k_3), \text{ if } \{k_1, k_2, k_3\} > \{l_1, l_2, l_3\} \\ l_1^*, \text{ if } \{k_1, k_2, k_3\} = \{l_1, l_2, l_3\}, \\ m(k_1, k_2, k_3), \text{ if } \{k_1, k_2, k_3\} \not\geq \{l_1, l_2, l_3\} \end{cases},$$

this is a permitted homomorphism, and it corresponds to an outer median, as an inner median must map (l_1, l_2, l_3) to the class of either $m(l_1, l_2, l_3)$ or $M(l_1, l_2, l_3)$.

Now suppose that (l_1, l_2, l_3) and (l_4, l_5, l_6) both generate sublattices isomorphic to $\mathbf{N}_5$. We can assume that $(l_1, l_2, l_3) \not\leq (l_4, l_5, l_6)$, and define the mapping

$$\mathcal{T}_{\mathbf{L}} \to \mathbf{L} : \{k_1, k_2, k_3\} \mapsto \begin{cases} M(k_1, k_2, k_3), \text{ if } \{k_1, k_2, k_3\} \geq \{l_1, l_2, l_3\} \\ m(k_1, k_2, k_3), \text{ if } \{k_1, k_2, k_3\} \not\geq \{l_1, l_2, l_3\} \end{cases},$$

again, this is a permitted homomorphism. The corresponding median maps $\{l_1, l_2, l_3\}$ to $M(l_1, l_2, l_3)$ and $\{l_4, l_5, l_6\}$ to $m(l_4, l_5, l_6)$, and so must be an outer median.

By Proposition 10, the above argument also works if we only assume both (l_1, l_2, l_3) and (l_4, l_5, l_6) to generate a lattice that generates the same variety as $\mathbf{N}_5$.

There is one outstanding case: when there are elements l_1, l_2, l_3 generating a sublattice $\mathbf{S}$ that is nondistributive, but generates a different variety than $\mathbf{N}_5$. That variety is contained in the one generated by $\mathbf{L}$, so it cannot contain $\mathbf{M}_3$. By [6], it must cover one of fifteen subdirectly irreducible lattices $(\mathbf{L}_1, \dots, \mathbf{L}_{15})$. By Jónsson's Lemma, this means that there is a $1 \leq i \leq 15$ so that $\mathbf{L}_i$ is isomorphic to $\mathbf{T}/\nu$ for a sublattice $\mathbf{T}$ of $\mathbf{S}$ and a congruence ν of $\mathbf{T}$.

It is easy to check that all the $\mathbf{L}_i$ contain at least two sublattices isomorphic to $\mathbf{N}_5$. Notice that if t_1/ν, t_2/ν and t_3/ν generate a sublattice of $\mathbf{T}/\nu$ isomorphic to $\mathbf{N}_5$, then there are elements $t_1', t_2', t_3' \in T$ such that $t_j'/\nu = t_j/\nu$ for all $j = 1, 2, 3$, and t_1, t_2 and t_3 generate a sublattice of $\mathbf{T}$ isomorphic to $\mathbf{N}_5$. (If $t_1/\nu < t_3/\nu$ in $\mathbf{T}/\nu$, then let t_3' be the smallest element if the ν-class of t_3, t_1' the largest element of t_1/ν smaller than t_3', and $t_2' = t_2$.) Thus, $\mathbf{T}$, and so $\mathbf{L}$, contains at least two sublattices isomorphic to $\mathbf{N}_5$. As we have seen, this means that there is an outer median of $\mathbf{L}$.

- (3) $\Rightarrow$ (1): If $\mathbf{L}$ is distributive, then $\operatorname{OutMed} \mathbf{L}$ is trivial. Otherwise, $\mathcal{T}_{\mathbf{L}}$ has only one element, denote it by $\{a, b, c\}$ with $a < c$. Notice that if $a < c' < c$, then $\{a, b, c'\}$ also generates a sublattice isomorphic to $\mathbf{n}_5$, contradicting the assumption. Therefore $a \prec c$, so the premitted interval of the only element of $\mathcal{T}_{\mathbf{L}}$ is a two-element interval. Hence, $|\operatorname{OutMed} \mathbf{L}| = 2$.

□

3. The outer and inner medians of 6 element lattices

In this case, $\mathcal{T}_{\mathbf{M}_4}$ is a four-element antichain, containing *abc*, *abd*, *acd*, *bcd*. The permittes interval of each of these triples is $\mathbf{M}_4$ itself. Hence, $\operatorname{OutMed} \mathbf{M}_4 \cong \mathbf{M}_4^4$.

Each of the 3-generated sublattices of $\mathbf{M}_4$ is either distributive or isomorphic to $\mathbf{M}_3$. Thus, $\operatorname{InnMed} \mathbf{M}_4 \cong \operatorname{InnMed} \mathbf{M}_3 \cong \mathbf{2}$.

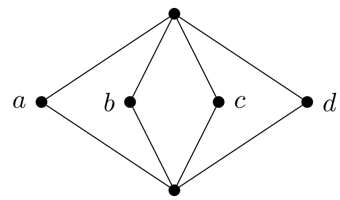

FIGURE 2. The lattice $\mathbf{M}_4$

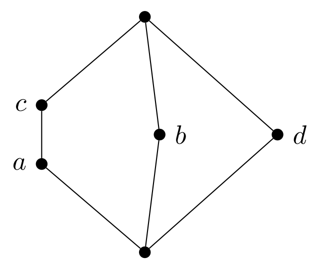

FIGURE 3. The lattice $\mathbf{A}_1$

The elements of $\mathcal{T}_{\mathbf{A}_1}$ are *abd*, *cbd*, *acb* and *acd*. Only the first two of these are comparable. The permitted interval for *abd* and *cbd* is $\{a, c\}$, and for *acb* and *acd* the whole $\mathbf{A}_1$. Thus, $\operatorname{OutMed}\mathbf{A}_1 \cong \mathbf{3} \times \mathbf{A}_1^2$.

The 3-generated sublattices of $\mathbf{A}_1$ generate the same variety as $\{\mathbf{M}_3, \mathbf{N}_5\}$. As the inner medians of $\mathbf{M}_3$ and $\mathbf{N}_5$ are independent, $\operatorname{InnMed}\mathbf{A}_1 \cong \operatorname{InnMed}\mathbf{M}_3 \times \operatorname{InnMed}\mathbf{N}_5 \cong \mathbf{2}^2$.

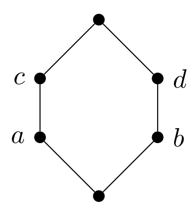

FIGURE 4. The lattice $\mathbf{A}_2$

The elements of $\mathcal{T}_{\mathbf{A}_2}$ are *abc*, *acd*, *abd* and *bcd*. Only the first two and the last two of these are comparable. The permitted interval for *abc* and *acd* is $\{a, c\}$, and for *abd* and *bcd* it is $\{b, d\}$. Thus, $\operatorname{OutMed}\mathbf{A}_1 \cong \mathbf{3}^2$.

As $\mathbf{A}_2$ generates the same variety as $\mathbf{N}_5$, $\operatorname{InnMed}\mathbf{A}_2 \cong \operatorname{InnMed}\mathbf{N}_5 \cong \mathbf{2}$.

$\mathcal{T}_{\mathbf{A}_2}$ is a chain containing the triples $abd < acd < bcd$, and the permitted intervals of these triples are $\{a, b\}$, $\{a, b, c\}$ and $\{b, c\}$ in this order. This enables us to calculate its outer median lattice, which will be an 8 element lattice.

As $\mathbf{A}_3$ generates the same variety as $\mathbf{N}_5$, $\operatorname{InnMed}\mathbf{A}_3 \cong \operatorname{InnMed}\mathbf{N}_5 \cong \mathbf{2}$.

$\mathcal{T}_{\mathbf{L}_4}$ is a three-element poset: it has smallest element *abc* with permitted interval $\{0, a, b, d\}$, and two incomparable elements, *acd* and *bcd* with permitted intervals $\{a, d\}$ and $\{b, d\}$, respectively. The outer median lattice of $\mathbf{L}_4$ turns out to be isomorphic to $\mathbf{3}^2$.

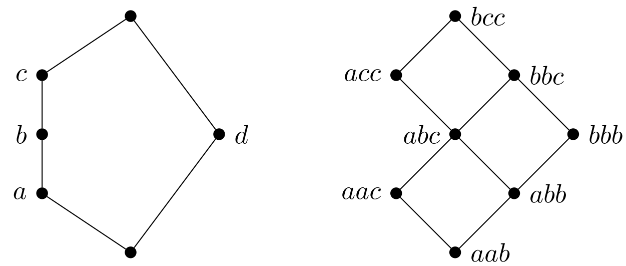

FIGURE 5. The lattice $\mathbf{A}_3$ and its outer median lattice

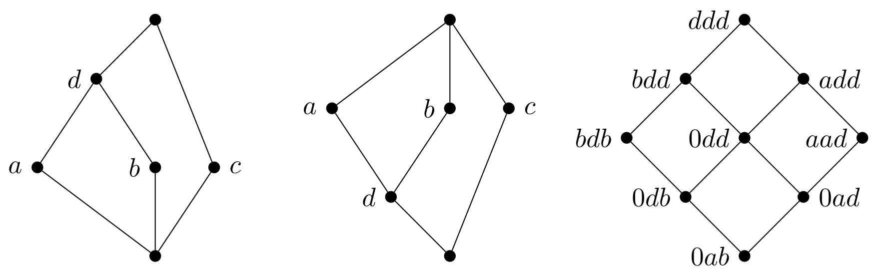

FIGURE 6. The lattices $\mathbf{L}_4$ and $\mathbf{L}_5$, and the poset $\mathcal{T}_{\mathbf{L}_4}$

The smallest and largest elements of $\mathbf{L}_4$, $0ab$ and ddd are necessarily inner medians. Consider the term

$$t(x, y, z) := \bigwedge_{\pi} (x \vee (y \wedge (x \vee z))),$$

As $\bar{t}(a, b, c) = 0$ and $\bar{t}(a, c, d) = \bar{t}(b, c, d) = d$ hold, $\bar{t}$ ensures that $0dd$ is an inner median.

In $\mathbf{L}_4$, both $\{a, c, d\}$ and $\{b, c, d\}$ generate sublattices isomorphic to $\mathbf{N}_5$. An inner median must act the same on both of these, so if must map either both or neither to d. Hence, $0db$, $0ad$, bdb and aad are outer medians.

Finally, as $\mathbf{L}_4$ has an automorphism swapping a and b, and $\{a, b, c\}$ is invariant under this, any inner median must map $\{a, b, c\}$ to an element fixed by this automorphism. Therefore, bdd and add are outer medians, and $\operatorname{InnMed} \mathbf{L}_4 \cong \mathbf{3}$.

As $\mathbf{L}_5$ is the dual of $\mathbf{L}_5$, $\operatorname{OutMed} \mathbf{L}_5 \cong \mathbf{3}^2$ and $\operatorname{InnMed} \mathbf{L}_5 \cong \mathbf{3}$.

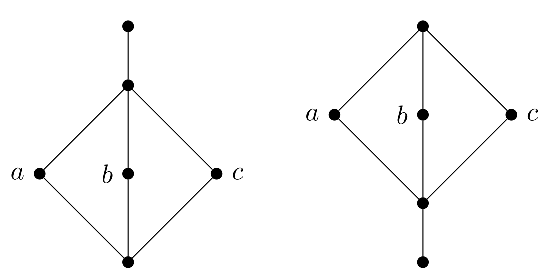

FIGURE 7. The lattices $\mathbf{B}_1$ and $\mathbf{B}_2$

By Proposition 9, OutMed $\mathbf{B}_1 \cong$ OutMed $\mathbf{B}_2 \cong$ OutMed $\mathbf{M}_3 \cong \mathbf{M}_3$, and by Proposition 10, InnMed $\mathbf{B}_1 \cong$ InnMed $\mathbf{B}_2 \cong$ InnMed $\mathbf{M}_3 \cong \mathbf{2}$.

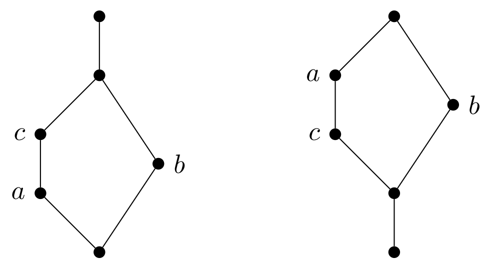

Figure 8. The lattices $\mathbf{B}_3$ and $\mathbf{B}_4$

By Proposition 9, OutMed $\mathbf{B}_3 \cong$ OutMed $\mathbf{B}_4 \cong$ OutMed $\mathbf{N}_5 \cong \mathbf{2}$, and by Proposition 10, InnMed $\mathbf{B}_3 \cong$ InnMed $\mathbf{B}_4 \cong$ InnMed $\mathbf{N}_5 \cong \mathbf{2}$.

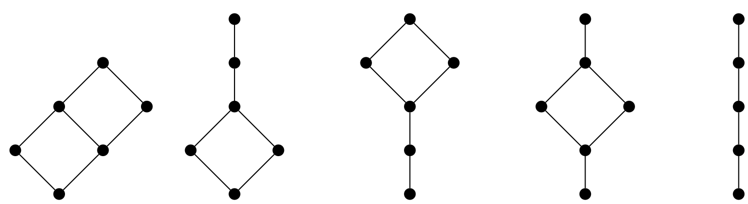

Figure 9. The 6 element distributive lattices

These latices have trivial outer and inner median lattices.

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