ON THE DISTRIBUTION OF PAIRS OF RANDOM POINTS FROM A SPHERICAL SHELL

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Abstract. We study the distribution of the distance of a pair of independent random points from a concentric spherical shell in $\mathbb{R}^d$ selected according to certain beta-type distributions. This model includes, as a special case, the uniform distribution.

1. Introduction

We investigate geometric models based on certain beta-type distributions in $\mathbb{R}^d$. Let $\mathbb{1}(\cdot)$ denote the indicator function of a set and $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^d$. We consider the d-dimensional beta-type distributions $\mu_{d,\beta}$ in the unit ball B^d with the following density function with respect to the Lebesgue measure

$$f_{d,\beta}(x) = c_{d,\beta}(1 - \|x\|^2)^\beta \mathbb{1}(0 \leq \|x\| \leq 1)$$

for $\beta > -1$ with

$$c_{d,\beta} = \frac{\Gamma\left(\frac{d}{2} + \beta + 1\right)}{\pi^{\frac{d}{2}}\Gamma(\beta + 1)}.$$

Important features of $\mu_{d,\beta}$ are that, on the one hand, $\mu_{d,0}$ is the uniform distribution in B^d, and, on the other hand, as $\beta \to -1^+$, $\mu_{d,\beta}$ converges weakly to the uniform distribution on the unit sphere S^{d-1}.

There have been several papers published recently on random polytope models based on $\mu_{d,\beta}$, see, for example, Grote, Kabluchko and Thäle [6], Gusakova and Kabluchko [7], Kabluchko, Temesvári and [10], Thäle Kabluchko, Thäle and Zaporozhets [13], Kabluchko and Panzo [9], and Kabluchko and Steinerberger [11]. We refer to a history, recent results, and further references to [6, 7, 9–11, 13] and to the upcoming book by Kabluchko, Steinerberger and Thäle [12].

Let $\mathrm{cl}(\cdot)$ denote the closure of a set in $\mathbb{R}^d$. For $0 \leq R < 1$, let $B_R = \mathrm{cl}(B^d \setminus RB^d)$ denote the closed region between the two concentric balls B^d and RB^d. We call B_R the spherical shell of inner radius R and outer radius 1.

For $0 \leq R < 1$, consider the restriction $\mu_{d,\beta,R}$ of $\mu_{d,\beta}$ to the spherical shell B_R normalized such that it is a probability distribution. Then $\mu_{d,\beta,R}$ is concentrated in B_R with the following density function with respect to the Lebesgue measure:

$$f_{d,\beta,R}(x) = c_{d,\beta,R}(1 - \|x\|^2)^\beta \mathbb{1}(R \leq \|x\| \leq 1) \tag{1.1}$$

for $\beta > -1$ with a suitable normalizing constant $c_{d,\beta,R}$. We note that $\mu_{d,\beta,0}$ is the beta-type distribution $\mu_{d,\beta}$ in B^d, and $\mu_{d,0,R}$ is the uniform distribution in B_R for any $0 \leq R < 1$.

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Let x_1 and x_2 be independent, identically distributed (i.i.d.) random points from B_R chosen according to $\mu_{d,\beta,R}$. We study the density $g^*_{d,\beta,R}(r)$ of the random variable $r = \|x_1 - x_2\|$. Our argument uses characteristic functions, and it is based on Lord [16]. We compute the functions $g^*_{d,\beta}(r)$ and $g^*_{d,0,R}(r)$ explicitly, and we also show that the analogous computation can be carried out for $g^*_{d,\beta,R}$ for any integer β and sufficiently large d, depending on β.

The motivation for examining the distribution of pairs of random points comes, in part, from the phenomenon that in the theory of random polytopes the significant part of many asymptotic formulas is generated by points in a thin shell close to the boundary; this explains the choice of the truncation of $\mu_{d,\beta}$. For more information on random polytopes see, for example, Hug [14], Reitzner [25] and Schneider [29].

Investigations of the distribution of the distance of two i.i.d. uniform random points in a convex body K (compact convex set with non-empty interior) go back to the first half of the twentieth century. The density functions for various bodies have been determined by different methods. In the particular case where K is a ball of dimension d, the density function was found by Borel [2] for $d = 2$, by Deltheil [4] for $d = 3, 5, 7, 9$ by Crofton's Theorem and by Boursin [3] for $d = 11, 13$. The general case was solved by Hammersley [8], and alternative methods were given by Lord [16], among others. One of the methods described in [16] uses characteristic functions and can be (in theory) applied to radially symmetric distributions that have a density with respect to the Lebesgue measure. We will use this method in this paper. The density function was also determined for some other specific bodies such as cubes, cylinders, etc. For the early history of the topic, see, for example, Kendall and Moran [15], for more recent references Mathai [19, Section 2.6.3].

For general K, Piefke [24] established a connection between the distribution of random chord lengths and distances of pairs in d-dimensions, extending earlier results for $d = 2$ and 3.

Fairthorne [5] considered the random model in which two uniform random points are selected from two concentric circular discs, such that one point is from the smaller disc and the other one is from the larger one. He determined the density function of the distance of the two random points. This result was extended to d-dimensions by Ruben [26].

Although we concentrate only on random distances in this paper, we note that more general models have also been investigated extensively. One such model is when one takes $1 \leq r \leq d$ i.i.d. random points from B^d according to a beta-type probability distribution. Such random points almost surely span an r-dimensional simplex. This more general model naturally includes both the uniformly distributed case and also the case of random distances of pairs of points (when $r = 1$). For results on mean values and integer moments of the volume of such random r-dimensional simplices see, for example, Miles [20] and Ruben and Miles [27]. The exact density function of the r-volume was given by Mathai [18]), and it is expressed in several forms, using hypergeometric functions, G-functions, H-functions, series expansions, etc. See also, for example, Pederzoli [21–23]. We refer for more detailed information, history and references to Mathai [19, Section 4.3 and pp. 427–428] and the upcoming book by Kabluchko, Steinerberger and Thäle [12].

We will use the symbol $\langle \cdot, \cdot \rangle$ for the usual Euclidean scalar product in $\mathbb{R}^d$ whose induced norm is $\| \cdot \|$. The volume of B^d is $\kappa_d = \pi^{\frac{d}{2}} / \Gamma(\frac{d}{2} + 1)$, where $\Gamma(\cdot)$ is Euler's gamma function (see Artin [1]), and the surface volume of S^{d-1} is $\omega_d = d\kappa_d$, see, for example, Schneider [28].

We will use Gauss's hypergeometric function ${}_2F_1(a,b,c,z)$, which is defined by the following series for complex numbers with $|z|<1$,

$$ {}_2F_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

where $a,b,c \in \mathbb{R}$ except when c is a nonpositive integer. The symbol $(x)_k$ is the rising factorial. Furthermore, let

$$ B(z;a,b) = \int_0^z u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \frac{z^a}{a}\,{}_2F_1(a,1-b,a+1,z). $$

be the incomplete beta function. If $z=1$, then we get the (complete) beta function, which we denote by $B(a,b)$.

The function $g^*_{d,0,0}(r)$ was determined by Hammersley [8], see also Lord [16,17],

$$ g^*_{d,0,0}(r) = \frac{d\,\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} r^{d-1} B\left(1-\frac{r^2}{4}; \frac{d+1}{2}, \frac{1}{2}\right), \quad 0 \le r \le 2. \tag{1.2} $$

In this paper we study the model in which two i.i.d. random points x_1 and x_2 are chosen according to the beta-type distribution $\mu_{d,\beta,R}$ from a spherical shell B_R. We study the density $g^*_{d,\beta,R}(r)$ of the random variable $r = \|x_1 - x_2\|$ using characteristic functions. We calculate explicitly, $g^*_{d,0,R}$ as follows.

Theorem 1.1. *Let* $C_d = \frac{d^2\Gamma\left(\frac{d}{2}\right)}{2\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}$. *For* $R \in [0,1)$ *and* $r \in [0,2]$, *let*

$$ g_1(d,r,R) = \frac{C_d}{2^{\frac{d}{2}}} \frac{r^{d-1}}{(1-R^d)^2} \int_{\arccos\left(\frac{2-r^2}{2}\right)}^{\pi} \frac{\sin^d\varphi}{(1-\cos\varphi)^{\frac{d}{2}}}\,\mathrm{d}\varphi, $$

$$ g_2(d,r,R) = -2C_d \frac{R^d r^{d-1}}{(1-R^d)^2} \times \int_{\arccos\left(\frac{1+R^2-r^2}{2R}\right)}^{\pi} \frac{\sin^d\varphi}{(1+R^2-2R\cos\varphi)^{\frac{d}{2}}}\,\mathrm{d}\varphi \cdot \mathbb{1}(1-R<r\le 1+R), $$

$$ g_3(d,r,R) = -2C_d \frac{R^d r^{d-1}}{(1-R^d)^2} \int_0^{\pi} \frac{\sin^d\varphi}{(1+R^2-2R\cos\varphi)^{\frac{d}{2}}}\,\mathrm{d}\varphi \cdot \mathbb{1}(0<r\le 1-R), $$

$$ g_4(d,r,R) = \frac{C_d}{2^{\frac{d}{2}}} \frac{R^d r^{d-1}}{(1-R^d)^2} \int_{\arccos\left(\frac{2R^2-r^2}{2R^2}\right)}^{\pi} \frac{\sin^d\varphi}{(1-\cos\varphi)^{\frac{d}{2}}}\,\mathrm{d}\varphi \cdot \mathbb{1}(0<r\le 2R). $$

Then

$$ g^*_{d,0,R}(r) = g_1(d,r,R) + \cdots + g_4(d,r,R). \tag{1.3} $$

We note that Theorem 1.1 may also be obtained via Ruben's method [26], besides other techniques; however, our argument is short and very direct. Moreover, it can also be carried out in a straightforward, although laborious, way for integer β and sufficiently large d depending on β, see Section 7.

We note that in all dimensions, $g^*_{d,0,0}(r) = g^*_{d,0}(r)$, and if $R \to 1^-$, then $g^*_{d,0,R}(r)$ tends to the density of the distance of two i.i.d. uniform random points from S^{d-1}. We also note that the functions $g_i(d,r,R)$, $i=1,\dots,4$ can be expressed in terms of incomplete beta integrals and incomplete Gaussian hypergeometric functions by standard substitutions; for details, see Section 8.

The paper is organized as follows. In Section 2, we describe the general method, and in Section 3 we collect some tools from the theory of Bessel functions. We illustrate the method by determining the density function $g^*_{d,\beta}$ in B^d in Section 4. We prove Theorem 1.1 in Section 5, and we provide explicit formulas for $g^*_{d,0,R}$ for $d = 2, 3$ cases as examples in Section 6. In Section 7, we show how this calculation can be carried out for positive integer β and sufficiently large dimension d. Finally, in Section 8, we show how one can express the functions in Theorem 1.1 in terms of incomplete beta integrals and incomplete Gaussian hypergeometric functions.

2. The method of characteristic functions

In this section, we recall the main points of the method we use to determine the density $g^*_{d,\beta,R}(r)$. For more details, we refer to Lord [16, 17] and Mathai [19].

Let x be a random point in $\mathbb{R}^d$ with a spherically symmetric distribution. Assume that this distribution has a continuous density $f(x)$ with respect to the Lebesgue measure. Then the characteristic function of $f(x)$ is

$$\phi(y) = \int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\,\mathrm{d}x,$$

where y is an arbitrary point of $\mathbb{R}^d$. It is well-known that if $x_1, \ldots, x_n$ are independent random points with characteristic functions $\phi(y_1), \cdots, \phi(y_n)$, then the characteristic function of $x_1 + \ldots + x_n$ is $\phi(y_1)\ldots\phi(y_n)$. Since f is assumed to be spherically symmetric, it only depends on $s = \|x\|$. We will use the notation $f(x) = h(s)$ for the d-dimensional density of x as a function of s. The corresponding one-dimensional density is denoted by $h^*(s)$. The connection between $h(s)$ and $h^*(s)$, by the virtue of the spherical symmetry of the distribution of x, is

$$h^*(s) = \omega_d s^{d-1} h(s).$$

Let $\varrho = \|y\|$, and let $\psi(\varrho)$ be the characteristic function of f in terms of ϱ. Then (for details see, for example, Lord [16] or Mathai [19, pp. 289–292])

$$\phi(y) = \psi(\varrho) = (2\pi)^{\frac{d}{2}} \varrho^{-\frac{d}{2}+1} \int_0^\infty s^{\frac{d}{2}} h(s) J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s,$$

where $J_\alpha(z)$ denotes a Bessel function of the first kind defined by the following series

$$J_\alpha(z) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m+\alpha}. \tag{2.1}$$

Using the inverse Fourier transform, one obtains that

$$h^*(s) = \frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty (s\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(s\varrho)\psi(\varrho)\mathrm{d}\varrho. \tag{2.2}$$

Since x has a spherically symmetric distribution, $-x$ has the same density function as x. Thus, if x_1 and x_2 are independent, then the densities of $x_1 + x_2$ and $x_1 - x_2$ are the same.

Therefore, the characteristic function of $z = x_1 - x_2$ is $\phi(z) = \phi(y_1)\phi(y_2) = \psi_1(\varrho)\psi_2(\varrho)$. If $r = \|z\|$, then using (2.2), we obtain that the one-dimensional density $g^*(r)$ is

$$g^*(r) = \frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty (r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\psi_1(\varrho)\psi_2(\varrho)\mathrm{d}\varrho.$$

We are going to show that both $\psi_{d,\beta,R}(\varrho)$ and $g^*_{d,\beta,R}(r)$ can be evaluated for certain combinations of β and d using known properties of Bessel functions.

In particular, if x_1 and x_2 are i.i.d. random points from B_R distributed according to $\mu_{d,\beta,R}$, then

$$\phi_{d,\beta,R}(y) = \psi_{d,\beta,R}(\varrho) = (2\pi)^{\frac{d}{2}} \varrho^{-\frac{d}{2}+1} \int_0^\infty s^{\frac{d}{2}} h_{d,\beta,R}(s) J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s.$$

where

$$h_{d,\beta,R}(s) = c_{d,\beta,R}(1-s^2)^\beta \mathbb{1}(R \le s \le 1).$$

Then the one-dimensional density $g^*_{d,\beta,R}(r)$ is

$$g^*_{d,\beta,R}(r) = \frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty (r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,\beta,R}(\varrho)\mathrm{d}\varrho.$$

3. Tools from the theory of Bessel function

In this section, we collect some tools from the theory of Bessel functions that we use in our arguments. For more detailed information and references, the reader may consult Watson's book [30]. Our main tool is

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)}{\varrho^\lambda}\,\mathrm{d}\varrho,$$

the so-called discontinuous integral of Weber and Schafheitlin. It is assumed that $0 < a, b$ so that the improper integral converges at ∞.

Lemma 3.1. *Assume that $\mu + \nu + 1 > \lambda > -1$ and $0 < b < a$. Then the integral on the left-hand side converges and the following holds*

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)}{\varrho^\lambda}\mathrm{d}\varrho = \frac{b^\nu \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\lambda + \frac{1}{2}\right)}{2^\lambda a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)} \times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1, \frac{b^2}{a^2}\right). \quad (3.1)$$

Formula (3.1) (see [30, (2) on p. 401]) was obtained by Sonine (1887) and Schafheitlin (1888) (for a historical discussion, we refer to [30, p. 398]).

The following formula, involving the product of three Bessel functions in the integral, can be obtained from the Weber–Schafheitlin integral by substitution, see [30, 2nd equation in Section 13.4].

Lemma 3.2. *Assume that $\nu > -\frac{1}{2}$, $\mu + \nu + 1 > \lambda > -1$. Then*

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)J_\nu(c\varrho)}{\varrho^{\lambda+\nu}}\mathrm{d}\varrho = \frac{\left(\frac{1}{2}bc\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\infty \int_0^\pi \frac{J_\mu(a\varrho)J_\nu(\varpi\varrho)}{\varpi^\nu \varrho^\lambda} \sin^{2\nu}\varphi\,\mathrm{d}\varphi\mathrm{d}\varrho, \quad (3.2)$$

where $\varpi = \sqrt{b^2+c^2-2bc\cos\varphi}$, and the integral on the right-hand side is absolutely convergent.

We will use the following formulas for evaluating indefinite integrals involving Bessel functions. The recursion formula (3.3) is originally from Lommel, see [30, (4) on p. 133],

$$\int z^{\mu+1} J_\nu(z)\,\mathrm{d}z = - (\mu^2-\nu^2)\int z^{\mu-1}J_\nu(z)\,\mathrm{d}z + (z^{\mu+1}J_{\nu+1}(z) + (\mu-\nu)z^\mu J_\nu(z)). \quad (3.3)$$

In the case when $\mu=\nu$, (3.3) reduces to

$$\int z^{\nu+1}J_\nu(z)\,\mathrm{d}z = z^{\nu+1}J_{\nu+1}(z),$$

see also [30, (1) on p. 132]. This yields, with a simple substitution of $r\varrho$, that

$$\int r^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}r = r^{\frac{d}{2}}\varrho^{-1}J_{\frac{d}{2}}(r\varrho). \quad (3.4)$$

4. The density $g^*_{d,\beta}(r)$

First, we demonstrate the method by calculating the density in the case when the independent random points x_1 and x_2 are distributed in B^d according to $\mu_{d,\beta}$. Since $f_{d,\beta}(x)$ is rotationally symmetric, the density of x can be written as a function of $s=\|x\|$, that is,

$$h_{d,\beta}(s) = c_{d,\beta}(1-s^2)^\beta \mathbb{1}(0\le s\le 1).$$

Using the (2.1) expansion of Bessel functions, we obtain that

$$\begin{aligned}
\psi_{d,\beta}(\varrho) &= (2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_0^\infty s^{\frac{d}{2}} h_{d,\beta}(s) J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s\\
&= (2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\frac{\Gamma\left(\frac{d}{2}+\beta+1\right)}{\pi^{\frac{d}{2}}\Gamma(\beta+1)}\\
&\quad\times\int_0^1 s^{\frac{d}{2}}(1-s^2)^\beta\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})}\left(\frac{s\varrho}{2}\right)^{2m+\frac{d}{2}-1}\mathrm{d}s\\
&= 2\frac{\Gamma\left(\frac{d}{2}+\beta+1\right)}{\Gamma(\beta+1)}\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})}\left(\frac{\varrho}{2}\right)^{2m}\int_0^1(1-s^2)^\beta s^{2m+d-1}\,\mathrm{d}s\\
&= \Gamma\left(\frac{d}{2}+\beta+1\right)\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2}+\beta+1)}\left(\frac{\varrho}{2}\right)^{2m}\\
&= \Gamma\left(\frac{d}{2}+\beta+1\right)2^{\frac{d}{2}+\beta}\varrho^{-\frac{d}{2}-\beta}J_{\frac{d}{2}+\beta}(\varrho).
\end{aligned}$$

Then

$$\begin{aligned}
g^*_{d,\beta}(r) &= \frac{2^{1-\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty (r\varrho)^{\frac{d}{2}}J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,\beta}(\varrho)\mathrm{d}\varrho\\
&= \frac{2^{\frac{d}{2}+2\beta+1}\Gamma^2\left(\frac{d}{2}+\beta+1\right)}{\Gamma\left(\frac{d}{2}\right)}r^{\frac{d}{2}}\int_0^\infty\varrho^{-\frac{d}{2}-2\beta}J_{\frac{d}{2}-1}(r\varrho)J^2_{\frac{d}{2}+\beta}(\varrho)\,\mathrm{d}\varrho.
\end{aligned}$$

First, we use (3.2) with $\mu=\frac{d}{2}-1, \nu=\frac{d}{2}+\beta, \lambda=\beta, a=r, b=c=1$. Note that as $d\ge 2$ and $\beta>-1$, it holds that $\nu=\frac{d}{2}+\beta>-\frac{1}{2}$ and $\mu+\nu+2=d+\beta+1>\beta+1=$

$\lambda+1>0$, so the conditions of Lemma 3.2 are satisfied. Next, we apply (3.1) with $\mu=\frac{d}{2}+\beta, \nu=\frac{d}{2}-1, \lambda=\beta, a=\varpi, b=r$. As $\mu+\nu+1=d+\beta>\beta=\lambda>-1$, the conditions of Lemma 3.1 are also satisfied. Assuming that $0<r<\varpi$, we get that

$$\int_0^\infty \varrho^{-\frac{d}{2}-2\beta} J_{\frac{d}{2}-1}(r\varrho) J^2_{\frac{d}{2}+\beta}(\varrho)\,\mathrm{d}\varrho = \frac{r^{\frac{d}{2}-1}}{2^{\frac{d}{2}+2\beta}\Gamma(\frac{1}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})} \int_{A_1}^{\pi} \frac{\sin^{d+2\beta}\varphi}{\varpi^d}\left(1-\frac{r^2}{\varpi^2}\right)^\beta \mathrm{d}\varphi,$$

where $\varpi=\sqrt{2(1-\cos\varphi)}$ and $A_1=\arccos(\frac{2-r^2}{2})$. Here, in the last step, we also use Euler's transformation

$${}_2F_1(a,b,c;z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z).$$

Thus

$$\begin{aligned} g^*_{d,\beta}(r) &= \frac{2^{2\beta+1}\Gamma^2\left(\frac{d}{2}+\beta+1\right)}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})} r^{d-1} \\ &\qquad \times \int_{A_1}^{\pi} \cos\left(\frac{\varphi}{2}\right)^{d+2\beta}\left(\sin^2\left(\frac{\varphi}{2}\right)-\frac{r^2}{4}\right)^\beta \mathrm{d}\varphi \\ &= \frac{2\Gamma^2\left(\frac{d}{2}+\beta+1\right)}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})} r^{d-1}(4-r^2)^\beta \\ &\qquad \times \int_0^{1-\frac{r^2}{4}} u^{\frac{d+2\beta-1}{2}}(1-u)^{-\frac{1}{2}}\left(1-\frac{4}{4-r^2}u\right)^\beta \mathrm{d}u. \end{aligned} \tag{4.1}$$

We obtained the last form by first substituting $t=\cos(\varphi/2)$, then $u=t^2$. Note that in the case when $\beta=0$, (4.1) reduces to Hammersley's formula (1.2). We note that the integral in (4.1) is the incomplete Gaussian hypergeometric function.

5. Proof of Theorem 1.1

Let $R\in[0,1)$, and let the independent random points x_1 and x_2 be chosen from the spherical shell B_R according to the uniform probability distribution $\mu_{d,0,R}$. Thus, x_1 and x_2 have identical (d-dimensional) densities

$$h_{d,0,R}(s)=\frac{1}{\kappa_d(1-R^d)}\mathbb{1}(R\le s\le 1).$$

The common characteristic function of x_1 and x_2 is

$$\psi_{d,0,R}(\varrho)=\frac{1}{\kappa_d(1-R^d)}(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_R^1 s^{\frac{d}{2}} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s.$$

Thus, by (3.4),

$$\psi_{d,0,R}(\varrho)=\frac{2^{\frac{d}{2}-1}d\,\Gamma(\frac{d}{2})}{1-R^d}\varrho^{-\frac{d}{2}}\left(J_{\frac{d}{2}}(\varrho)-R^{\frac{d}{2}}J_{\frac{d}{2}}(R\varrho)\right). \tag{5.1}$$

Let $r=\|x_2-x_1\|$, as before. Then, by (2.2), we obtain the (one-dimensional) density $g^*_{d,0,R}(r)$ as follows

$$g^*_{d,0,R}(r)=\frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty (r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,0,R}(\varrho)\,\mathrm{d}\varrho$$

$$= \frac{2^{\frac{d}{2}-1} d^2\, \Gamma(\frac{d}{2})}{(1-R^d)^2} r^{\frac{d}{2}} \int_0^\infty \varrho^{-\frac{d}{2}} \left(J_{\frac{d}{2}}(\varrho) - R^{\frac{d}{2}} J_{\frac{d}{2}}(R\varrho) \right)^2 J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho. \quad (5.2)$$

By expanding the square in (5.2), we get that $g^*_{d,0,R}(r)$ is the sum of the following three terms:

$$g^*_{d,0,R}(r) = \frac{2^{\frac{d}{2}-1} d^2\, \Gamma(\frac{d}{2})}{(1-R^d)^2} r^{\frac{d}{2}} \Bigg($$

$$\int_0^\infty \varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho \quad (5.3)$$

$$- 2R^{\frac{d}{2}} \int_0^\infty \varrho^{-\frac{d}{2}} J_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho \quad (5.4)$$

$$+ R^d \int_0^\infty \varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho \Bigg). \quad (5.5)$$

We evaluate (5.3)–(5.5), with the help of Lemmas 3.1 and 3.2. Since the integrals in (5.3) and (5.5) are very similar, we work out only (5.5) in detail.

We use Lemma (3.2) with the choice $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = c = R$ and $\lambda = 0$. Since $\nu = \frac{d}{2} > -\frac{1}{2}$, and $\mu + \nu + 2 = d + 1 > \lambda + 1 = 1 > 0$, the conditions of Lemma 3.2 are satisfied. Therefore, we obtain that

$$\begin{aligned}(5.5) &= R^d \int_0^\infty \varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho \\ &= \frac{2^{-\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^{2d} \int_0^\pi \frac{\sin^d \varphi}{\varpi_4^{\frac{d}{2}}} \int_0^\infty J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_4 \varrho)\, \mathrm{d}\varrho \mathrm{d}\varphi, \quad (5.6)\end{aligned}$$

where $\varpi_4 = \sqrt{2R^2(1-\cos\varphi)}$.

Next, we apply formula (3.1) with $\mu = \frac{d}{2}$, $\nu = \frac{d}{2} - 1$, $a = \varpi_4$, $b = r$, $\lambda = 0$. As $d \geq 2$, it holds that $\nu = \frac{d}{2} - 1 > -\frac{1}{2}$ and $\mu + \nu + 1 = d > \lambda = 0 > -1$, thus the conditions of Lemma 3.1 are satisfied. The condition $0 < r < \varpi_4$ holds precisely when $0 < r \leq 2R$, and then the calculation yields

$$\begin{aligned}(5.6) &= \frac{2^{-\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^{2d} \int_{A_4}^\pi \frac{\sin^d \varphi}{\varpi_4^{\frac{d}{2}}} \frac{r^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)}{\varpi_4^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\Gamma(1)}\, {}_2F_1\left(\frac{d}{2}, 0, \frac{d}{2}, \frac{r^2}{\varpi_4^2}\right) \mathrm{d}\varphi \\ &= \frac{2^{-\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^{2d} r^{\frac{d}{2}-1} \int_{A_4}^\pi \left(\frac{\sin\varphi}{\varpi_4}\right)^d \mathrm{d}\varphi, \quad (5.7)\end{aligned}$$

with $A_4 = \arccos\left(\frac{2R^2 - r^2}{2R^2}\right)$. It is also clear that if $r \to 2R^-$, then (5.7) tends to 0.

If we apply (3.1) with $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = \varpi_4$, $\lambda = 0$, then $0 < \varpi_4 < r$ is satisfied when $r > 2R$, and the calculation yields that $(5.6) = 0$.

By a similar calculation, we obtain from (5.3) that

$$\int_0^\infty \varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}-1}(r\varrho)\, \mathrm{d}\varrho = \frac{2^{-\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} r^{\frac{d}{2}-1} \int_{A_1}^\pi \left(\frac{\sin\varphi}{\varpi_1}\right)^d \mathrm{d}\varphi, \quad (5.8)$$

with $A_1 = \arccos\left(\frac{2-r^2}{2}\right)$ and $\varpi_1 = \sqrt{2(1-\cos\varphi)}$. This formula is valid for all $r \in [0, 2]$ and $R \in [0, 1)$.

Now, we turn to the evaluation of the integral (5.4). We use (3.2) with the choice $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = R$, $c = 1$, $\lambda = 0$. Since μ, ν and λ are the same as when we used (3.2) in the evaluation of (5.5), the conditions of Lemma 3.2 are satisfied. We obtain that

$$\begin{aligned}(5.4) &= -2R^{\frac{d}{2}} \int_0^\infty \varrho^{-\frac{d}{2}} J_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho) \mathrm{d}\varrho \\ &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^d \int_0^\infty \int_0^\pi \frac{J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_2 \varrho)}{\varpi_2^{\frac{d}{2}}} \sin^d \varphi \, \mathrm{d}\varphi \mathrm{d}\varrho \\ &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^d \int_0^\pi \frac{\sin^d \varphi}{\varpi_2^{\frac{d}{2}}} \int_0^\infty J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_2 \varrho) \, \mathrm{d}\varrho \mathrm{d}\varphi \qquad (5.9)\end{aligned}$$

where $\varpi_2 = \sqrt{1 + R^2 - 2R\cos\varphi}$.

Finally, we use (3.1) with $\mu = \frac{d}{2}$, $\nu = \frac{d}{2} - 1$, $a = \varpi_2$, $b = r$ and $\lambda = 0$. Again, as μ, ν and λ are the same as in the evaluation of (5.5) by (3.1), the conditions of Lemma 3.1 are satisfied. The condition $0 < r < \varpi_2$ holds precisely when $0 < r \le 1 + R$.

If $1 - R < r \le 1 + R$, then we get that

$$\begin{aligned}(5.9) &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} R^d \int_{A_2}^\pi \frac{r^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(1)} \, {}_2F_1\left(\frac{d}{2}, 0, \frac{d}{2}, \frac{r^2}{\varpi_2^2}\right) \left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi \\ &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} r^{\frac{d}{2}-1} R^d \int_{A_2}^\pi \left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi, \qquad (5.10)\end{aligned}$$

where $A_2 = \arccos\left(\frac{1+R^2-r^2}{2R}\right)$. If $r \to 1 + R^-$, then $(5.10) \to 0$.

If $r \le 1 - R$, then

$$(5.9) = -\frac{2^{-\frac{d}{2}+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)} r^{\frac{d}{2}-1} R^d \int_0^\pi \left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi. \qquad (5.11)$$

If $r \to 1 - R^+$, then (5.10) tends to (5.11) evaluated at $r = 1 - R$.

If we use (3.1) with $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = \varpi_2$ and $\lambda = 0$, then the condition $0 < \varpi_2 < r$ holds when $r > 1 + R$. In this case we get that $(5.9) = 0$.

Now, if we define the functions $g_i(d, r, R)$, $i = 1, \ldots, 4$ as in Theorem 1.1, then (5.7), (5.8), (5.10) and (5.11) show that the density function $g^*_{d,0,R}(r)$ is the sum of the $g_i(d, r, R)$, $i = 1, \ldots, 4$.

This finishes the proof of Theorem 1.1.

6. Examples

We provide, as examples, the explicit formulas for the planar ($d = 2$) and the 3-dimensional cases.

6.1. The $d = 2$ case.

Direct calculations of the functions $g_i(2, r, R)$, $i = 1, \ldots, 4$ yield the following.

$$g_1(2, r, R) = \frac{2r}{\pi(1-R^2)^2} \left(\pi - \arccos\left(\frac{2-r^2}{2}\right) - \sqrt{1 - \left(\frac{2-r^2}{2}\right)^2}\right),$$

$$g_2(2, r, R) = -\frac{8R^2 r}{\pi(1-R^2)^2} \cdot \left(\frac{R^2-1}{2R^2} \cdot \frac{\pi}{2} + \frac{(R-1)^2\pi}{4R^2} + \frac{\pi}{2R} - \frac{\sin(a(r, R))}{2R}\right.$$

$$-\frac{R^2-1}{2R^2}\arctan\left(\frac{R+1}{1-R}\tan\left(\frac{a(r,R)}{2}\right)\right)-\frac{(R-1)^2}{4R^2}a(r,R)-\frac{a(r,R)}{2R}\right),$$

$$\text{where } a(r,R)=\arccos\left(\frac{R^2+1-r^2}{2R}\right),$$

$$g_3(d,r,R)=-\frac{4R^2r}{(1-R^2)^2},$$

$$g_4(d,r,R)=\frac{2R^2r}{\pi(1-R^2)^2}\left(\pi-\arccos\left(\frac{2R^2-r^2}{2R^2}\right)-\sqrt{1-\left(\frac{2R^2-r^2}{2R^2}\right)^2}\right).$$

The graph of $g^*_{2,0,R}$ is shown in Figure 1 for a few values of R. The $R=1$ case represents the density function of two i.i.d. random points from S^1 chosen according to the normalized arc-length.

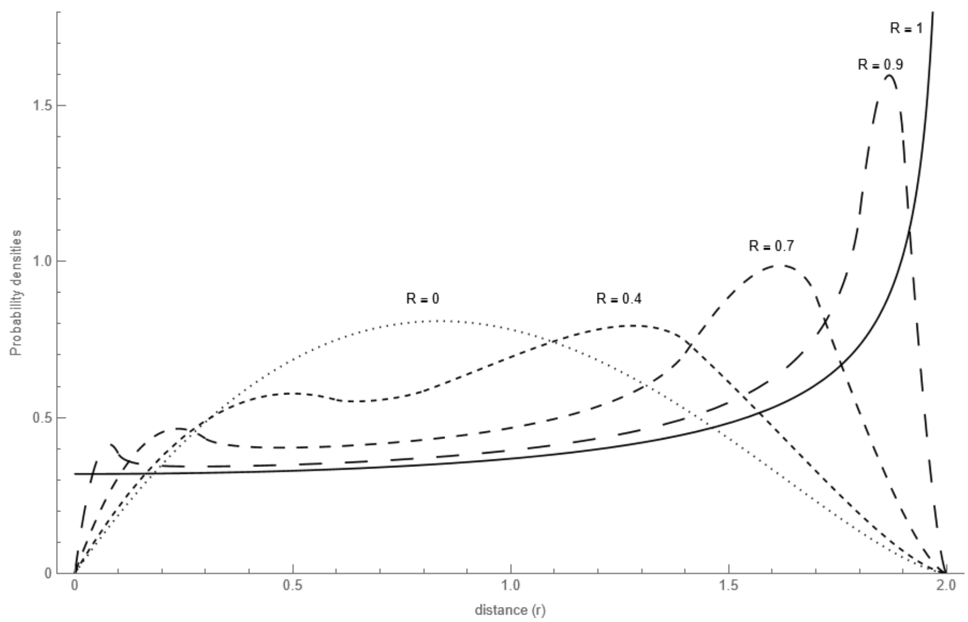

FIGURE 1. The graph of $g^*_{2,0,R}(r)$ for various values of R drawn by *Mathematica*.

6.2. **The $d=3$ case.** If $d=3$ then we obtain the following formulas.

$$g_1(3,r,R)=\frac{9}{4}\frac{r^2}{(1-R^3)^2}\left(\frac{4}{3}+\frac{1}{12}r^3-r\right),$$

$$g_2(3,r,R)=-\frac{3}{2}\frac{r^2}{(1-R^3)^2}\left(2+2R^3-\frac{3R^4+3-r^4-6R^2+6R^2r^2+6r^2}{4r}\right),$$

$$g_3(3,r,R)=-6\frac{R^3r^2}{(1-R^3)^2},$$

$$g_4(3,r,R)=\frac{9}{4}\frac{R^3r^2}{(1-R^3)^2}\left(\frac{4}{3}+\frac{1}{12}\frac{r^3}{R^3}-\frac{r}{R}\right).$$

The graph of the function $g^*_{3,0,R}$ is drawn in Figure 2 for a few specific values of R. We note that the solid line represents the density of the distance of two i.i.d. random points chosen from the surface S^2 according to the normalized spherical Lebesgue measure; this is marked with $R = 1$ in the figure. In this case, the density function is linear in r.

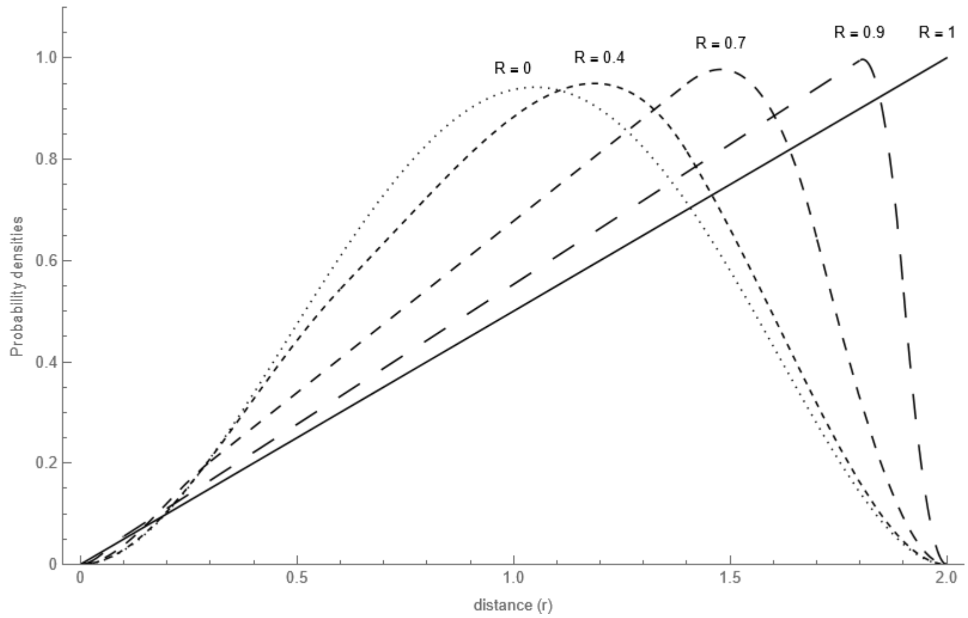

FIGURE 2. The function $g^*_{3,0,R}(r)$ for different values of R drawn by *Mathematica*.

Not surprisingly, the 3-dimensional density functions are better behaved than in two dimensions; they have a single maximum in the interval $[0, 2]$.

7. Beta type distributions in spherical shells

We return to the case of general truncated beta type distributions $\mu_{d,\beta,R}$ in spherical shells B_R for $0 \le R < 1$, and we show how the density $g^*_{d,\beta,R}$ can be determined explicitly for integer values of β and sufficiently large d, depending on β. Let $s = \|x\|$ be as before. Then the normalizing constant $c_{d,\beta,R}$ is

$$c_{d,\beta,R} = \frac{1}{\int_{\mathbb{R}^d} f_{d,\beta,R}(x)\,\mathrm{d}x},$$

where

$$\begin{aligned} c^{-1}_{d,\beta,R} &= \int_{\mathrm{cl}(B^d \setminus RB^d)} (1 - \|x\|^2)^\beta \,\mathrm{d}x = \omega_d \int_R^1 (1 - s^2)^\beta s^{d-1}\,\mathrm{d}s \\ &= \frac{\omega_d}{2}\left(B\left(1; \frac{d}{2}, \beta + 1\right) - B\left(R^2; \frac{d}{2}, \beta + 1\right)\right) \\ &= \frac{1}{c_{d,\beta}} - \frac{\omega_d}{2} B\left(R^2; \frac{d}{2}, \beta + 1\right). \end{aligned}$$

Now, as $f_{d,\beta,R}$ is rotationally symmetric, the d-dimensional density in terms of s is

$$h_{d,\beta,R}(s) = c_{d,\beta,R}(1-s^2)^\beta \mathbb{1}(R \le s \le 1),$$

and we have that

$$\begin{aligned}\psi_{d,\beta,R}(\varrho) &= (2\pi)^{\frac{d}{2}} \varrho^{-\frac{d}{2}+1} \int_0^\infty s^{\frac{d}{2}} h_{d,\beta,R}(s) J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s \\ &= (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} \int_R^1 s^{\frac{d}{2}} (1-s^2)^\beta J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s.\end{aligned}$$

If β is a non-negative integer, then the Binomial Theorem yields that

$$\psi_{d,\beta,R}(\varrho) = (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} \sum_{k=0}^{\beta} (-1)^k \binom{\beta}{k} \int_R^1 s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s. \tag{7.1}$$

Applying (3.3) recursively to each term $\int s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s$ in (7.1), we arrive in k steps to the indefinite integral $\int s^{\frac{d}{2}} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s$, which can be evaluated by (3.4).

In particular, for $k = 1, \dots, \beta$, the substitution $z = s\varrho$ and repeated application of (3.3) yield, for some constants $c_0, \dots c_k$ and $e_1, \dots, e_k$ depending on d and i, that

$$\begin{aligned}&\int s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s \\ &= \varrho^{-\frac{d}{2}-2k-1}\left(c_k z^{\frac{d}{2}+2k} + c_{k-1} z^{\frac{d}{2}+2(k-1)} \cdots + c_0 z^{\frac{d}{2}}\right) J_{\frac{d}{2}}(z) \\ &\quad + \varrho^{-\frac{d}{2}-2k-1}\left(e_k z^{\frac{d}{2}+2k-1} + e_{k-1} z^{\frac{d}{2}+2(k-1)-1} \cdots + e_1 z^{\frac{d}{2}+1}\right) J_{\frac{d}{2}-1}(z) \\ &= \left(c_k \frac{s^{\frac{d}{2}+2k}}{\varrho} + c_{k-1} \frac{s^{\frac{d}{2}+2(k-1)}}{\varrho^3} + \cdots + c_0 \frac{s^{\frac{d}{2}}}{\varrho^{2k+1}}\right) J_{\frac{d}{2}}(s\varrho) \\ &\quad + \left(e_k \frac{s^{\frac{d}{2}+2k-1}}{\varrho^2} + e_{k-1} \frac{s^{\frac{d}{2}+2(k-1)-1}}{\varrho^4} + \cdots + e_1 \frac{s^{\frac{d}{2}+1}}{\varrho^{2k}}\right) J_{\frac{d}{2}-1}(s\varrho) \\ &= \sum_{i=0}^{k} c_i \frac{s^{\frac{d}{2}+2i}}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(s\varrho) + \sum_{j=1}^{k} e_j \frac{s^{\frac{d}{2}+2j-1}}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(s\varrho)\end{aligned}$$

Thus, after evaluating the definite integrals in (7.1), we get an explicit formula for $\psi_{d,\beta,R}(\varrho)$ in which each term contains a power of ϱ and a Bessel function $J_\nu(\varrho)$ or $J_\nu(R\varrho)$ for $\nu \in \{\frac{d}{2}, \frac{d}{2}-1\}$. Therefore

$$\begin{aligned}\psi_{d,\beta,R}(\varrho) = (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} &\left(c_0 \frac{s^{\frac{d}{2}}}{\varrho^{2k+1}} + \sum_{k=1}^{\beta} \binom{\beta}{k} (-1)^k\right. \\ &\times \left(\sum_{i=1}^{k} c_i \frac{1}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(\varrho) - \sum_{i=1}^{k} c_i \frac{R^{\frac{d}{2}+2i}}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(R\varrho)\right. \\ &\left.\left. + \sum_{j=1}^{k} e_j \frac{1}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(\varrho) - \sum_{j=1}^{k} e_j \frac{R^{\frac{d}{2}+2j-1}}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(R\varrho)\right)\right).\end{aligned}$$

Now, substituting $\psi_{d,\beta,R}(\varrho)$ into

$$g^*_{d,\beta,R}(r) = \frac{2^{1-\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty (r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,\beta,R}(\varrho)\mathrm{d}\varrho, \tag{7.2}$$

then expanding the square $\psi^2_{d,\beta,R}(\varrho)$, one obtains a sum of integrals, each containing the product of a power of ϱ, $J_{\frac{d}{2}-1}(r\varrho)$ and two Bessel functions from among $J_{\frac{d}{2}-1}(\varrho)$, $J_{\frac{d}{2}-1}(R\varrho)$, $J_{\frac{d}{2}}(\varrho)$, $J_{\frac{d}{2}}(R\varrho)$. First, we want to apply Lemma 3.2 to each integral. In each integral, at least two of the Bessel functions have the same order and the exponent of ϱ is between 0 and $-\frac{d}{2}-4\beta$. So, $\mu, \nu \in \{\frac{d}{2}-1, \frac{d}{2}\}$, and $0 \le \lambda < 4\beta+1$ in Lemma 3.2. Then $\nu > -\frac{1}{2}$ and $\lambda > -1$ are satisfied. The condition $\mu+\nu+1 > \lambda$ puts a lower bound $d \ge 4\beta+1$ on the dimension. If this is satisfied, then Lemma 3.2 can be used. The applicability of Lemma 3.1 also follows as the conditions on μ, ν and λ are weaker than in Lemma 3.2. This process is a straightforward, albeit tedious computation that yields an explicit formula for $g^*_{d,\beta,R}(r)$ in the form of a sum of functions each one of which comes from an integral in (7.2).

8. Concluding remarks

As mentioned after Theorem 1.1, the functions $g_i(d,r,R)$ can be transformed by standard substitutions in the following way. Substituting first $t = \cos(\varphi/2)$, then $u = t^2$, we obtain

$$\int \left(\frac{\sin\varphi}{\sqrt{1-\cos\varphi}}\right)^d \mathrm{d}\varphi = -2^{\frac{d}{2}} \int u^{\frac{d-1}{2}} (1-u)^{-\frac{1}{2}}\,\mathrm{d}u,$$

$$\int \left(\frac{\sin\varphi}{\sqrt{1+R^2-2R\cos\varphi}}\right)^d \mathrm{d}\varphi = -\frac{2^d}{(R+1)^d} \int u^{\frac{d-1}{2}} (1-u)^{\frac{d-1}{2}} \left(1-\frac{4R}{(R+1)^2}u\right)^{-\frac{d}{2}} \mathrm{d}u.$$

Thus, for $R \in [0,1)$ and $r \in [0,2]$,

$$g_1(d,r,R) = 2^{\frac{d}{2}} C_d \frac{r^{d-1}}{(1-R^d)^2} B\left(1-\frac{r^2}{4}; \frac{d+1}{2}, \frac{1}{2}\right),$$

$$g_2(d,r,R) = -2^{d+1} C_d \frac{R^d r^{d-1}}{(1-R^d)^2(1+R)^d} \cdot \mathbb{1}(1-R < r \le 1+R) \times \int_0^{\frac{(R+1)^2-r^2}{4R}} u^{\frac{d-1}{2}} (1-u)^{\frac{d-1}{2}} \left(1-\frac{4R}{(R+1)^2}u\right)^{-\frac{d}{2}} \mathrm{d}u,$$

$$g_3(d,r,R) = -2^{d+1} C_d \frac{R^d r^{d-1}}{(1-R^d)^2(1+R)^d} B\left(\frac{d+1}{2}, \frac{d+1}{2}\right) \times {}_2F_1\left(\frac{d}{2}, \frac{d+1}{2}, d+1, \frac{4R}{(R+1)^2}\right) \cdot \mathbb{1}(0 < r \le 1-R),$$

$$g_4(d,r,R) = 2^{\frac{d}{2}} C_d \frac{R^d r^{d-1}}{(1-R^d)^2} B\left(1-\frac{r^2}{4R^2}; \frac{d+1}{2}, \frac{1}{2}\right) \cdot \mathbb{1}(0 < r \le 2R),$$

where, for $g_3(d,r,R)$, we also used Euler's integral formula

$$B(b, c-b)\, {}_2F_1(a,b,c,z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,\mathrm{d}x,$$

which holds for $c > b > 0$ and $z < 1$.

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