# ON THE VARIANCES OF SPHERICAL AND HYPERBOLIC RANDOM POLYTOPES

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ABSTRACT. We prove asymptotic upper bounds on the variances of the volume and vertex number of spherical random polytopes in spherical convex bodies, and hyperbolic random polytopes in convex bodies in hyperbolic space. We also consider a circumscribed model on the sphere.

### 1. INTRODUCTION AND RESULTS

Random polytopes in non-Euclidean geometries have recently attracted much attention. It is a natural problem to try to transfer statements from Euclidean theory to more general settings. For an overview of such results, see, for example, Besau, Ludwig, and Werner [BLW18], Besau and Thäle [BT20], Kabluchko and Panzo [KP25], Schneider [Sch22], and the references therein.

For  $d \geq 2$ , let  $\mathcal{M}^d$  denote one of the following spaces: Euclidean *d*-space  $\mathbb{R}^d$ , the unit sphere  $\mathbb{S}^d$  in  $\mathbb{R}^{d+1}$ , or the hyperbolic space  $\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} : x_1^2 + \ldots x_d^2 - x_{d+1}^2 = -1, x_{d+1} > 0\}$ . Endow each space with the corresponding geodesic distance: the metric *d* induced by the Euclidean scalar product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}^d$ ,  $d_S(x, y) = \arccos\langle x, y \rangle$  for  $x, y \in \mathbb{S}^d$ , and  $\cosh d_H(x, y) = x_{d+1}y_{d+1} - x_1y_1 - \ldots - x_dy_d$  for  $x, y \in \mathbb{H}^d$ .

We call a set K convex in  $\mathcal{M}^d$  if, together with any two of its points, the unique geodesic segment connecting them is also contained in K. A convex body is a compact convex set in  $\mathcal{M}^d$  with non-empty interior. We note that a spherical convex body is always contained in an open hemisphere.

The volume (Lebesgue measure) of a measurable set in  $\mathcal{M}^d$  is denoted by  $\operatorname{Vol}_{\mathcal{M}^d}(\cdot)$ , or specifically  $V(\cdot)$  in  $\mathbb{R}^d$  for short. Let  $\mathcal{K}(\mathcal{M}^d)$  denote the family of convex bodies in  $\mathcal{M}^d$ . The convex hull of a closed set  $X \subset \mathcal{M}^d$  is the intersection of all closed half-spaces containing X if  $\mathcal{M}^d = \mathbb{R}^d$  or  $\mathbb{H}^d$ , and the intersection of all closed hemispheres containing X if  $\mathcal{M}^d = \mathbb{S}^d$ .

We consider the following probability model. Let  $K \in \mathcal{K}(\mathcal{M}^d)$ , and let  $x_1, \ldots, x_n$ be *n* independent random points chosen according to the uniform distribution in *K*. The convex hull of the points  $x_1, \ldots, x_n$  is a random polytope  $K_n$  in *K*. Let  $H_{d-1}^{\mathcal{M}^d}(x)$  denote the generalized Gauss–Kronecker curvature at a boundary

Let  $H_{d-1}^{\mathcal{M}^{\circ}}(x)$  denote the generalized Gauss–Kronecker curvature at a boundary point  $x \in \partial K$ . The spherical and hyperbolic cases of the following theorem were proved by Besau, Ludwig, and Werner [BLW18, Theorems 2.2, 3.2, and Corollaries 2.3, 3.3]. The Euclidean case is due to Schütt [Sch94] extending results of Bárány [Bár92].

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**Theorem 1.** Let  $K \in \mathcal{K}(\mathcal{M}^d)$ . If  $K_n$  is the convex hull of n random points chosen uniformly in K, then

$$\lim_{n \to \infty} \mathbb{E}(\operatorname{Vol}_{\mathcal{M}^d}(K \setminus K_n)) \cdot n^{\frac{2}{d+1}} = \beta_d \operatorname{Vol}_{\mathcal{M}^d}(K)^{\frac{2}{d+1}} \int_{\partial K} H_{d-1}^{\mathcal{M}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x,$$
$$\lim_{n \to \infty} f_0(K_n) \cdot n^{-\frac{d-1}{d+1}} = \beta_d \operatorname{Vol}_{\mathcal{M}^d}(K)^{-\frac{d-1}{d+1}} \int_{\partial K} H_{d-1}^{\mathcal{M}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x.$$

where  $\beta_d$  is an explicitly known constant that depends only on d.

For real sequences f and g, we write  $f \ll g$  if there exists a constant  $\gamma > 0$  such that  $|f(n)| \leq \gamma g(n)$  for all  $n \in \mathbb{N}$ . If  $f \ll g$  and  $g \ll f$ , we write  $f \approx g$ .

We say that a ball of radius r > 0 rolls freely in K (K slides freely in a ball of radius R > 0, resp.) if for any  $x \in \partial K$  there exists a ball of radius r (R, resp.) containing x on its boundary and contained in K (containing K, resp.). If K has a rolling ball and slides freely in a ball at the same time, then  $\partial K$  is  $C^1$  and strictly convex. However,  $\partial K$  is not necessarily  $C^2$ . One of our main results is the following theorem.

**Theorem 2.** Let  $K \in \mathcal{K}(\mathcal{M}^d)$  that has a rolling ball and which slides freely in a ball. Then

$$\operatorname{Var} \operatorname{Vol}_{\mathcal{M}^d}(K_n) \ll n^{-\frac{d+3}{d+1}},$$
$$\operatorname{Var} f_0(K_n) \ll n^{\frac{d-1}{d+1}}.$$

The variance upper bound of the volume implies a strong law of large numbers that can be proved by standard arguments (see, for instance, [BFV10], [Rei03]).

**Corollary 1.** Under the same assumptions as in Theorem 2, it holds with probability one, that

$$\lim_{n \to \infty} (\operatorname{Vol}_{\mathcal{M}^d}(K \setminus K_n)) \cdot n^{\frac{2}{d+1}} = \beta_d \operatorname{Vol}_{\mathcal{M}^d}(K)^{\frac{2}{d+1}} \int_{\partial K} H_{d-1}^{\mathcal{M}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x.$$

The number of vertices is not a monotone function of n, but using the fact that it may be increased by at most one when another point is added, for  $d \ge 4$ , similarly to [Rei03], a strong law of large numbers can be proved for  $f_0$  as well.

**Corollary 2.** Under the same assumptions as in Theorem 2, for  $d \ge 4$ , it holds with probability one, that

$$\lim_{n \to \infty} f_0(K_n) \cdot n^{-\frac{d-1}{d+1}} = \beta_d \operatorname{Vol}_{\mathcal{M}^d}(K)^{-\frac{d-1}{d+1}} \int_{\partial K} H_{d-1}^{\mathcal{M}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x$$

Since Besau and Thäle [BT20] proved in  $\mathbb{S}^d$  and  $\mathbb{H}^d$  that if K has  $C^2_+$  boundary, then  $\operatorname{Var} \operatorname{Vol}_{\mathcal{M}^d}(K_n) \gg n^{-\frac{d+3}{d+1}}$ , Theorem 2 yields the following.

**Theorem 3.** Let  $K \in \mathcal{K}(\mathcal{M}^d)$  with  $C^2_+$  boundary. Then

$$\operatorname{Var} \operatorname{Vol}_{\mathcal{M}^d}(K_n) \approx n^{-\frac{d+3}{d+1}}$$

For  $\mathcal{M}^d = \mathbb{R}^d$ , both the lower and the upper bounds are due to Reitzner [Rei03, Rei05].

Our proof of Theorem 2 uses results of a weighted model described in Subsection 2.1, and thus it is indirect. We also give a direct proof of Theorem 2 in Section 4 via a non-euclidean version of the economical cap theorem that avoids using spherical integral geometry.

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1.1. Circumscribed model. The probability model discussed in this section considers circumscribed spherical polytopes containing a convex body K. This model is naturally connected to the inscribed ones discussed via spherical polarity. The role of random points inside of K is replaced by random closed hemispheres containing K, and the intersection of n such hemispheres is a random polytope containing K. This model was studied, for example, by Besau, Ludwig and Werner [BLW18].

Let  $\mathcal{H}$  denote the space of closed hemispheres in  $\mathbb{S}^d$ . Each point  $x \in \mathbb{S}^d$  is the pole of a unique closed hemisphere  $H^-(x) = \{y \in \mathbb{S}^d : \langle x, y \rangle \leq 0\}$ . We define the measure of a Borel set  $A \subset \mathcal{H}$  as

$$\mu(A) = \frac{1}{\omega_d} \int_{\mathbb{S}^d} \mathbf{1}(H^-(x) \in A) \, \mathrm{d}x,$$

where  $\omega_d$  is the surface volume of  $\mathbb{S}^d$  and integration is with respect to spherical Lebesgue measure.

Let  $\mathcal{H}_K = \{H^- \in \mathcal{H} : K \subset H^-\}$ . Choose *n* i.i.d. random hemispheres containing *K* according to the uniform distribution with the probability measure  $\mu_K = \mu/\mu(\mathcal{H}_K)$ . The intersection of these hemispheres is a random polytope containing *K*, denoted by  $K^{(n)}$ .

The spherical polar  $K^*$  of a convex body K is defined as  $K^* = \bigcap_{x \in K} H^-(x)$ . Since polarity reverses set inclusion, a hemisphere  $H^-(x)$  contains K if and only if  $x \in \mathcal{K}^*$ . The polar body  $(K^{(n)})^*$  is a polytope contained in  $K^*$ , and it is the convex hull of those n i.i.d. random points in  $K^*$  that are the poles of the random hemispheres containing K. This provides a direct connection between the inscribed and circumscribed models.

Based on this connection, Besau, Ludwig, and Werner [BLW18] proved asymptotic formulas for the expectation of the spherical mean width and the number of facets  $f_{d-1}$  of  $K^{(n)}$ . The spherical mean width  $U_1(K)$  is defined as

$$U_1(K) = \frac{1}{2} \int_{G(d+1,d)} \chi(K \cap H) \,\mathrm{d}\nu(H),$$

where G(d+1, d) denotes the Grassmannian of the *d*-dimensional linear subspaces of  $\mathbb{R}^{d+1}$ ,  $\nu$  is the unique rotation invariant probability measure on G(d+1, d), and  $\chi$  denotes the Euler characteristic.

**Theorem 4** ([BLW18], Corollary 2.6). Let  $K \in \mathcal{K}(\mathbb{S}^d)$ . If  $K^{(n)}$  is the intersection of n random hemispheres containing K and chosen uniformly according to  $\mu_K$ , then

$$\lim_{n \to \infty} \mathbb{E}_{\mu_K} (U_1(K^{(n)}) - U_1(K)) \cdot n^{\frac{2}{n+1}} = \frac{\beta_d}{\omega_d} \operatorname{Vol}_{\mathbb{S}^d} (K^*)^{\frac{2}{d+1}} \int_{\partial K} H_{d-1}^{\mathbb{S}^d} (K, x)^{\frac{d}{d+1}} \mathrm{d}x,$$
$$\lim_{n \to \infty} \mathbb{E}_{\mu_K} f_{d-1}(K^{(n)}) \cdot n^{-\frac{d-1}{n+1}} = \beta_d \operatorname{Vol}_{\mathbb{S}^d} (K^*)^{-\frac{d-1}{d+1}} \int_{\partial K} H_{d-1}^{\mathbb{S}^d} (K, x)^{\frac{d}{d+1}} \mathrm{d}x.$$

Our variance upper bounds in Theorem 2 yield the following upper bounds on the variances of  $U_1(K^{(n)})$  and  $f_{d-1}(K^{(n)})$  due to spherical polarity.

**Corollary 3.** Let  $K \in \mathcal{K}(\mathbb{S}^d)$  that has a rolling ball and which slides freely in a ball. Then

$$\operatorname{Var} U_1(K^{(n)}) \ll n^{-\frac{d+3}{d+1}},$$
$$\operatorname{Var} f_{d-1}(K^{(n)}) \ll n^{\frac{d-1}{d+1}}.$$

### 2. Geometric tools

2.1. Weighted non-uniform models in  $\mathbb{R}^d$ . Let  $K \in \mathcal{K}(\mathbb{R}^d)$  and let  $\varrho : K \to (0, \infty)$  be a probability density function that is continuous in a neighbourhood of the boundary of K (in K). Then  $\mathbb{P}_{\varrho}(A) := \int_A \varrho(x) \, dx$  for any measurable set  $A \subset K$ . We denote the expectation and variance with respect to  $\mathbb{P}_{\varrho}$  by  $\mathbb{E}_{\varrho}$  and  $\operatorname{Var}_{\varrho}$ , respectively. Let  $x_1, \ldots, x_n$  be i.i.d. random points from K distributed according to  $\mathbb{P}_{\varrho}$ , and let  $K_{(n)}$  be their convex hull, a random polytope in K.

Furthermore, let  $\lambda : K \to (0, \infty)$  be a weight function that is integrable in K and continuous in a neighborhood of  $\partial K$  and let  $V_{\lambda}(A) = \int_{A} \lambda(x) \, dx$  for all measurable subsets  $A \subset K$ . For  $\varrho \equiv 1/V(K)$  and  $\lambda \equiv 1$ , we obtain the uniform model.

In this model, with no smoothness condition on K, Böröczky, Fodor and Hug [BFH10] studied the asymptotic behavior of the expectation of the weighted volume of missed part of K and the number of vertices  $f_0(K_{(n)})$  of the random polytope  $K_{(n)}$ , and proved that

$$\lim_{n \to \infty} \mathbb{E}_{\varrho}(V_{\lambda}(K_{(n)})) \cdot n^{\frac{2}{d+1}} = \beta_d \int_{\partial K} \varrho(x)^{\frac{-2}{d+1}} \lambda(x) H_{d-1}^{\mathbb{R}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x$$
$$\lim_{n \to \infty} \mathbb{E}_{\varrho}(f_0(K_{(n)})) \cdot n^{-\frac{d-1}{d+1}} = \beta_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+1}} H_{d-1}^{\mathbb{R}^d}(K, x)^{\frac{1}{d+1}} \mathrm{d}x.$$

Assuming that  $\partial K$  is  $C_{+}^{2}$ , Besau and Thäle [BT20] proved the following asymptotic variance lower bound and a central limit theorem:

$$\operatorname{Var}_{\varrho}(V_{\lambda}(K_{(n)})) \gg n^{-\frac{d+3}{d+1}},\tag{1}$$

The lower bounds in [BT20] for  $\mathbb{S}^d$  and  $\mathbb{H}^d$  were deduced from (1) by choosing particular weight functions that come from the gnomonic projections.

Under weaker smoothness assumptions, Bakó-Szabó and Fodor [BSF24] proved matching asymptotic upper bounds for the variance of the weighted volume and the number of vertices.

**Theorem 5** ([BSF24], Theorem 1.1). For a convex body  $K \subset \mathbb{R}^d$  that has a rolling ball and which slides freely in a ball, it holds that

$$\operatorname{Var}_{\varrho}(V_{\lambda}(K_{(n)})) \ll n^{-\frac{d+3}{d+1}},$$
$$\operatorname{Var}_{\varrho}(f_0(K_{(n)})) \ll n^{\frac{d-1}{d+1}},$$

where the implied constants depend only on  $K, \varrho, \lambda$  and the dimension d.

We use Theorem 5 in our proof of Theorem 2.

2.2. **Gnomonic projection.** The gnomonic projection maps a *d*-dimensional open hemisphere from the origin radially to a tangent hyperplane. We refer to the point of tangency as the center of the projection. We may assume that the convex body  $K \subset \mathbb{S}^d$  is contained in the (upper) open hemisphere  $\mathbb{S}^d_+ = \{x \in \mathbb{S}^d : \langle x, e_{d+1} \rangle > 0\}$ , and the center of projection is  $e_{d+1}$ . Then the gnomonic projection  $g : \mathbb{S}^d_+ \to \mathbb{R}^d$  is

$$g(x) = \frac{x}{\langle x, e_{d+1} \rangle} - e_{d+1},$$

and the hyperplane  $\{x \in \mathbb{R}^{d+1} : \langle x, e_{d+1} \rangle = 0\}$  is identified with  $\mathbb{R}^d$ . The map g is bijective and  $C^{\infty}$ . Geodesic arcs of  $\mathbb{S}^d_+$  are mapped into straight line segments in  $\mathbb{R}^d$ , thus the image of a spherical convex body is a convex body in the Euclidean sense. The gnomonic image of a set X is often denoted by  $\overline{X} = g(X)$ .

The gnomonic projection can be defined in the hyperbolic model similarly:  $h : \mathbb{H}^d \to \mathbb{R}^d$  maps the points of the hyperboloid centrally to the tangent hyperplane  $H = \{x \in \mathbb{R}^{d+1} : \langle x, e_{d+1} \rangle = 1\}$ , then identify H with  $\mathbb{R}^d$ , i.e. identify

$$h(x) = \left(\frac{x_1}{x_{d+1}}, \dots, \frac{x_d}{x_{d+1}}\right).$$

As ||h(x)|| < 1 for all  $x \in \mathbb{H}^d$ , the image  $g(\mathbb{H}^d)$  is the open unit ball of  $\mathbb{R}^d$ . The map h is  $C^{\infty}$ , geodesic arcs are mapped into straight line segments, and it is bijective between  $\mathbb{H}^d$  and int  $B^d$ . The image of a hyperbolic convex body is a convex body in  $\mathbb{R}^d$ .

2.3. Non-euclidean economic cap covering theorem. Let  $K \in \mathcal{K}(\mathcal{M}^d)$  and let  $H^-$  be a closed half-space if  $\mathcal{M}^d = \mathbb{R}^d$  or  $\mathbb{H}^d$ , and a closed hemisphere if  $\mathcal{M}^d = \mathbb{S}^d$ . Let  $H^+$  denote the other closed half-space (hemisphere) determined by  $H^-$ . For t > 0, we define the convex floating body of K as

$$K_{[t]} = \bigcap \{ H^- \colon \operatorname{Vol}_{\mathcal{M}^d}(K \cap H^+) \le t \}.$$

$$\tag{2}$$

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Euclidean floating bodies were defined by Bárány and Larman [BL88], Schütt and Werner [SW04], the spherical floating body was introduced by Besau and Werner [BW16], and hyperbolic floating bodies were defined by Besau and Werner [BW18].

The closure of the complement of  $K_{[t]}$  in K is called the wet part of K with parameter t, and it is denoted by K(t). It was proved in [BW16, Theorem 2.1] that

$$\operatorname{Vol}_{\mathbb{S}^d}(K(t)) \approx t^{\frac{2}{d+1}}, \text{ as } t \to 0^+.$$
(3)

The economic cap covering theorem in  $\mathbb{R}^d$  was proved by Bárány and Larman [BL88] and Bárány [Bár89].

**Theorem 6** ([BL88], [Bár89]). Assume that  $K \subset \mathbb{R}^d$  is a convex body with V(K) = 1 and  $0 < t < t_0 = (2d)^{-2d}$ . Then there exist caps  $C_1, \ldots, C_m$  and pairwise disjoint convex sets  $C'_1, \ldots, C'_m$  such that  $C'_i \subset C_i$  for each i, and

(i) 
$$\bigcup_{1}^{m} C'_{i} \subset K(t) \subset \bigcup_{1}^{m} C_{i}$$
,

(ii)  $V(C'_i) \gg t$  and  $V(C_i) \ll t$  for each i,

(iii) for each cap C with  $C \cap K_{[t]} = \emptyset$  there is a  $C_i$  containing C.

We will use the following lemma as a technical tool.

**Lemma 1.** The gnomonic projection preserves the order of magnitude of the volume of caps.

*Proof.* Assume that  $K \in \mathcal{K}(\mathbb{S}^d)$  and  $e_{d+1}$  is the center of the minimum radius ball (circumball) containing K. Note that the radius  $R_K$  of the circumball is less than  $\pi/2$ . Consider a cap C of K with  $\operatorname{Vol}_{\mathbb{S}^d}(C) = t$ . Then

$$V(g(C)) \le t \cdot \frac{1}{\langle v, e_{d+1} \rangle^{d+1}},$$

where v is the farthest point of C from the center of projection,  $e_{d+1}$ . On the one hand, V(g(C)) > t. On the other hand, the quantity  $\frac{1}{\langle v, e_{d+1} \rangle^{d+1}}$  is bounded from above by a constant  $c_{K,d}$  depending only on K and d. Thus,  $V(g(C)) \leq c_{K,d} \cdot t$ .

For the hyperbolic case, assume again that the center of the circumball of  $K \in \mathcal{K}(\mathbb{H}^d)$  is  $e_{d+1}$ . Let D be a hyperbolic cap of K with  $\operatorname{Vol}_H(D) = t$ . Then V(h(D)) < t, and  $V(h(D)) \ge t \cdot \frac{1}{\langle v, e_{d+1} \rangle^{d+1}}$ , where  $\frac{1}{\langle v, e_{d+1} \rangle^{d+1}}$  is bounded from below, similarly to the spherical case.

**Theorem 7.** Let  $K \in \mathcal{K}(\mathbb{S}^d)$  (or  $\mathcal{K}(\mathbb{H}^d)$ , resp.) and  $0 < t < (2d)^{-2d} \cos^{d+1} R_K$ (or  $0 < t < (2d)^{-2d}$  resp.). Then there exist caps  $C_1, \ldots, C_m$  and pairwise disjoint convex sets  $C'_1, \ldots, C'_{m'}$ , with  $m \approx m'$ , such that

- (i)  $\bigcup_{i=1}^{m'} C'_i \subset K(t) \subset \bigcup_{i=1}^m C_i,$ (ii)  $\operatorname{Vol}_{\mathcal{M}^d} C_i \ll t, \ (i = 1, \dots, m) \ and \ \operatorname{Vol}_{\mathcal{M}^d} C'_i \gg t, \ (i = 1, \dots, m'),$
- (iii) for every cap C of volume at most t, there is a  $C_i$  containing C,
- (iv) every convex set  $C'_i$  is contained in some  $C_j$ .

*Proof.* We only give the proof for the case of  $\mathbb{S}^d$ , the hyperbolic variant can be shown essentially in the same way. The wet part K(t) is the union of caps  $C_{\alpha}$ of spherical volume at most t. The images of the caps  $C_{\alpha}$  under the gnomonic projection g are the caps  $\overline{C}_{\alpha} = g(C_{\alpha})$  in  $\overline{K} = g(K)$  with possibly different volumes of  $\bar{t}_{\alpha}$ . By compactness, there exists a minimal and a maximal volume among  $\bar{t}_{\alpha}$ , let us denote these by  $\bar{t}_{\min}$  and  $\bar{t}_{\max}$ , respectively. Consider the (Euclidean) wet parts in  $\overline{K}$  with parameters  $\overline{t}_{\min}$  and  $\overline{t}_{\max}$ . Then

$$\overline{K}(\overline{t}_{\min}) \subset g(K(t)) \subset \overline{K}(\overline{t}_{\max}).$$

Apply Theorem 6 to both wet parts. Denote the caps and the convex sets contained in them provided by Theorem 6 corresponding to  $\bar{t}_{\max}$  by  $D_i$  and  $D'_i$ (i = 1, ..., m) and those corresponding to  $\bar{t}_{\min}$  by  $E_i$  and  $E'_i$  (i = 1, ..., m). Let  $C_i = g^{-1}(D_i), (i = 1, ..., m) \text{ and } C'_i = g^{-1}(E'_i) \text{ for some indices } i = 1, ..., m'$ specified later.

The large caps  $D_i$  cover  $\overline{K}(\overline{t}_{\max})$  so they cover g(K(t)) as well. Furthermore, for any spherical cap C of (spherical) volume at most t, its image g(C) has volume at most  $\bar{t}_{\max}$ , therefore, it is contained in some  $D_i$ .

The sets  $E'_i$  are disjoint and contained in caps of volume  $\bar{t}_{\min}$ , therefore they are contained in g(K(t)), and there are caps  $D_{j(i)}$  such that  $E'_i \subset D_{j(i)}$  for each  $i \in \{1, \ldots, m\}$ . However, the same index j can belong to multiple i's, so some of the sets  $E'_i$  can be dropped to avoid multiplicity. The number of these sets cannot be too large due to the criteria on the volumes of  $D_i$  and  $E'_i$ . The preimages of the remaining sets will be the spherical sets  $C'_i$ , (i = 1, ..., m) with  $m \approx m'$ .

Lemma 1 and the corresponding part of the Euclidean theorem yield (ii).

## 3. Proof of Theorem 2

3.1. The spherical case. Assume that a spherical ball of radius r rolls freely in K. Let  $e_{d+1}$  be the center of the circumsphere of K. Then K is contained in the upper hemisphere.

For a boundary point  $x \in \partial K$ , denote by  $B_x$  the ball of radius r for which  $x \in \partial B_x$  and  $B_x \subset K$ . Then g maps balls in  $\mathbb{S}^2_+$  to ellipsoids in  $\mathbb{R}^d$ , and those balls, whose center is  $e_{d+1}$ , are mapped to Euclidean balls in  $\mathbb{R}^d$ . Thus, for each  $x \in \partial K, \overline{B}_x = g(B_x)$  is an ellipsoid in  $\overline{K}$ . Due to compactness, there is a maximal  $\kappa_{\rm max}$  among the principal curvatures of all  $\overline{B}_x$ . Then, a Euclidean ball of radius  $1/\kappa_{\rm max}$  rolls freely in any  $\overline{B}_x$  (see [Sch14, Corollary 3.2.13.]) and, in turn, it rolls freely in  $\overline{K}$ , thus  $\overline{K}$  also has a rolling ball.

Now, assume that K slides freely in a spherical ball of radius R. Then its spherical polar body  $K^*$  has a rolling ball of radius  $(\pi/2 - R)$  This can be seen in the following way: For a boundary point x on  $\partial K^*$ , there is at least one supporting hypersphere  $G^*$ . Its polar G is a point on the boundary of K. Since K has a sliding ball, a ball  $B_G$  of radius R contains K, and G is on its boundary. Then  $B_G^*$  is a ball of radius  $(\pi/2 - R)$  contained in  $K^*$  and intersects  $G^*$  at a boundary point y of  $K^*$ . If  $y \neq x$  then  $\partial K^*$  contains a great circle segment, thus  $\partial K$  must have a point where it is not smooth. However, this contradicts the assumption that K has a rolling ball, therefore x = y, and  $K^*$  also has a rolling ball.

Note that the assumptions on K yield that the rolling ball inside  $K^*$  is in the lower open hemisphere for all  $x \in \partial K^*$ .

Let  $\tilde{g}$  be the gnomonic projection whose center is  $-e_{d+1}$ . Let  $(\cdot)^{\circ}$  denote Euclidean polarity in  $\mathbb{R}^d$ . It is known, see Schneider [Sch22, Lemma 3.2.2.], that  $g(K)^{\circ} = \tilde{g}(K^*)$ . By the above argument, the existence of a rolling ball of  $K^*$  implies that its image,  $\tilde{g}(K)^{\circ}$  has a rolling ball too. Hug [Hug00, Prop. 1.45.] proved that if a convex body  $L \subset \mathbb{R}^d$  has a rolling ball, then  $L^{\circ}$  slides freely in a ball of finite radius. Hence, g(K) slides freely in a ball.

The gnomonic projection maps  $\operatorname{Vol}_{\mathbb{S}^d}$  into the Lebesgue measure in  $\mathbb{R}^d$  with the density  $\psi(x) = (1 + ||x||^2)^{-(d+1)/2}$  (see [BW16, Proposition 4.2]), and  $\operatorname{Vol}_{\mathbb{S}^d}(K_n)$  has the same distribution as the weighted volume of the convex hull of n i.i.d. random points in  $\mathbb{R}^d$  (cf. [BT20, Section 5.1]). Thus, we may apply Theorem 5, as the boundary conditions are satisfied for  $\overline{K}$ .

3.2. The hyperbolic case. Assume that the center of a maximal radius inscribed ball  $B_1$  in K is the point  $e_{d+1}$ . Then  $h(e_{d+1}) = o$  and  $h(B_1) = \overline{B}_1 \subset \operatorname{int} B^d$  in  $\mathbb{R}^d$ .

Suppose that K slides freely in a ball of radius R. Let  $B_2$  denote the smallest ball centered at  $e_{d+1}$  that contains the union of all the sliding balls of radius R of K. The boundary of K is contained in the spherical shell determined by the concentric balls  $B_1$  and  $B_2$ , and  $\partial \overline{K}$  is in the (Euclidean) spherical shell determined by  $\overline{B}_1$ and  $\overline{B}_2$ .

The gnomonic image of a hyperbolic ball contained in  $B_2$  is an ellipsoid in  $\overline{B}_2$  (contained in int  $B^d$  in  $\mathbb{R}^d$ ). It is clear that in the images of such balls the ratio of the ellipsoid's largest and shortest axes is bounded from above.

For a fixed boundary point  $x \in \partial K$ , the image of the sliding ball  $B_x$  of K at x is an ellipsoid that slides freely in a Euclidean ball of radius  $R_x$ . By compactness, there is a maximum radius  $\tilde{R}$  among all such  $R_x$ . Then  $\overline{K}$  slides freely in a radius  $\tilde{R}$  ball.

Now suppose that a ball of radius r rolls freely in K. Then at every boundary point x, a ball inside of K intersects K at x. The gnomonic projection of this ball is an ellipsoid in  $\overline{K}$  for which the ratio of the axes is bounded. There is a maximal principal curvature  $\kappa_{\max}$  among all these ellipsoids, and a ball of radius  $1/\kappa_{\max}$ rolls freely in all of them. Consequently, it rolls freely in  $\overline{K}$ .

The hyperbolic volume of K has the same distribution as the weighted volume of the convex hull of n i.i.d. random points chosen by the probability density function  $\psi / \int_{\overline{K}} \psi(x) \, dx$  in  $\mathbb{R}^d$ , where  $\psi = (1 - ||x||^2)^{-(d+1)/2}$ ,  $x \in \text{int } B^d$  is the density of the image measure of  $\operatorname{Vol}_{\mathbb{H}^d}$ . For more information, see [BT20, Section 5.2].

Thus, the conditions of Theorem 5 are satisfied, so we may apply its statement, which yields the desired asymptotic variance upper bound for  $\operatorname{Vol}_{\mathbb{H}^d}(K_n)$ .

#### 4. Direct proof of Theorem 2

We only prove the spherical case, the hyperbolic case is similar. The proof is based on the argument in [BFV10] and [Rei03], with some modifications required by the intrinsic geometry of the sphere.

Denote the event that the spherical floating body  $K_{[\operatorname{Vol}_{\mathbb{S}^d}(K)(c \log n)/n)]}$  is contained in  $K_n$  by  $T_n$  for a suitable constant c that will be specified later. There is a constant  $\delta$  such that the probability of  $T_n^c$ , i.e., the complement of  $T_n$ , is at most  $n^{-\delta c}$ . This can be seen as follows.

Suppose that the floating body has a point x in  $K \setminus K_n$ . If there is a cap C containing x whose boundary great sphere supports  $K_n$ , then C does not contain any of the random points. If this is not the case, we show that there is a sufficiently large empty part of the cap C. We may assume that x is in the plane  $x_{d+1} = 0$ . Thus, the coordinate surfaces of the standard spherical polar coordinate system through x cut C into  $2^d$  pieces. If all of these parts contained a random point, then their convex hull would contain x. Thus, there must be at least one part of C that is empty and its volume is at least constant times that of the cap C. However, the definition of floating body yields  $\operatorname{Vol}_{\mathbb{S}^d}(C) \geq \operatorname{Vol}_{\mathbb{S}^d}(K)(c \log n)/n)$ . Thus, the probability of  $T_n^c$  is at most  $(1 - (\delta c \log n)/n)^n$ , which tends to  $n^{-\delta c}$  as  $n \to \infty$ . In the Euclidean case, more precise asymptotics of probabilities are proven in [BD97].

We estimate the variance from above by the Efron-Stein jackknife inequality [ES81]. We choose c to be sufficiently large, so the conditional expectation on the event  $T_n^c$  can be omitted.

$$\operatorname{Var}\operatorname{Vol}_{\mathbb{S}^{d}}(K_{n}) \ll n \cdot \mathbb{E}(\operatorname{Vol}_{\mathbb{S}^{d}}(K_{n+1}) - \operatorname{Vol}_{\mathbb{S}^{d}}(K_{n}))^{2} \\ \ll n \cdot \mathbb{E}[(\operatorname{Vol}_{\mathbb{S}^{d}}(K_{n+1}) - \operatorname{Vol}_{\mathbb{S}^{d}}(K_{n}))^{2} \mathbf{1}(T_{n})].$$
(4)

For an index set  $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ , let  $F_I$  denote the spherical convex hull of  $x_{i_1}, \ldots, x_{i_d}$ , which is a (d-1)-dimensional spherical simplex with probability 1. Note that  $K_{n+1} \setminus K_n$  is either empty or a union of spherical simplices determined by  $x_{n+1}$  and the facets of  $K_n$  that can be seen from  $x_{n+1}$ . We say that a facet F of  $K_n$  is visible from a point  $x \notin K_n$ , if the open geodesic arc between x and a point in the relative interior of F is disjoint from  $K_n$ . Let  $\mathcal{F}(x_{n+1})$  denote the set of facets of  $K_n$  that are visible from  $x_{n+1}$ , and let  $A_I$  denote the event that  $F_I \in \mathcal{F}(x_{n+1})$ .

Then we can estimate (4) as follows.

$$(4) = \frac{n}{\operatorname{Vol}_{\mathbb{S}^{d}}(K)^{n+1}} \int_{K^{n+1}} \left( \sum_{F \in \mathcal{F}(x_{n+1})} \operatorname{Vol}_{\mathbb{S}^{d}} \left( \operatorname{conv}\left[F, x_{n+1}\right] \right) \right)^{2} \mathbf{1}(T_{n}) \, \mathrm{d}x_{1} \dots \mathrm{d}x_{n+1} \\ \ll \frac{n}{\operatorname{Vol}_{\mathbb{S}^{d}}(K)^{n+1}} \int_{K^{n+1}} \left( \sum_{I} \mathbf{1}(A_{I})V_{+}(F_{I}) \right)^{2} \mathbf{1}(T_{n}) \, \mathrm{d}x_{1} \dots \mathrm{d}x_{n+1} \\ \ll \frac{n}{\operatorname{Vol}_{\mathbb{S}^{d}}(K)^{n+1}} \sum_{I,J} \int_{K^{n+1}} \mathbf{1}(A_{I})V_{+}(F_{I})\mathbf{1}(A_{J})V_{+}(F_{J})\mathbf{1}(T_{n}) \, \mathrm{d}x_{1} \dots \mathrm{d}x_{n+1},$$
(5)

where  $V_+(F_I)$  is the volume of the spherical cap  $C(F_I)$  determined by  $F_I$ , and the summation goes through all *d*-tuples *I* and *J*, so *I* and *J* might intersect. Fix the number of common elements of *I* and *J* to *k*.

For any given  $k \in \{0, 1, \dots, d\}$  let  $I = \{1, \dots, d\}$  and  $J = \{d - k + 1, \dots, 2d - k\}$ and set  $F = \operatorname{conv} \{x_i : i \in I\}$  and  $G = \operatorname{conv} \{x_j : j \in J\}$ . The corresponding terms in (5) are independent of the choice of  $i_1, \ldots, i_d$  and  $j_1, \ldots, j_d$ . Thus,

$$(5) \leq \frac{n}{\operatorname{Vol}_{\mathbb{S}^d}(K)^{n+1}} \sum_{k=0}^d \binom{n}{d} \binom{d}{k} \binom{n-d}{d-k} \int_{K^{n+1}} \mathbf{1}(F \in \mathcal{F}(x_{n+1})) V_+(F) \times \mathbf{1}(G \in \mathcal{F}(x_{n+1})) V_+(G) \mathbf{1}(T_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n+1}.$$
(6)

Due to the symmetric roles of F and G, we may restrict the integration to those pairs of F and G for which diam $(F) \ge \text{diam}(G)$ , where diam $(\cdot)$  denotes the spherical diameter of a set. Thus,

$$(6) \ll \frac{1}{\operatorname{Vol}_{\mathbb{S}^d}(K)^{n+1}} \sum_{k=0}^d n^{2d-k+1} \int_{K^{n+1}} \mathbf{1}(F \in \mathcal{F}(x_{n+1})) V_+(F) \times \mathbf{1}(G \in \mathcal{F}(x_{n+1})) V_+(G) \mathbf{1}(\operatorname{diam}(F) \ge \operatorname{diam}(G)) \mathbf{1}(T_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n+1}$$

Since C(F) and C(G) have at least  $x_{n+1}$  in common, replacing  $\mathbf{1}(G \in \mathcal{F}(x_{n+1}))$  to  $\mathbf{1}(C(F) \cap C(G) \neq \emptyset)$  in the integrand does not decrease the integral.

We estimate the terms in the sum separately for each k = 0, ..., d, denoted by  $\Sigma_k$ .

$$\Sigma_k \ll \frac{n^{2d-k+1}}{\operatorname{Vol}_{\mathbb{S}^d}(K)^{n+1}} \int_{K^{n+1}} \mathbf{1}(F \in \mathcal{F}(x_{n+1})) V_+(F) \mathbf{1}(C(F) \cap C(G) \neq \emptyset) V_+(G) \times \mathbf{1}(\operatorname{diam}(F) \ge \operatorname{diam}(G)) \mathbf{1}(T_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n+1}.$$
(7)

If  $F \in \mathcal{F}(x_{n+1})$  then all the points  $x_{2d-k+1}, \ldots, x_n$  must be contained in  $K \setminus C(F)$ and  $x_{n+1}$  has to be contained in C(F). Then we can integrate with respect to  $x_{2d-k+1}, \ldots, x_n, x_{n+1}$  and the condition  $T_n$  can be replaced by the condition  $W_n = \{V_+(F) \leq \operatorname{Vol}_{\mathbb{S}^d}(K)(c \log n)/n\}.$ 

$$(7) \ll n^{2d-k+1} \int_{K^{2d-k}} \left( 1 - \frac{V_+(F)}{\operatorname{Vol}_{\mathbb{S}^d}(K)} \right)^{n-2d+k} V_+(F)^2 \mathbf{1}(C(F) \cap C(G) \neq \emptyset) \times V_+(G) \mathbf{1}(\operatorname{diam}(F) \ge \operatorname{diam}(G)) \mathbf{1}(W_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_{2d-k}.$$
(8)

According to Lemma 1, the gnomonic images  $\overline{C(F)}$  and  $\overline{C(G)}$  are not too distorted. Although, the relation of their diameters may be reversed, their ratio is bounded by a constant depending only on K. Thus, there exists a constant  $\overline{\gamma}$  with the following property. Let  $C_{\overline{\gamma}}$  be the cap with the same vertex as  $\overline{C(F)}$  and height equal to that of  $\overline{C(F)}$  times  $\overline{\gamma}$ . Then the diameter of  $C_{\overline{\gamma}}$  is at least as large as that of  $\overline{C(G)}$ , and  $C_{\overline{\gamma}}$  is still sufficiently small. Then  $V(\overline{C(F)}) \approx V(C_{\overline{\gamma}})$ . We showed above that the conditions of the rolling and the sliding balls are preserved by g. In [BSF24], it was proved (see (8) and its proof on pp. 6–7) that under these conditions,  $V(\overline{C(G)}) \ll V(C_{\overline{\gamma}})$ , and, in turn,  $V(\overline{C(G)}) \ll V(\overline{C(F)})$ . Projecting these back to the sphere, we obtain  $V_+(G) \ll V_+(F)$ . Hence,

$$\int_{K^{d-k}} \mathbf{1}(C(F) \cap C(G) \neq \emptyset) \mathbf{1}(\operatorname{diam}(F) \ge \operatorname{diam}(G)) \times V_{+}(G) \mathbf{1}(W_{n}) \, \mathrm{d}x_{d+1} \dots \mathrm{d}x_{2d-k}$$
$$\ll V_{+}(F)^{d-k+1}. \tag{9}$$

Based on (9), we may estimate (8) from above.

$$(8) \ll n^{2d-k+1} \int_{K^d} \left( 1 - \frac{V_+(F)}{\operatorname{Vol}_{\mathbb{S}^d}(K)} \right)^{n-2d+k} V_+(F)^{d-k+3} \mathbf{1}(W_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_d.$$
(10)

Now we use the spherical economic cap covering theorem. Due to the condition  $W_n$ , every cap C(F) has volume at most  $\operatorname{Vol}_{\mathbb{S}^d}(K)(c\log n)/n$ . Let h be a (positive) integer with  $2^{-h} \leq (c\log n)/n$ . For each such h, let  $\mathcal{C}_h$  be a collection of caps  $\{C_1, \ldots, C_{m(h)}\}$  forming the economic cap covering of the wet part of K with  $t = \operatorname{Vol}_{\mathbb{S}^d}(K)2^{-h}$  (we assume that n is sufficiently large). Then  $\operatorname{Vol}_{\mathbb{S}^d}(C_i) \ll \operatorname{Vol}_{\mathbb{S}^d}(K)2^{-h}$ . Consider an arbitrary  $(x_1, \ldots, x_d)$  with the corresponding C(F) having volume at most  $\operatorname{Vol}_{\mathbb{S}^d}(K)(c\log n)/n$ , and associate with  $(x_1, \ldots, x_d)$  the maximal h such that for some  $C_i \in \mathcal{C}_h$ ,  $C(F) \subset C_i$ . Such an h clearly exists. Then

$$V_+(F) \le \operatorname{Vol}_{\mathbb{S}^d}(C_i) \ll 2^{-h}.$$

On the other hand, by the maximality of h,

$$\frac{V_+(F)}{\operatorname{Vol}_{\mathbb{S}^d}(K)} \ge 2^{-(h+1)}$$

Let us now calculate the integral over  $K^d$  under the condition  $W_n$  by integrating each  $(x_1, \ldots, x_d)$  on its associated  $C_i$ . We can estimate the integrand in (10) as

$$\left(1 - \frac{V_+(F)}{\operatorname{Vol}_{\mathbb{S}^d}(K)}\right)^{n-2d+k} V_+(F)^{d-k+3} \ll \left(1 - 2^{-(h+1)}\right)^{n-2d+k} \cdot 2^{-h(d-k+3)},$$

which yields that the integral on  $(C_i)^d$   $(C_i \in \mathcal{C}_h)$  is bounded by

$$\exp\left(-(n-2d+k)\cdot 2^{-(h+1)}\right)\cdot 2^{-h(d-k+3)}\cdot \operatorname{Vol}_{\mathbb{S}^d}(C_i)^d \\ \ll \exp\left(-(n-2d+k)\cdot 2^{-(h+1)}\right)\cdot 2^{-h(d-k+3)}\cdot 2^{-hd}.$$

The last piece needed for the proof is to calculate the number of elements of  $C_h$ . By (3), the volume of the wet part of K with parameter  $2^{-h}$  is  $\operatorname{Vol}_{\mathbb{S}^d}(K(2^{-h})) \approx (2^{-h})^{\frac{2}{d+1}}$ . (The  $\approx$  notation makes sense, since  $h \to \infty$  as  $n \to \infty$ ). Therefore,

$$|\mathcal{C}_h| \ll \frac{2^{-\frac{2h}{d+1}}}{2^{-h}} = 2^{\frac{h(d-1)}{d+1}}$$

Assembling the pieces estimating (10), we obtain with  $h_0 = \lfloor c \log n/n \rfloor$ ,

$$(10) \ll n^{2d-k+1} \sum_{h=h_0}^{\infty} \exp\left(-(n-2d+k) \cdot 2^{-(h+1)}\right) \cdot 2^{-h(d-k+3)} \cdot 2^{-hd} \cdot |\mathcal{C}_h|$$
$$= n^{2d-k+1} \sum_{h=h_0}^{\infty} \exp\left(-(n-2d+k) \cdot 2^{-(h+1)}\right) \cdot 2^{-h\left(2d-k+1+\frac{d+3}{d+1}\right)}.$$
(11)

The sum in (11) can be calculated the same way as in [BFV10, p. 617], its order of magnitude is  $n^{-2d+k-1}n^{-\frac{d+3}{d+1}}$ , therefore

$$\Sigma_k \ll n^{2d-k+1} n^{-2d+k-1} n^{-\frac{d+3}{d+1}} = n^{-\frac{d+3}{d+1}}.$$

This holds for all k = 0, ..., d, thus the sum of them is of the same order of magnitude, which proves the theorem.

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